MIGSAA advanced Ph.D. course

Two-dimensional statistical hydrodynamics

Tadahiro Oh

Spring 2019

Contents:

LECTURES 1-3

ANDREIA CHAPOUTO

Contents

1. Navier-Stokes equations

Consider a fluid moving with velocity $\vec{u}$ and f a property of the flow. One can understand the change in f in two ways, depending on the coordinates,

$$\text{Euler coordinates:} \quad \frac{\partial f}{\partial t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t, x) - f(t,x)}{\Delta t},$$

$$\text{Lagrange coordinates:} \quad \frac{\mathrm{D} f}{\mathrm{D} t} := \lim_{\Delta t \to 0} \frac{f(t+\Delta t, x + \vec{u}\Delta t) - f(t,x)}{\Delta t}.$$

The first represents the usual time derivative while the second captures the change of f with respect to the flow. It is often called *material derivative*, but can also be mentioned as *advective*, *hydrodynamic*, *Lagrangian* or *Stokes derivative*.

In this course, we focus on the *incompressible Navier-Stokes equations* (NSE), with $u = (u_1, u_2, u_3) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ the velocity field and $p : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ the pressure, satisfying

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \Delta u + f, \quad t > 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}, \tag{1.1}$$

with a forcing f.

The Navier-Stokes equations are of great interest in physics, used to describe the motion of viscous fluids. They are also of great interest in mathematics and have been extensively studied.

Note that the left-hand side of the first equation in (1.1) corresponds to the material derivative of u, as $\frac{\mathrm{D}u}{\mathrm{D}t} = \partial_t u + (u \cdot \nabla)u$, while the second equation imposes the incompressibility of the fluid. Moreover, the system has 4 equations and 4 unknowns, and can be written as follows

$$\begin{cases} \partial_t u_j + \sum_{k=1}^{3} u_k \partial_k u_j = -\partial_j p + \Delta u_j + f_j, \quad j = 1,2,3 \\ \sum_{k=1}^{3} \partial_k u_k = 0. \end{cases}$$

We start by introducing the Helmhotlz decomposition in order to simplify the equation.

Definition 1.1 (Helmhotlz decomposition)**.** *Let $u \in H^s(\mathbb{R}^d)$, $s \geq 0$. Then, there exist functions $A : \mathbb{R}^d \to \mathbb{R}^d$ and $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that*

$$u = \underbrace{\nabla \times A}_{\text{divergence free}} + \underbrace{\nabla \phi}_{\text{curl free}} .$$

This is called the Helmholtz decomposition of u.

Remark 1.2. (i) A similar decomposition can be defined on the torus, with the addition of a harmonic term, the *Hodge decomposition*. Such term can be removed by imposing the mean zero condition to u.
(ii) The Helmhotlz decomposition in $L^p(\mathbb{R})$ for $p > 2$ requires more care.

We want to apply the Helmholtz decomposition to the initial data u_0.

Let v_0 divergence free and w_0 such that u_0 has the following Helmholtz decomposition $u_0 = v_0 + \nabla w_0$.

Taking divergence of u_0, we obtain

$$-\Delta w_0 = -\mathrm{div} u_0 \implies w_0 = -\nabla(-\Delta)^{-1}\nabla \cdot u_0.$$

Therefore, we can write the divergence free part as follows

$$\begin{aligned} v_0 &= u_0 - \nabla w_0 \\ &= (\mathrm{Id} + \nabla(-\Delta)^{-1}\nabla \cdot)u_0 \\ &:= \Pi u_0, \end{aligned}$$

with the operator Π denoted as the *Leray projection*. Component-wise, v_{0j}, for $j = 1,2,3$, is defined as follows

$$\begin{aligned} v_{0j} &= \sum_{k=1}^{3} (\delta_{jk} + \partial_j(-\Delta)^{-1}\partial_k) u_{0k} \\ &= \mathcal{F}^{-1}\Bigg(\sum_{k=1}^{3} \bigg(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \bigg) \widehat{u}_{0k}(\xi) \Bigg). \end{aligned}$$

Remark 1.3. Recall that the Riesz transform is defined as $\frac{i\xi_j}{|\xi|}$ on the Fourier side, and can be interpreted as the higher dimensional analogue of the Hilbert transform,

$$\begin{aligned} \mathrm{H}f(x) &= \text{p.v.} \int \frac{f(x-y)}{y} dy \ : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \quad 1 < p < \infty, \\ \mathcal{F}(\mathrm{H}f)(\xi) &= i\mathrm{sgn}(\xi)\widehat{f}(\xi). \end{aligned}$$

In addition,

$$\begin{aligned} R_j f(x) &= \int \frac{x_j - y_j}{|x-y|^{d+1}} f(y) \ fy, \quad R_j : L^p \to L^p, \quad 1 < p < \infty, \\ \sum_{j=1}^{d} R_j^2 &= -\mathrm{Id}. \end{aligned}$$

We want to apply the Leray projection to the equation and study only u, not p. For simplicity, assume that u is divergence free.

Applying the Leroy projection to (1.1) gives

$$\begin{cases} \partial_t u + \Pi\big((u\cdot\nabla)u\big) = Lu + \Pi f \\ \Pi u = u \\ u|_{t=0} = u_0 \end{cases}, \tag{1.2}$$

since $\Pi(\nabla p) = 0$ and defining $L := \Pi\Delta$.

If u is a solution of NSE, then it satisfies the following *Duhamel formulation* (or *mild formulation*)

$$u(t) = e^{tL}u_0 - \int_0^t e^{(t-t')L}\Pi\big((u\cdot\nabla)u\big)(t') \ dt' + \int_0^t e^{(t-t')L}\Pi f(t') \ dt'. \tag{1.3}$$

Proposition 1.4 (L^p-L^q estimate)**.** *Let $1 \leq p \leq 1 \leq \infty$, the following estimates hold*

$$\big\|e^{t\Delta}f\big\|_{L^q_x} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p_x}, \tag{1.4}$$

$$\big\|D^\alpha(e^{t\Delta}f)\big\|_{L^q_x} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\alpha}{2}}\|f\|_{L^p_x}, \tag{1.5}$$

for all $t > 0$, $\alpha \geq 0$.

The previous estimates hold on the real line and in the periodic setting. On the real line, we will use a scaling argument. However, we require a different approach on the torus, namely the Poisson summation formula.

Lemma 1.5. *Let f be a periodic function. Then the following holds*

$$\sum_{n\in\mathbb{Z}^d} \widehat{f}(n)e^{in\cdot x} = \sum_{n\in\mathbb{Z}^d} f(n+x).$$

Proof. Let

$$F(x) = \sum_{n\in\mathbb{Z}^d} f(x+n) = \sum_{n\in\mathbb{Z}^d} \widehat{F}(n)e^{in\cdot x},$$

where

$$\begin{aligned}\widehat{F}(n) &= \int_{\mathbb{T}^d} F(x)e^{-in\cdot x}\ dx \\ &= \int_{\mathbb{T}^d} \sum_{m\in\mathbb{Z}^d} f(x+m)e^{-in\cdot x}\ dx \\ &= \sum_{m\in\mathbb{Z}^d}\int_{\mathbb{T}^d} f(y)e^{-in\cdot(y-m)}\ dy \\ &= \int_{\mathbb{R}^d} f(x)e^{-in\cdot x}\ dx - \widehat{f}(n).\end{aligned}$$

□

Proof of Proposition 1.4. Since

$$e^{t\Delta}f(x) = \int K_t(x-y)f(y)\ dy,$$

where the kernel K_t is given by a Gaussian, we have

$$\left\|e^{t\Delta}f\right\|_{L^q_x} \lesssim \|K_t\|_{L^r_x}\|f\|_{L^p_x},$$

with $\frac{1}{q}+1=\frac{1}{r}+\frac{1}{p}$ using Young's inequality.

Now we want to evaluate the norm of K_t.

We first show the estimate on the real line. Since

$$\widehat{K}_t(\xi) = e^{-t|\xi|^2} = \widehat{K}\big(t^{\frac{1}{2}}\xi\big),$$

for $K := K_1$, it follows that

$$K_t(x) = \frac{1}{t^{\frac{d}{2}}}K\left(\frac{x}{t^{\frac{1}{2}}}\right),$$

hence

$$\|K_t\|_{L^r_x} = t^{-\frac{d}{2}}\left\|K\left(\frac{x}{t^{\frac{1}{2}}}\right)\right\|_{L^r_x} = t^{-\frac{d}{2}}t^{\frac{d}{2}\frac{1}{r}}C_k \sim t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

To show the second estimate, assume that $D=\sqrt{-\Delta}$, otherwise the proof follows by scaling as before. We have that

$$D^\alpha\big(e^{t\Delta}f\big) = D^\alpha(K_t * f) = \big(D^\alpha K_t\big) * f.$$

Similarly, on the Fourier side, we have

$$\mathcal{F}\{D^\alpha K_t\}(\xi) = |\xi|^\alpha e^{-t|\xi|^2} = t^{-\frac{\alpha}{2}}\underbrace{\big(t^{\frac{1}{2}}|\xi|\big)^\alpha e^{-t|\xi|^2}}_{\widehat{G}_t}.$$

Let $G=G_1\in\mathcal{S}(\mathbb{R}^d)$, $G_t(x)=t^{-\frac{d}{2}}G\Big(\frac{x}{t^{\frac{1}{2}}}\Big)$. It follows that

$$\left\|D^\alpha K_t\right\|_{L^r_x} = t^{-\frac{\alpha}{2}}\|G_t\|_{L^r_x} = t^{-\frac{\alpha}{2}}t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, \tag{1.6}$$

and the result follows.

Now focus on the periodic setting. We cannot use a scaling argument, thus we must use the Poisson summation formula (1.6).

In this case we have $e^{t\Delta}f = K_t * f$, with $\widehat{K}_t(n) = e^{-t|n|^2}$. Thus, we want to estimate the L^r-norm of K_t. Using the Poisson summation formula (1.6)

$$\begin{aligned}\|K_t\|_{L^r_x(\mathbb{T}^d)} &= \Big\| \sum_n \widehat{K}_t(n) e^{in\cdot x} \Big\|_{L^r_x} \\ &= \Big\| \sum_n K_t(x+n) \Big\|_{L^r_x}.\end{aligned}$$

Using Hölder's inequality, it follows that

$$\begin{aligned}\|K_t\|_{L^r_x(\mathbb{T}^d)} &\lesssim \Big\| \Big(\sum_n \langle n\rangle^{-\beta r'} \Big)^{\frac{1}{r'}} \|\langle n\rangle^{\beta} K_t(x+n)\|_{\ell^r_n} \Big\|_{L^r_x(\mathbb{T}^d)} \\ &\lesssim \|\langle x\rangle^{\beta} K_t(x)\|_{L^r_x(\mathbb{R}^d)} \\ &\lesssim \frac{1}{t^{\frac{d}{2}}} \Big\| \langle x t^{-\frac{1}{2}}\rangle^{\beta} K\Big(\frac{x}{t^{\frac{1}{2}}}\Big) \Big\|_{L^r_x(\mathbb{R}^d)} = \tilde{c}_k t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)},\end{aligned}$$

for $\beta r' > d$.

A similar computation holds for $\|D^\alpha K_t\|_{L^r_x(\mathbb{T}^d)}$. □

Remark 1.6. The estimate on the torus is only valid for $0 < t \leq 1$.
On the real line, for $t \gg 1$, $e^{-t|\xi|^2}$ has exponential decay, but weaker for $|\xi| \ll 1$. However, on the torus we cannot expect decay without imposing the mean zero condition.

The following linear estimates follow from Proposition 1.4

Corollary 1.7. *Let $1 < p \leq q < \infty$ (or $1 < p < q = \infty$). Then the following estimate holds*

$$\left\| D^\alpha e^{tL} f \right\|_{L^q_x} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\alpha}{2}} \|f\|_{L^p_x}. \tag{1.7}$$

We now focus on the scaling invariance of the equation. Let (u,p) be a solution to (1.1) and $\lambda > 0$. Then, (u^λ, v^λ) defined as follows

$$\begin{cases} u^\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x) \\ p^\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x) \end{cases},$$

is also a solution. Note that

$$\|u^\lambda\|_{L^q_t L^r_x(\mathbb{R}_+\times\mathbb{R}^d)} = \lambda^{1-\frac{d}{r}-\frac{2}{q}} \|u\|_{L^q_t L^r_x}.$$

Hence the scaling invariant indices are given by $\frac{2}{q}+\frac{d}{r} = 1$ for u and $\frac{2}{q}+\frac{d}{r} = 2$ for p.

For instance, for $d = 3$, $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$, where

$$\begin{aligned}\|f\|_{\dot{H}^s} &= \Big(\int |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \Big)^{\frac{1}{2}}, \\ \|f\|_{W^{s,p}} &= \|\mathcal{F}^{-1}(\langle\xi\rangle^s \widehat{f})\|_{L^p_x}.\end{aligned}$$

The following quantity is conserved for (1.1)

$$\int |u(T)|^2 dx + \int_0^T \int |\nabla u(t)|^2 \, dx \, dt = \int |u_0|^2 \, dx,$$

however it is too weak to control the $L^3_x(\mathbb{R}^3)$-norm.

2. Small data global well-posedness

In this section, we show global well-posedness of homogeneous NSE in $L^3(\mathbb{R}^3)$ and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Thus, let $f \equiv 0$.

Theorem 2.1. (i) *There exists $\delta > 0$ such that if $\|u_0\|_{L^3_x(\mathbb{R}^3)} < \delta$, then there exists a unique solution u to* (1.1) *in $C([0,\infty); L^3_x) \cap C((0,\infty); W^{1,3}_x)$. Furthermore, the flow map depends continuously on the initial data.*
(ii) *The same result holds in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.*

To prove local well-posedness, we want to use the mild formulation to define the solution map

$$\Gamma(u)(t) = e^{tL}u_0 - \int_0^t e^{(t-t')L}\Pi\big((u\cdot\nabla)u\big)(t')\ dt'$$

and show that it is a contraction.

Using Proposition 1.4

$$\begin{aligned}
\|\Gamma u(t)\|_{L^3_x} &\le C\|u_0\|_{L^3_x} + C\int_0^t (t-t')^{-\frac{1}{2}}\|(u\cdot\nabla)u(t')\|_{L^{\frac{3}{2}}_x}dt' \\
&\le C\|u_0\|_{L^3_x} + C\int_0^t (t-t')^{-\frac{1}{2}}\|u(t')\|_{L^3_x}\|\nabla u(t')\|_{L^3_x}dt' \\
&\le C\|u_0\|_{L^3_x} + C\int_0^t (t-t')^{-\frac{1}{2}}(t')^{-\frac{1}{2}}dt'\|u\|_{L^\infty_t((0,t);L^3_x)}\sup_{t'\in(0,t)}(t')^{\frac{1}{2}}\|\nabla u(t')\|_{L^3_x}.
\end{aligned}$$

Recall the definition of the *beta function*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt,$$

with $\mathrm{Re}(p), \mathrm{Re}(q) > 0$. Note that

$$\int_0^t (t-t')^{-\frac{1}{2}}(t')^{-\frac{1}{2}}dt' = \int_0^1 (1-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}d\tau = B\left(\frac{1}{2},\frac{1}{2}\right) < \infty.$$

It remains to estimate the $W^{1,3}$-norm, using Proposition 1.4 with $p=2, q=3$,

$$\|\nabla\Gamma u(t)\|_{L^3_x} \le Ct^{-\frac{1}{2}}\|u_0\|_{L^3_x} + \int_0^t (t-t')^{-\frac{3}{4}}\|(u\cdot\nabla)u(t')\|_{L^2_x}dt'.$$

Focusing on the second term, we have

$$\begin{aligned}
\|(u\cdot\nabla)u(t')\|_{L^2_x} &\lesssim \|u(t')\|_{L^6_x}\|\nabla u(t')\|_{L^3_x} \\
&\lesssim \||\nabla|^{\frac{1}{2}}\|_{L^3_x}\|\nabla u(t')\|_{L^3_x} \\
&\lesssim \|u(t')\|^{\frac{1}{2}}_{L^3_x}\|\nabla u(t')\|^{\frac{3}{2}}_{L^3_x}.
\end{aligned}$$

Hence,

$$t^{\frac{1}{2}}\|\nabla\Gamma u(t)\|_{L^3_x} \le C\|u_0\|_{L^\infty_t([0,\infty);L^3_x)} + Ct^{\frac{1}{2}}\int_0^t (t-t')^{-\frac{3}{4}}(t')^{-\frac{3}{4}}dt'\big(\sup_{t'}\|u(t')\|_{L^3_x}\big)^{\frac{1}{2}} \cdot \big(\sup_{t'}(t')^{\frac{1}{2}}\|\nabla u(t')\|_{L^3_x}\big)^{\frac{1}{2}}.$$

Let $X = C_t\big([0,\infty);L^3_x\big)\cap L^\infty_t\big((0,\infty);W^{1,3}_x\big)$, restricted to divergence free functions, for simplicity. Consider the spaces defined by the following norms

$$\|u\|_Y := \|u\|_{L^\infty((0,\infty);L^3_x)},$$
$$\|u\|_Z := \sup_{t\in(0,\infty)} t^{\frac{1}{2}}\|\nabla u(t)\|_{L^3_x}.$$

One can also define the spaces restricted to the time interval $[0,T]$, for some $T>0$,

$$\|u\|_{Y_T} := \inf\big\{\|v\|_Y:\ v\in Y,\ v|_{[0,T]=u}\big\},$$

with infimum taken over all extensions $v\in Y$ of u. The spaces X_T and Z_T are defined in a similar manner.

2.1. **Small data global well-posedness in $L^3(\mathbb{R}^3)$.** We already showed

$$\|\Gamma u\|_Y \le C_0\|u_0\|_{L^3_x} + C_1\|u\|_Y\|u\|_Z,$$
$$\|\Gamma u\|_Z \lesssim \|u_0\|_{L^3_x} + \|u\|_Y^{\frac{1}{2}}\|u\|_Z^{\frac{3}{2}}.$$

Similarly,

$$\|\Gamma u-\Gamma v\|_Y \lesssim \|u-v\|_Y\|u\|_Z + \|v\|_Y\|u-v\|_Z,$$
$$\|\Gamma u-\Gamma v\|_Z \lesssim \|u-v\|_Y^{\frac{1}{2}}\|u-v\|_Z^{\frac{1}{2}}\|u\|_Z + \|v\|_Y^{\frac{1}{2}}\|v\|_Z^{\frac{1}{2}}\|u-v\|_Z.$$

Let $\|u_0\|_{L^3_x}\ll 1$, and consider a closed ball of radius η, $B_\eta\subset X$, for $\eta = 10C_0\|u_0\|_{L^3_x}\ll 1$.
Then, combining the previous estimates

$$\|\Gamma u\|_X \le 2C_0\|u_0\|_{L^3_x} + C_1\|u\|_X^2 < \eta,$$
$$\|\Gamma u-\Gamma v\|_X \le C_2\big(\|u\|_X+\|v\|_X\big)\|u-v\|_X \le 2C_2\eta\|u-v\|_X,$$

hence we must choose $2C_2\eta\le\frac{1}{2}$.

Using Banach fixed point theorem, there exists u such that $\Gamma u = u$ in $B_\eta\subset X$, and small data global well-posedness in $L^3(\mathbb{R}^3)$ follows.

Remark 2.2. Uniqueness in $L^3_x(\mathbb{R}^3)$ follows from a continuity argument.
To show uniform continuity, the same estimates give $\|u-v\|_X\lesssim\|u_0-v_0\|_{L^3_x}$.

It remains to compute pressure p from the solution u. Note that

$$\partial_t u - \Delta u + (u\cdot\nabla)u - \nabla p = 0 \implies \nabla p = \partial_t u - \Delta u + (u\cdot\nabla)u =: G(t).$$

Hence,

$$\Pi\big(G(t)\big) = \partial_t u - Lu - \Pi\big((u\cdot\nabla)u\big)$$
$$\implies G(t) = \text{ curl free } = \nabla\phi$$
$$\implies p = \phi + \text{const}.$$

2.2. **Small data global well-posedness in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.** Focus on the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$-norm of the solution map

$$\begin{aligned}\|\Gamma u(t)\|_{\dot{H}^{\frac{1}{2}}} &\leq C\|u_0\|_{\dot{H}^{\frac{1}{2}}} + C\int_0^t (t-t')^{-\frac{1}{2}}\|(u\cdot\nabla)u(t')\|_{L_x^{\frac{3}{2}}}dt' \\ &\leq C\|u_0\|_{\dot{H}^{\frac{1}{2}}} + C\int_0^t (t-t')^{-\frac{1}{2}}\|u(t')\|_{L_x^3}\|\nabla u(t')\|_{L_x^3}dt' \\ &\leq C\|u_0\|_{\dot{H}^{\frac{1}{2}}} + C\int_0^t (t-t')^{-\frac{1}{2}}\|u(t')\|_{\dot{H}^{\frac{1}{2}}}\|\nabla u(t')\|_{L_x^3}dt'.\end{aligned}$$

Moreover,

$$\|\Gamma u(t)\|_{\dot{H}^{\frac{1}{2}}} \leq C\|u_0\|_{\dot{H}^{\frac{1}{2}}} + C\int_0^t (t-t')^{-\frac{1}{2}}(t')^{-\frac{1}{2}}dt'\|u\|_{L_t^\infty \dot{H}^{\frac{1}{2}}}\|\nabla u\|_Z.$$

Similarly,

$$\begin{aligned}\|\nabla\Gamma u(t)\|_{L_x^3} &\leq CT^{-\frac{1}{2}}\big\||\nabla|^{-\frac{1}{2}}u_0\big\|_{L_x^2} + \|u\|_{L_t^\infty L_x^3}^{\frac{1}{2}}\|u\|_Z^{\frac{3}{2}} \\ &\lesssim CT^{-\frac{1}{2}}\big\||\nabla|^{-\frac{1}{2}}u_0\big\|_{L_x^2} + \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}^{\frac{1}{2}}\|u\|_Z^{\frac{3}{2}},\end{aligned}$$

from Sobolev inequality.

Therefore, running a contraction mapping argument in $\tilde{X} = C_t\dot{H}^{\frac{1}{2}} \cap Z$ yields small data global wll-posedness of (1.1) in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.

3. Large data local well-posedness

We now focus on showing local well-posedness of NSE for large data. We start by showing the following lemma.

Lemma 3.1. *Let $1 \leq p \leq q \leq \infty$, $\alpha \geq 0$, K compact in L^p. Then, there exists $F(t) : (0,1] \to \mathbb{R}_+$, such that $\lim_{t\to 0^+} F(t) = 0$ and*

$$t^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{\alpha}{2}}\|D^\alpha e^{t\Delta}f\|_{L^q} \leq F(t),$$

$\forall t \in (0,1]$, $\forall f \in K$.

Proof. Suppose $K = \{f\}$ and let $\theta := \frac{d}{2}\big(\frac{1}{p} - \frac{1}{q}\big) + \frac{\alpha}{2}$. Then,

$$\begin{aligned}t^\theta\|D^\alpha e^{t\Delta}f\|_{L^q} &\leq t^\theta\|D^\alpha e^{t\Delta}(f-g)\|_{L^q} + t^\theta\|D^\alpha e^{t\Delta}g\|_{L^q} \\ &\lesssim \|f-g\|_{L^p} + t^\theta\|D^\alpha e^{t\Delta}g\|_{L^q},\end{aligned}$$

for all $g \in \mathcal{S}$.

Given $j \geq 1$, there exists $g_j \in \mathcal{S}$ such that

$$\begin{aligned}t^\theta\|D^\alpha e^{t\Delta}f\|_{L^q} &\leq \frac{1}{2j} + t^\theta\|D^\alpha e^{t\Delta}g_j\|_{L^q} \\ &\leq \frac{1}{j},\end{aligned}$$

for all $0 < t \leq t_j$.

Hence, let $F(t) = \inf_j \left(\frac{1}{j}\right) + t$.

Now, consider the general case. Given j,

$$K \subset \bigcup_{k=1}^{N_j} B_{\frac{1}{2j}}(g_k^j)$$

for some $g_k^j \in \mathcal{S}$. Then,

$$t^\theta \|D^\alpha e^{t\Delta} f\|_{L^q} \leq \frac{1}{2j} + t^\theta \|D^\alpha e^{t\Delta} g_k^j\|_{L^q},$$

for $f \in B_{\frac{1}{2j}}(G_k^j)$.

It follows that

$$t^\theta \|D^\alpha e^{t\Delta} f\|_{L^q} \leq \frac{1}{2j} \max_k (t^\theta \|D^\alpha e^{t\Delta} g_k^j\|_{L^q}),$$

for all $f \in K$. Then, take infimum in j to define $F(t)$. □

3.1. **Large data local well-posedness for in L_x^3.** In order to show local well-posedness we want to run a contraction mapping argmument in $X_T = Y_T \cap Z_T$ on $B_{R,\eta} = \big\{\|u\|_{Y_T} \leq R, \|u\|_{Z_T} \leq \eta\big\}$.

We have seen that

$$\|\Gamma u\|_{Y_T} \leq C_0 \|u_0\|_{L_x^3} + C_1 \|u\|_{Y_T} \|u\|_{Z_T},$$

but this is not enough fo the difference estimate. Hence, consider the following modified estimate

$$\begin{aligned}
\|\Gamma u\|_{Y_T} &\leq C_0 \|u_0\|_{L_x^3} + C \int_0^t (t-t')^{-\frac{1}{4}} \|u \cdot \nabla(t')\|_{L_x^2} dt' \\
&\leq C_0 \|u_0\|_{L_x^3} + C_1 \underbrace{\int_0^t (t-t')^{-\frac{1}{4}} (t')^{-\frac{3}{4}} dt'}_{B\left(\frac{3}{4},\frac{3}{4}\right)<\infty} \|u\|_{Y_T}^{\frac{1}{2}} \|\nabla u\|_{Z_T}^{\frac{3}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\Gamma u\|_{Y_T} &\leq C_0 \|u_0\|_{L_x^3} + \|u\|_{Y_T}^{\frac{1}{2}} \|u\|_{Z_T}^{\frac{3}{2}}, \\
\|\Gamma u\|_{Z_T} &\leq \|e^{tL} u_0\|_{Z_T} + \|u\|_{Y_T}^{\frac{1}{2}} \|u\|_{Z_T}^{\frac{3}{2}}.
\end{aligned}$$

Let $u \in B_{R,\eta}$, with $R = 2C_0 \|u_0\|_{L_x^3}$, thus

$$\|\Gamma u\|_{Y_T} \leq \frac{1}{2} R + C_1 R^{\frac{1}{2}} \eta^{\frac{3}{2}} \leq R.$$

By Lemma 3.1, there exists $T = T(u_0) > R$ such that $\|e^{tL} u_0\|_{Z_T} \leq \frac{1}{2}\eta$, which implies that

$$\|\Gamma u\|_{Z_T} \leq \frac{1}{2}\eta + C_2 R^{\frac{1}{2}} \eta^{\frac{3}{2}} \leq \eta,$$

therefore $\Gamma u \in B_{R,\eta}$, with $\eta = \eta(R) = \eta(\|u_0\|_{L_x^3}) \ll 1$.

Regarding the difference estimate, we have

$$\begin{aligned}
\|\Gamma u - \Gamma v\|_{X_T} &\leq C\big(R^{\frac{1}{2}} + \eta^{\frac{1}{2}}\big)\eta^{\frac{1}{2}} \|u - v\|_{X_T} \\
&\leq \frac{1}{2} \|u - v\|_{X_T},
\end{aligned}$$

by choosing $\eta = \eta(R) \ll 1$. Thus, using Banach fixed point argument, local well-posedness in $L^3_x(\mathbb{R}^3)$ (or $\mathbb{T}^3$) follows.

3.2. **Large data local well-posedness in $\dot{H}^{\frac{1}{2}}_x$.** Similarly, let $\tilde{Y}_T = C_T \dot{H}^{\frac{1}{2}}_x$ and $\tilde{X}_T = \tilde{Y}_T \cap Z_T$.

It remains to estimate the $\tilde{Y}_T$-norm,

$$\|\Gamma u\|_{\tilde{Y}_T} \leq C_0 \|u_0\|_{\dot{H}^{\frac{1}{2}}} + C \int_0^t (t-t')^{-\frac{1}{4}} \|u \cdot \nabla u(t')\|_{L^2_x} dt'.$$

The integral term is controlled as follows

$$\begin{aligned} \int_0^t (t-t')^{-\frac{1}{4}} \|u \cdot \nabla u(t')\|_{L^2_x} dt' &\lesssim \|u\|^{\frac{1}{2}}_{Y_T} \|u\|^{\frac{3}{2}}_{Z_T} \\ &\lesssim \|u\|^{\frac{1}{2}}_{\tilde{Y}_T} \|u\|^{\frac{3}{2}}_{Z_T}, \end{aligned}$$

using Sobolev inequality. The result follows from previous arguments.

4. Navier-Stokes equations with forcing

We want to extend the analysis to the non homogeneous NSE. Therefore, consider the forced NSE equation, with deterministic or stochastic forcing,

$$\text{forced NSE:} \quad \partial_t u - \Delta u + (u \cdot \nabla) u - \nabla p = f, \quad f \text{ deterministic}, \tag{4.1}$$

$$\text{stochastic NSE:} \quad \partial_t u - \Delta u + (u \cdot \nabla) u - \nabla p = \zeta, \tag{4.2}$$

with ζ stochastic forcing, white in time (or kick force in time), smooth in x.

The stochastic forcing ζ is defined as follows

$$\zeta = \phi \xi,$$

with ξ space-time white noite and ϕ a smoothing operator in x, for example Hilbert-Shmidt from L^2_x to H^S_x.

Before proceeding, we must introduce some stochastic analysis.

4.1. **Basic stochastic analysis.** Let $W(t) = (W^1(t), \dots, W^d(t))$ denote a L^2-cylindrical Wiener process, with

$$W^j(t,x) = \sum_{n \in \mathbb{Z}^d} \beta^j_n e^{in\cdot x},$$

where $\{\beta^j_n\}_{\substack{n \in \mathbb{Z}^d \\ j \in \{1,\dots,d\}}}$ a family of independent complex-valued Brownian motions (BM),

$$\beta^j_n = \operatorname{Re}(\beta^j_n) + i \operatorname{Im}(\beta^j_n),$$

with real and imaginary parts independent real-valued BMs.

Let $(\Omega, \mathcal{F}, P)$ a probability space. A Brownian motion (BM) B on $\mathbb{R}_+$ is a stochastic process such that

(i) $B(0) = 0$, a.s.

(ii) $B(t) - B(s) \sim \mathbb{N}(0, t-s)$, $t > s$.

(iii) independent increment on disjoint time intervals: $B(t_1) - B(s_1), B(t_2) - B(s_2)$ are independent, $t_2 > s_2 > t_1 > s_2$.

The Brownian motion B satisfy the following properties.

- $\mathbb{E}\big(|B(t)-B(s)|^{2k}\big) = \frac{(2k)!}{2^k k!}(t-s)^k$
- $\mathbb{E}\big(|B(t)-B(s)|^{p}\big) \sim_p |t-s|^{\frac{p}{2}}$, with implicit constant $C_p \le p^{\frac{p}{2}}$

We will need the following result.

Lemma 4.1 (Kolmogorov continuity criterion)**.** *Let $\{X_t\}$ a stochastic process with values in a metric space S. Suppose there exists $p \ge 1, \alpha > 0$ such that $\mathbb{E}(d(X_t, X_s)^p) \lesssim |t-s|^{1+\alpha}$ for all t, s. Then,*

$$P\left(\sup_{t\neq s} \frac{d(X_t, X_0)}{|t-s|^{\frac{\alpha}{p}-\gamma}} \ge \lambda\right) \le \frac{C}{\lambda^p}, \quad \forall 0<\gamma<\frac{\alpha}{p},$$

which implies that X_t is a.s. $\big(\frac{\alpha}{p}-\varepsilon\big)$-Hölder continuous. In particular, a.s. continuous.

Remark 4.2. *The proof follows from the Borel-Cantelli Lemma.*

Since $\mathbb{E}\big(|B(t)-B(s)|^p\big) \lesssim |t-s|^{1+\left(\frac{p}{2}-1\right)}$ for all finite p, using Kolmogorov's continuity criterion, we get $\frac{\alpha}{p} = \frac{1}{2} - \frac{1}{p} \to \frac{1}{2}$ as $p \to \infty$, which implies that BM is a.s. $\frac{1}{2}$-Hölder continuous.

We want to study the regularity of Brownian motion. Therefore, we must introduce the following notation and function spaces. Let $j \in \mathbb{Z}$ and

$$Q_j(f) = \int \varphi\left(\frac{|\xi|}{2j}\right)\widehat{f}(\xi)e^{i\xi\cdot x}d\xi,$$

for some nice bump function $\varphi \in C_c^\infty$, supported on $\big[\frac{1}{2}, 2\big]$, with $\sum_{j\in\mathbb{Z}} \phi\Big(\frac{|\xi|}{2j}\Big)$. Moreover, let $p_j = Q_j$ for $j \ge 1$ and $p_0 = \sum_{j\le 0} Q_j$ the projection onto $\{|\xi| \lesssim 1\}$.

Consider the Besov spaces defined by the norm

$$\|f\|_{B^s_{p,q}} = \big\|2^{js}\|p_j(f)\|_{L^p_x}\big\|_{\ell^q_j(\mathbb{Z}_{\ge 0})},$$

when $p=q=2$ corresponds to $B^s_{2,2} = H^s$. In addition, we can introduce the homogeneous space with the norm

$$\|f\|_{\dot{B}^s_{p,q}} = \big\|2^{js}\|Q_j(f)\|_{L^p_x}\big\|_{\ell^q_j(\mathbb{Z}_{\ge 0})}.$$

Note that for $0<s<1$, $\dot{C}^s = \dot{B}^s_{\infty,\infty}$ and $C^s = \dot{C}^s \cap L^\infty = B^s_{\infty,\infty}$.

Now, we can focus on the regularity of Brownian motion. Locally in time BM $\in B^{\frac{1}{2}-}_{\infty,\infty} \supset W_t^{\frac{1}{2}-,p}$, $1 \le p \le \infty$, and BM $\in B^{\frac{1}{2}-}_{p,q}$ for $1 \le p \le q \le \infty$.

Regarding the covariance, we have

$$\mathbb{E}\big(B(t)B(s)\big) = t \wedge s,$$
$$\mathbb{E}\big((B(t)-B(s))B(s)\big) + \mathbb{E}(B^2(s)) = s, \quad t > s.$$

Recall the definition of the *Wiener integral*

$$I(f) = \int_a^b f(t)\ dB(t),$$

$f \in L^2([a,b])$ deterministic.

Step 1: Step function $f(t) = \sum_{j=1}^n a_{j-1}\mathbf{1}_{[t_{j-1},t_j]}(t)$, with deterministic functions a_j. Define $I(f) = \sum_{j=1}^n a_{j-1}(B(t_j) - B(t_j - t_{j-1}))$. Then,

$$\begin{aligned}
\mathbb{E}\big(I(f)\big) &= 0,\\
\mathbb{E}\big(I(f)\big)^2 &= \sum_{j=1}^n\sum_{k=1}^n a_{j-1}a_{k-1}\mathbb{E}\big((B(t_j) - B(t_j - t_{j-1}))(B(t_k) - B(t_{k-1}))\big)\\
&= \sum_{j=1}^n a_{j-1}^2(t_j - t_{j-1})\\
&= \|f\|^2_{L^2([a,b])}.
\end{aligned}$$

Step 2: General $f \in L^2([a,b])$. Approximate f by step functions f_n in $L^2([a,b])$ and define $I(f) = \lim\limits_{n\to\infty} I(f_n)$. Thus, the conditions on the mean and variance apply and $I : L^2([a,b]) \to L^2(\Omega)$ is an isometry (onto the image).

Remark 4.3. If B is complex-valued,

$$\mathbb{E}|I(f)|^2 = 2\|f\|^2_{L^2([a,b])}.$$

4.2. **Deterministic forcing.** Consider NSE with deterministic forcing f

$$\begin{cases}\partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = f\\ \text{div} u = 0\\ u|_{t=0} = u_0\end{cases}.$$

The Duhamel formula gives

$$u(t) = \Gamma_{u_0,f}u(t) = e^{tL}u_0 - \int_0^t e^{(t-t')L}\Pi\big((u\cdot\nabla)u\big)(t')\,dt' + \int_0^t e^{(t-t')L}\Pi f(t')\,dt'.$$

Denote the last term by $F(t)$. Considering the previous analysis, it suffices to control F in the relevant norms. Namely,

$$\begin{aligned}
&\|F\|_{Y_T} \le \|f\|_{L^1_T L^2_x},\\
&\|F\|_{\tilde{Y}_T} \le \|f\|_{L^1_T \dot{H}^{\frac{1}{2}}_x},\\
&\|F\|_{Z_T},
\end{aligned}$$

where the last one must be made small by choosing $T \ll 1$.

Note that

$$t^{\frac{1}{2}}\left\|\nabla\int_0^t e^{(t-t')L}\Pi f(t')\,dt'\right\|_{L^3_x} \lesssim t^{\frac{1}{2}}\int_0^t (t-t')^{-\frac{1}{2}}\|f(t')\|_{L^3_x}dt'.$$

This quantity can be controlled by the two following quantities

$$\begin{aligned}
&\text{(LHS)} \lesssim t^{\frac{1}{2}}\int_0^t (t-t')^{-\frac{1}{2}}(t')^{-\frac{1}{2}}dt' \sup_{t'\in(0,t)}\|f(t')\|_{L^3_x},\\
&\text{(LHS)} \lesssim \|f\|_{L^q_T L^2_x},
\end{aligned}$$

for some $q \ge 2$.

Remark 4.4. We can take a rougher forcing, by imposing higher integrability in time, note that

$$\|F\|_{Y_T} \lesssim \|f\|_{L^q_T W_x^{-2+,3}}, \quad q \gg 1$$
$$\|F\|_{\tilde{Y}_T} \lesssim \|f\|_{L^q_T H_x^{-\frac{3}{2}+}}.$$

4.3. **Stochastic Navier-Stokes equations.** Consider NSE with stochastic forcing

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = \phi\xi \\ \text{div} u = 0 \\ u|_{t=0} = u_0 \end{cases},$$

with ξ space-time white noise and ϕ a smoothing operator in x.

Applying Π to the equation gives

$$\partial_t u - Lu + \Pi\big((u\cdot\nabla)u\big) = \Pi(\phi\xi).$$

Consider the mild formulation

$$u(t) = e^{tL}u_0 - \int_0^t e^{(t-t')L}\Pi\big((u\cdot\nabla)u\big)(t')\,dt' + \Pi\bigg(\int_0^t e^{(t-t')L}\phi\,dW(t')\bigg),$$

denoting the last therm as $\Psi = (\Psi_1, \Psi_2, \Psi_3)$, the stochastic convolution.

In the periodic setting, we define the stochastic convolution as follows

$$\Psi_j = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} e^{in\cdot x}\int_0^t e^{-(t-t')|n|^2}\phi_n\,d\beta_n^j(t')dt',$$

where $\beta_{-n} = \bar{\beta}_n^j$ and $\phi_{-n} = \phi_n$, for $j = 1, 2, 3$.

For simplicity, we will drop the j in the following.

Proposition 4.5. *Let $\phi \in HS(L^2; H^s)$/ Then,*

$$\Psi \in C_t^{\frac{\alpha}{2}-}W_x^{s+1-\alpha,r}(\mathbb{T}^d),\ \text{a.s.}\ , \quad r \le \infty$$
$$\Psi \in C_t W_x^{s+1-\varepsilon,r}(\mathbb{T}^d).$$

Proof. Let $t \le \tau$. Calculating the space-time covariance

$$\begin{aligned}
&\mathbb{E}\big(\Psi(t,x)\overline{\Psi(\tau,y)}\big) \\
&\quad = \mathbb{E}\Bigg(\bigg(\sum_n e^{in\cdot x}\int_0^t e^{-(t-t')|n|^2}\phi_n\,d\beta_n^j(t')dt'\bigg)\bigg(\sum_m e^{im\cdot y}\int_0^\tau e^{-(\tau-t')|n|^2}\phi_m\,d\beta_m^j(t')dt'\bigg)\Bigg) \\
&\quad = 2\sum_n e^{in\cdot(x-y)}|\phi_n|^2\int_0^t e^{-(t-t')|n|^2}e^{(\tau-t')|n|^2}\,dt' \\
&\quad = \sum_{n\neq 0} e^{in\cdot(x-y)}\frac{|\phi_n|^2}{|n|^2}\underbrace{\Big(e^{(t-\tau)|n|^2} - e^{-(t+\tau)|n|^2}\Big)}_{C_n(t,\tau)\le 1}.
\end{aligned}$$

□

Applying $\langle\nabla_x\rangle^{s+1}$, $\langle\nabla_y\rangle^{s+1}$

$$\mathbb{E}\big(\langle\nabla_x\rangle^{s+1}\Psi(t,x)\langle\nabla_y\rangle^{s+1}\Psi(\tau,y)\big) \lesssim \sum_n e^{in\cdot(x-y)}\langle n\rangle^{2s}|\phi_n|^2 C_n(t,\tau).$$

Setting $t=\tau$, $x=y$, *since* Ψ *is a Gaussian random variable, we have that*

$$\begin{aligned}\mathbb{E}\big(|\langle\nabla\rangle^{s+1}\Psi(t,x)|^p\big) &\leq p^{\frac{p}{2}}\mathbb{E}\big(|\langle\nabla\rangle^{s+1}\Psi(t,x)|^2\big)^{\frac{p}{2}}\\ &\leq p^{\frac{p}{2}}\|\phi\|^p_{HS(L^2;H^s)}.\end{aligned}$$

Let $r<\infty$. *Then, for all* $r\leq p<\infty$

$$\begin{aligned}\big\|\|\Psi(t)\|_{W^{s+1,r}_x}\big\|_{L^p(\Omega)} &\leq \big\|\|\langle\nabla\rangle^{s+1}\Psi(t,x)\|_{L^p_x(\Omega)}\big\|_{L^r_x(\mathbb{T}^d)}\\ &\lesssim p^{\frac{1}{2}}\|\phi\|_{HS(L^2;H^s)}.\end{aligned}$$

For $r=\infty$, *use Sobolev inequality in* x

$$\|\Psi(t)\|_{W^{s+1-\varepsilon,\infty}_x} \lesssim \|\Psi(t)\|_{W^{s+1,r}_x}, \quad r<\infty.$$

Fix $t>0$, *then*

$$\begin{aligned}&\Psi(t)\in W^{s+1,r}_x \ \text{a.s.} \quad r<\infty\\ &\Psi(t)\in W^{s+1-\varepsilon,\infty}_x \ \text{a.s.}.\end{aligned}$$

Given $h\in\mathbb{R}$ *such that* $t+h>0$, $\delta_h\Psi(t,x)=\Psi(t+h,x)-\Psi(t,x)$. *Then,*

$$\begin{aligned}\mathbb{E}\big(\delta_h\Psi(t,x)\delta_h\Psi(t,y)\big) &= E\big(\Psi(t+h,x)\Psi(t+h,y)\big) - E\big(\Psi(t+h,x)\Psi(t,y)\big)\\ &\quad - E\big(\Psi(t,x)\Psi(t+h,y)\big) + E\big(\Psi(t,x)\Psi(t,y)\big)\\ &= \sum_n e^{in\cdot(x-y)}\frac{|\phi_n|^2}{|n|^2}\big(C_n(t+h,t+h)-C_n(t+h,t)\\ &\qquad\qquad - C_n(t,t+h)+C_n(t,t)\big).\end{aligned}$$

Note that

$$\begin{aligned}|C_n(t+h,t+h)-C_n(t+h,t)| &= \big|1-e^{-2(t+h)|n|^2}-e^{-h|n|^2}+e^{-(2t+h)|n|^2}\big|\\ &= \Big|\big(1-e^{-h|n|^2}\big)\big(1+e^{-(2t+h)|n|^2}\big)\Big|\\ &\lesssim |h|^\alpha|n|^{3\alpha},\end{aligned}$$

using mean value theorem, for all $\alpha\in[0,1]$.

It follows that

$$\begin{aligned}\big\|\langle\nabla\rangle^{s+1-\alpha}\delta_h\Psi(t,x)\big\|_{L^p(\Omega)} &\leq p^{\frac{1}{2}}\big\|\langle\nabla\rangle^{s+1-\alpha}\delta_h\Psi(t,x)\big\|_{L^2(\Omega)}\\ &\lesssim p^{\frac{1}{2}}|h|^{\frac{\alpha}{2}}\|\phi\|_{HS(L^2;H^s)}.\end{aligned}$$

Hence,

$$\big\|\|\delta_h\Psi(t)\|_{W^{s+1-\alpha,r}_x}\big\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}}|h|^{\frac{\alpha}{2}}\|\phi\|_{HS(L^2;H^s)}.$$

Using the Kolmogorov continuity criterion, we have $\psi\in C_t^{\frac{\alpha}{2}-}W^{s+1-\alpha,r}_x$, *a.s.* $r<\infty$ *(and* $r=\infty$*).*

Remark 4.6. *In the periodic setting, from the previous results it follows that SNSE is locally well-posed in* $\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)$*, since* $\Psi \in \tilde{Y}_T = L^\infty H_x^{\frac{1}{2}}$ *and* $\Psi \in Z_T$*. Similarly, it is locally well-posed in* $L_x^3(\mathbb{T}^3)$*, as* $\Psi \in Y_T$*.*

TWO-DIMENSIONAL STATISTICAL HYDRODYNAMICS

GUOPENG LI

Contents

1. Lecture 3

Notation 1.1. Define the function space

$$\mathcal{H} := L^2(\mathbb{T}^2; \mathbb{R}^2),$$

which is the space of divergence free and mean 0 functions.
Define $\mathbb{Z}^2_+ := \{(n_1, n_2) : n_1 > 0 \text{ or } n_1 = 0 \text{ and } n_2 > 0\}$, note that $\mathbb{Z}^2_+ \cup (-\mathbb{Z}^2_+) = \mathbb{Z}^2 \setminus \{0\}$.
Next, we define the orthonormal basis on $\mathcal{H}$:

$$e_n := \begin{cases} c_n n^\perp sin(nx), n \in \mathbb{Z}^2_+ \\ c_n n\perp cos(nx), n \in -\mathbb{Z}^2_- \end{cases}$$

where $c_n = \frac{1}{\sqrt{2}\pi|n|}$, $n = (n_1, n_2)$ and $n^\perp = (-n_2, n_1)$.
Finally, we define the following function space:

$$\mathcal{H}_T := \{u \in L^2_T H^1, \partial_t \in L^2_T H^{-1}\},$$

with the norm

$$\|u\|_{\mathcal{H}_T} := \big(\|u\|_{L^2_T H^1} + \|\partial_t u\|_{L^2_T H^{-1}}\big)^{\frac{1}{2}}.$$

Notice, if we have $u \in \mathcal{H}_T$, then we have the mapping

$$t \mapsto \langle \partial_t u(t), u(t)\rangle_{L^2_x} \in L^1_T.$$

Also, we have $\mathcal{H}_T \subset C_T L^2_x$ due to the following:

$$\int_0^T \langle \partial_t u, u\rangle_{L^2_x} dt \leq \|\partial_t u\|_{L^2_T H^{-1}_x} \|u\|_{L^2_T H^1_x},$$

which implies $\|u(t)\|^2_{L^2}$ is (absolute) continuous, therefore the we have claim above.

1.1. **Compactness.** All we need now is some kind of compact embedding theorem, of the type of the Rellich Lemma but for vector valued functions.

We first see one lemma, which will used in our proof of the compactness proposition.

Lemma 1.1. *Given $\varepsilon > 0$, there exists $c_\varepsilon > 0$ so that for all $x \in X_1$, we have*

$$\|x\|_0 \leq \varepsilon\|x\|_1 + c_\varepsilon\|x\|_{-1}$$

Proof. Suppose for a contradiction, if we do NOT have the claim. There exist $\{x_n\} \subset x_1$ such that

$$\|x_n\|_0 \geq \varepsilon\|x_n\|_1 + c_\varepsilon\|x_n\|_{-1}.$$

Let $y_n = \frac{x_n}{\|x_n\|_1}$. Then,

$$\begin{aligned} &\|y_n\|_0 \geq \varepsilon + n\|y_n\|_{-1}, \\ &\|y_n\|_1 = 1. \end{aligned} \tag{1.1}$$

By assumption X_1 is separable and reflexive, we have some ball $B_1 \subset X_1$ is weakly compact. Hence, there exist subsequence, we denote by y_n so that

$$y_n \rightharpoonup y \quad \text{in } X_1.$$

Since the inclusion $X_1 \subset\subset X_0$ is compact, $y_n \to y$ in X_0. By (1.1) we have

$$\|y_n\|_{-1} \leq \frac{1}{n}\|y_n\|_0 \leq \frac{c}{n} \to 0.$$

So we have $y = 0$, but by (1.1) again $\|y\|_0 \geq \varepsilon$. We arrive a contradiction. □

Proposition 1.2. *Let X_1, X_0 and X_{-1} be three separable reflexive Banach spaces with $X_0 \subset\subset X \subset X1$, the inclusion $X_1 \subset\subset X_0$ is compact and the inclusion $X_0 \subset X_{-1}$ is continuous. Let u_n be a sequence of function satisfying*

$$\begin{aligned} &\{u_n\} \textit{ is bounded in } L_T^{p_1}X_1, \\ &\{\partial_t u_n\} \textit{ is bounded in } L_T^{p_2}X_{-1}, \end{aligned}$$

for $1 < p_1 < p_2 < \infty$. Then there exist subsequence u_{n_j} of u_n which is convergent in $L_T^{p_1}X_0$

Proof. Without loss of generality, we assume there exists subsequence $u_n \rightharpoonup 0$ in $L_T^{p_1}X_1$. The goal here is to show $u_n \to 0$ in $L_T^{p_1}X_0$ (strongly). We now claim: By Lemma 1.1 it suffices to show $u_n \to 0$ in $L_T^{p_1}X_{-1}$. If we suppose for now the claim is true, then

$$\begin{aligned} \int_0^T \|u_n\|_{X_0}^{p_1} dt &\leq \varepsilon \sup_n \int_0^T \|u_n(t)\|_{X_1}^{p_1} dt + C_\varepsilon p_1 \int_0^T \|u_n(t)\|_{X^{-1}}^{P_1} dt \\ &\leq C\varepsilon + o(1), \end{aligned}$$

as $n \to \infty$. This will implies $u_n \to 0$ in $L_T^{p_1}X_0$.
Now, we prove the claim. Let $I \subset \mathbb{R}$ be the time interval in $\mathbb{R}$, which is bounded. $L \in (X_1)^*$, $(X_1)^*$ denotes the dual of X_1, and $\chi_I(t)L \in (L_T^{p_1}X_1)^*$. Then, we have

$$\langle u_n, \chi_I L\rangle = \int_I \langle u_n(t), L\rangle dt = \langle \int_I u_n(t)dt, L\rangle,$$

for all L. Noticing that $\langle u_n, \chi_I L\rangle \to 0$, because $u_n \rightharpoonup 0$ in $L^{p_1}_T X_1$. So that we have $\int_I u_n(t)dt$ weakly convergent in X_1 to 0. Hence, by assumption $X_1 \subset\subset X_0$, up to a subsequence we have

$$\int_I u_n(t)dt \to 0 \text{ in } X_0, \tag{1.2}$$

therefore it is convergent in X_{-1}.
Fix $t \in [0,T]$ and write $u_n(t) - u_n(t_1) = \int_t^{t_1} \frac{du_n}{ds} ds$. Average in t_1 over $[t-\varepsilon, t]$, we have

$$u_n = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} u_n(t_1)dt_1 + \frac{1}{\varepsilon}\int_t^{t-\varepsilon} (s-t+\varepsilon)\frac{du_n}{ds}(s)ds.$$

Also, by Hölder

$$\left\| \frac{1}{\varepsilon}\int_t^{t-\varepsilon} (s-t+\varepsilon)\frac{du_n}{ds}(s)ds \right\|_{X_{-1}} \lesssim \varepsilon^{\frac{1}{p_2'}} \|\frac{du_n}{ds}\|_{L^{p_2}_T X_{-1}} \lesssim \varepsilon^{\frac{1}{p_2'}}.$$

Given $\varepsilon_0 > 0$, choose $\varepsilon > 0$ small so that $C\varepsilon^{\frac{1}{p_2'}} \le \frac{\varepsilon_0}{2}$. By (1.2) we obtain

$$\|u_n(t)\|_{X_{-1}} \le \frac{\varepsilon_0}{2} + \frac{1}{\varepsilon}\left\| \int_t^{t-\varepsilon} u_n(t_1)dt_1 \right\|_{X_{-1}} \to 0.$$

On the other hand, by the fundamental theorem of calculus we have

$$\|u_n(t_1) - u_n(t_2)\|_{X^{-1}} \le C|t_1 - t_2|^{\frac{1}{p_2'}}$$

for all $n \ge 1$. For fixed $\varepsilon > 0$,

$$\sup_{t\in[0,T]} \|u_n(t)\|_{X_{-1}} \le \max_{j=1,\dots,[\frac{T}{\varepsilon}]} \|u_n(j\varepsilon)\|_{X_{-1}} + C\varepsilon^{\frac{1}{p_2'}} \le \varepsilon_0$$

for all $n \ge N(\varepsilon_0)$. Then, implies

$$\sup_{t\in[0,T]} \|u_n(t)\|_{X_{-1}} \to 0,$$

therefore $u_0 \to 0$ in $L^{p_1}_T X_{-1}$. This we complete the proof. □

By using Proposition 1.2 we have

$$\mathcal{H}_T \subset\subset L^2_T H^s_x, \quad s < 1.$$

2. Lecture 4

In the remaining part of the notes, we will focus on $\mathbb{T}^2$.

2.1. Basic properties of the bilinear form. We first recall some basic properties of the function spaces. From Rellich compactness lemma, we have

$$H^1(\mathbb{T}^2) \subset\subset H^s, \quad s < 1.$$

We have $\mathcal{H}_T \subset C_T L^2_x$, then by Aubin-Lions compactness lemma we have

$$\mathcal{H}_T \subset\subset L^2_t H^s_x([0,T]\times\mathbb{T}^2), \quad -1 < s < 1.$$

Finally, we recall the bilinear form $B(u,v) = \Pi\big((u\cdot\nabla)v\big)$, we simply write $B(u) = B(u,u)$ and the Leary projection $L = \Pi\Delta$.

Proposition 2.1. *Let $u, v, w \in \mathcal{H} \cap C^\infty$, then we have*

$$(i) \quad \langle B(u,v), v\rangle_{L^2_x} = 0$$
$$(ii) \quad \langle B(u,v), w\rangle = -\langle B(u,w), v\rangle$$

Proof. By the divergent free of u, and integration by part we have

$$B(u,v), v\rangle_{L^2_x} = \int_{\mathbb{T}^2} u^j \partial_j (v^k v^k) dx = \frac{1}{2}\int_{\mathbb{T}^2} u^j \partial_j (|v|^2) dx$$
$$= -\frac{1}{2}\int_{\mathbb{T}^2} \partial_j u^j (|v|^2) dx = 0.$$

This finish the part (i). Then, by using part (i) and the bilinearity,

$$0 = \langle B(u, v+w), v+w\rangle = \langle B(u,v), w\rangle + \langle B(u,w), v\rangle.$$

This complete the proof. □

Proposition 2.2. *Let $u, v, w \in \mathcal{H} \cap C^\infty$, we then have*

$$(i)|\langle B(u,v), w\rangle| \lesssim \|u\|_{H_x^{\frac{1}{2}}} \|v\|_{H_x^{\frac{1}{2}}} \|w\|_{H^1_x}$$
$$(ii)\|B(u,v)\|_{H_x^{-1}} \lesssim \|u\|_{H_x^{\frac{1}{2}}} \|v\|_{H_x^{\frac{1}{2}}}$$

Proof. We first observe that the first inequality is true if and only if the second inequality is true, so we only prove the first one, by duality. By using Proposition 2.1, Sobolev embedding $H^{\frac{1}{2}}(\mathbb{T}^2) \subset L^4(\mathbb{T}^2)$, Hölder's inequality

$$|\langle B(u,v), w\rangle| = |\langle B(u,w), v\rangle| \leq \int_{\mathbb{T}^2} |u||\nabla w||v| dx$$
$$\lesssim \|u\|_{H_x^{\frac{1}{2}}} \|v\|_{H_x^{\frac{1}{2}}} \|u\|_{H^1_x}.$$

□

Proposition 2.3. *For $u, v \in \mathcal{H} \cap C^\infty$, we have*

$$\|B(u,v)\|_{H_x^{-3}} \lesssim \|u\|_{L^2_x} \|v\|_{L^2_x}.$$

Proof. For $w \in H^3_x$,

$$|\langle B(u,v), w\rangle| = |\langle B(u,w), v\rangle| = \int_{\mathbb{T}^2} |u| \cdot |\nabla w||v| dx$$
$$\lesssim \|u\|_{L^2_x} \|v\|_{L^2_x} \|w\|_{H_x^{2+}}.$$

□

Proposition 2.4. *Let $u, v \in \mathcal{H}_T$, then we have*

$$\int_0^T \langle Lu(t), v(t)\rangle dt = -\int_0^T \langle \nabla u(t), \nabla v(t)\rangle dt,$$

and

$$\int_0^T \langle \partial_t u(t), u(t)\rangle dt = \frac{1}{2}\big(\|u\|^2_{L^2_x} - \|u(0)\|^2_{L^2_x}\big)$$

Proof. Integration by parts to get the claim. □

Proposition 2.5. *Let $u_j \in \mathcal{H}_T$ for $j = 1, 2, 3$. Then the following mapping holds*

$$(u_1, u_2, u_3) \mapsto \langle B(u_1(t), u_2(t), u_3(t)\rangle.$$

Proof. We claim $\mathcal{H}_T \subset L^4_T H^{\frac{1}{2}}_x$.

$$\begin{aligned}\|u\|^4_{L^4_T H^{\frac{1}{2}}_x} &= \int_0^T \|u\|^4_{H^{\frac{1}{2}}_x} dt \\ &\quad (\text{using } \|u\|^4_{H^{\frac{1}{2}}_x} \lesssim \|u(t)\|^2_{L^2_x} \|u(t)\|^2_{H^1_x}) \\ &\lesssim \|u\|^2_{L^\infty_T L^2_x} \|u\|^2_{L^2_T H^1_x} \lesssim \|u\|^4_{\mathcal{H}_T}.\end{aligned}$$

By using above, and Hölder's inequality

$$\begin{aligned}\int_0^T \langle B(u_1(t), u_2(t)), u_3(t)\rangle dt &\lesssim \|u_1\|_{L^4_T H^{\frac{1}{2}}_x} \|u_2\|_{L^4_T H^{\frac{1}{2}}_x} \|u_3\|_{L^2_T H^1_x} \\ &\lesssim \|u_1\|_{\mathcal{H}_T} \|u_2\|_{\mathcal{H}_T} \|u_3\|_{\mathcal{H}_T}.\end{aligned}$$

□

2.2. **Global well-posedness in $L^2(\mathbb{T}^2)$ of the Navier-Stokes equations via the energy method.** We consider the following Navier-Stokes equations on $\mathbb{T}^2$.

$$\begin{cases}\partial_t u - Lu + B(u) = f \\ u|_{t=0} = u_0 \in L^2_{df},\end{cases} \tag{2.1}$$

where $f = \Pi f \in L^2_T H^{-1}_x$. We will prove equation (2.1) is globally well-posed on $\mathbb{T}^2$ by using energy method.

Theorem 2.6 (Global well-posedness on $\mathbb{T}^2$)**.** *Given $u_0 \in \mathcal{H} = L^2_{df}$, there exists a unique global solution u to the Navier-Stokes equations* (2.1) *with $u|_{t=0} = u_0$, $u \in \mathcal{H}_T$ for all $T > 0$ and*

$$\sup_{0\le t\le T}\left(\|u\|^2_{L^2_x} + \int_0^t \|u(t')\|^2_{H^1_x} dt'\right) \le \|u_0\|^2_{L^2_x} + \int_0^T \|f(t)\|^2_{H^{-1}} dt, \quad \forall T > 0. \tag{2.2}$$

Proof. We fix $T > 0$ and work on $[0, T]$.

Uniqueness:

Suppose there exist two solutions $u, v \in \mathcal{H}_T$. Let $w = u - v$, substitute into equation (2.1), we have

$$\partial_t w - Lw + B(w, w) + B(v, w) = 0.$$

Multiply by w and integrate in x, t. Noticing from Proposition 2.1 and Proposition 2.2 we have the following

$$|\langle B(v, w), w\rangle| = 0$$

and

$$\begin{aligned}|\langle B(w, u), w\rangle| = |\langle B(w, w), u\rangle| &\le c\|w\|^2_{H^{\frac{1}{2}}_x} \|u\|_{H^1_x} \\ &\le c\|w\|_{L^2_x} \|w\|_{H^1_x} \|u\|_{H^1_x} \\ &\le \frac{1}{2}\|w\|^2_{H^1_x} + c\|w\|^2_{L^2_x} \|u\|^2_{H^1_x},\end{aligned}$$

and implies

$$\frac{d}{dt}\|w\|_{L_x^2}^2 + 2\|w\|_{H_x^1}^2 = -2\langle B(w,u), w\rangle$$
$$\|w\|_{H_x^1}^2 + c\|w\|_{L_x^2}^2\|u\|_{H_x^1}^2.$$

Hence, we have

$$\frac{d}{dt}\|w\|_{L_x^2}^2 \le c\|w\|_{L_x^2}^2\|u\|_{H_x^1}^2,$$

then by using Gronwall's inequality, and $w(0) = 0$

$$\|w\|_{L_x^2}^2 \le exp\left[c\int_0^t \|u(t')\|_{H_x^1}^2 dt'\right]\|w(0)\|_{L_x^2}^2 = 0.$$

This finish the uniqueness.

Existence:

For the existence, we will split into two steps. First we will a priori bound as in (2.2); then we construct a argument based on Galerkin approximation, we pass to the limit by using our priori bound.

STEP 1:A priori bound

Suppose u is a smooth solution to NSE (2.1). Multiply by u and integrate we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L_x^2}^2 &= \langle \partial_t u, u\rangle \\ &= \langle Lu, u\rangle - \langle B(u), u\rangle + \langle f, u\rangle \\ &\le -\|u\|_{\dot{H}^1}^1 + \|u\|_{H_x^1}\|f\|_{H_x^{-1}}\end{aligned}$$

$$\le -\frac{1}{2}\|u\|_{H_x^1}^2 + \frac{1}{2}\|f\|_{H^{-1}}^2 dt'.$$

Integrating in time[1]

$$\|u(t)\|_{L_x^2}^2 + \int_0^t \|u(t')\|_{H_x^1}^2 dt' = \|u(0)\|_{L_x^2}^2 + \int_0^t \|f(t')\|_{H_x^{-1}}^2 dt'.$$

Taking the supreme over time on $[0,T]$, we get

$$\|u\|_{L_T^\infty L_x^2} + \|u\|_{L_T^2 H_x^1} \le C(u_0, f),$$

for some constant depending on u_0 and f. By Proposition 2.2,

$$\|B(u)\|_{H_x^{-1}} \lesssim \|u\|_{H_x^{\frac{1}{2}}}^2 \lesssim \|u\|_{L_x^2}\|u\|_{H_x^1},$$

and by using our equation (2.1)

$$\begin{aligned}\|\partial_t u\|_{L_T^2 H_x^{-1}} &\le \|u\|_{L_T^2 H_x^1} + \|u\|_{L_T^\infty L_x^2}\|u\|_{L_T^2 H_x^1} + \|f\|_{L_T^2 H_x^{-1}} \\ &\le C(u_0, f),\end{aligned}$$

where $C(u_0, f)$ is a non-decreasing function. That is $\|u\|_{\mathcal{H}_T} \le C(u_0, f)$.

---

[1]

$$\|u(t)\|_{L_x^2}^2 + 2\int_0^t \|\nabla u(t')\|_{L_x^2}^2 dt' = \|u(0)\|_{L_x^2}^2 + 2\int_0^t \langle f(t'), u(t')\rangle dt'$$

STEP 2: Galerkin approximation
We first define the projection

$$p_N : L^2_{df} \to E_N = \text{span}\{e_n : |n| \le N\}.$$

Then apply p_N to the equation (2.1), we have

$$\partial_t p_N u - L p_N + p_N B(u) = p_N f.$$

Hence, one can reduce equation (2.1) into finite dimensional system of ODEs on the "Fourier" point of view.

$$\begin{cases} \partial_t u_N - L u_N + p_N B(u_N) = p_N f \\ u_N|_{t=0} = p_N u_0. \end{cases} \tag{2.3}$$

By the Cauchy-Lipschitz theorem, there exists one unique locally in time solution u_N to (2.3). Blow-up alternative:

$$u_N \text{ exists on } [0,T]$$

or $\exists T_N < T$ so that

$$\lim_{t \to T_N^-} \|u_N(t)\|_{L^2_x} = +\infty.$$

Multiply the equation (2.3) by u_N and integrate, by noticing $\langle p_N B(u_N), u_N\rangle = \langle B(u_N), u_N\rangle$ and same computation as in step 1, we have

$$\sup_{N \ge 1} \left(\|u\|_{\mathcal{H}_T} + \|u_N\|_{L^\infty_T L^2_x} \right) \le C(u_0, f).$$

We have that u_N exists on $[0,T]$ and

$$u_{N_j} \rightharpoonup u \quad \text{in } \mathcal{H}_T.$$

In $L^2_T H^{-1}_x$ both

$$\partial_t u_{N_j} \rightharpoonup \partial_t u$$
$$L u_{N_j} \rightharpoonup L u.$$

By Proposition 1.2 (Aubin-Lions compactness lemma), there exists subsequence u_{N_j} so that

$$u_{N_j} \to u \quad \text{in } L^2_T H^{\frac{1}{2}}_x,$$

by Proposition 2.2, we have

$$B(u_{N_j}) \to B(u) \quad \text{in } L^2_T H^1_x.$$

By definition of p_{N_j}, we have

$$u_{N_j}(0) = p_{N_j} u_0 \to u_0 \quad \text{in } L^2_x,$$

and

$$p_{Nj} f \to f \quad \text{in } L^2_T H^{-1}_x.$$

Now, we consider the (2.3) with p_{N_j},

$$\partial_t u_{N_j} - L u_{N_j} + p_{N_j} B(u_{N_j}) = p_{N_j} f. \tag{2.4}$$

From (2.4), wo notice that we cannot take the limit on $p_{N_j}B(u_{N_j})$. Therefore, one needs to apply p_m for a fixed m on (2.4), then $N_j \geq m$ (which holds for $j \gg 1$), we have

$$\partial_t p_m u_{N_j} - Lp_m u_{N_j} + p_m B(u_{N_j}) = p_m f. \tag{2.5}$$

By taking the limit in j, we have convergence of (2.5) in $L^1_T H^{-1}_x$, that is

$$\partial_t p_m u - Lp_m u + p_m B(u) = p_m f. \tag{2.6}$$

Notice that equation (2.6) holds for any $m \geq 1$, therefore we can take the limit in m to obtain

$$\begin{cases} \partial_t u - Lu + B(u) = f \\ u|_{t=0} = u_0 \in L^2_x, \end{cases}$$

this complete the proof. □

Remark 2.7. We can also start with a given subsequence u_{N_j} of u_N, then we show there exists one subsequence $u_{N_{j_k}}$ of subsequence u_{N_j} so that $u_{N_{j_k}} \to u$, where u is independent of choice of subsequence u_{N_j}. Finally, we have $u_N \to u$.

For the energy bound. By weak convergence, we have

$$\|u\|_{L^2_T H^1_x} \leq \liminf_{j\to\infty} \|u_{N_j}\|_{L^2_T H^1_x}$$

and the definition of p_{N_j}. Also the weak* convergent,

$$\|u(t)\|_{L^\infty_T L^2_x} \leq \liminf_{j\to\infty} \|u_{N_j}\|_{L^\infty_T L^\infty_x}.$$

References

[1] P. Constantin, C. Foias, *Navier-Stokes Equatuin* University of Chicago Press, 1988.

LECTURES 5 AND 6: EXISTENCE OF AN INVARIANT MEASURE FOR THE KICK-FORCED AND WHITE-FORCED TWO DIMENSIONAL NAVIER-STOKES EQUATION

TYPED BY W. J. TRENBERTH

Abstract. These notes are based on lectures 5 and 6 of the course Two-dimensional statistical hydrodynamics, taught by Hiro Oh.

1. Lecture 5: Existence of an invariant measure for the kick-forced two dimensional Naiver-Stokes equation

Recall that $B(u) = B(u,u) = \Pi((u \cdot \nabla)u)$ is the nonlinearity of the Naiver-Stokes equation. In a previous lecture we used integration by parts to show

$$\langle B(u,v), v\rangle = 0. \tag{1.1}$$

In the following Lemma we prove a similar result.

Lemma 1.1. *Suppose $u \in H \cap H^2$ where $H = L^2_{df,mean\ 0}$. Then $\langle B(u), \Delta u\rangle = 0$.*

Proof. First we claim that for $u \in H^k(\mathbb{T}^2)$ such that div $u = 0$ there exists a function $\psi \in H^{k+1}(\mathbb{T}^2)$, unique up to a constant, such that $u = \text{curl}\ \psi := (-\partial_2\psi, \partial_1\psi)$. Indeed as div $u = 0$,

$$\partial_1 u_1 = -\partial_2 u_2. \tag{1.2}$$

This implies

$$u_1 = \int \partial_2 u_2\, dx_1 + c(x_2)$$
$$u_2 = \int \partial_1 u_1\, dx_2 + c(x_1)$$

where the first equation comes from integrating (1.2) with respect to x_1 and the second equation comes from integrating (1.2) with respect to x_2. Together these equations show

$$\psi = -\int\int \partial_1 u_1\, dx_1 dx_2 + C$$

which proves the claim.

Since $u \in H^2$, from the above claim, there exists $\psi \in H^3$ such that

$$u = \text{curl}\ \psi \tag{1.3}$$

Also recall that $B(u) \in L^2_{df}$ and

$$\|B(u)\|_{L^2} \lesssim \|u\|_{L^\infty}\|\nabla u\|_{L^2} \lesssim \|u\|_{H^2} \tag{1.4}$$

where in the last inequality we used the Sobolev embedding $H^s(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2)$ for $s > 1$.

The Helmholtz decomposition $H^k = H^k_{df} \oplus H^{k+1}_{curl\,free}$ then implies that there exists $p \in H^1$ such that

$$B(u) = (u \cdot \nabla)u - \nabla p. \tag{1.5}$$

In the two dimensional setting we have curl $(u_1, u_2) = \partial_1 u_2 - \partial_2 u_1$. This seems unusual at first since in the three dimensional setting the curl is a vector quantity. However these different definitions are easily reconciled as,

$$\text{curl}(u_1, u_2, 0) = (\partial_1 u_2 - \partial_2 u_1)\widehat{k}.$$

Using (1.3), (1.5), integration by parts, the fact that curl$\nabla p = 0$ and the vector calculus identity curl $((u \cdot \nabla)u) = (u \cdot \nabla)$curl u we have,

$$\begin{aligned}
\langle B(u), \Delta u\rangle &= \int ((u \cdot \nabla)u - \nabla p) \cdot \text{curl } \Delta\psi \, dx \\
&= -\int (\text{curl } (u \cdot \nabla)u - \text{curl } \nabla p) \cdot \Delta\psi \, dx \\
&= -\int \text{curl } (u \cdot \nabla)u \cdot \Delta\psi \, dx \\
&= -\int (u \cdot \nabla) \text{ curl } u \cdot \Delta\psi \, dx.
\end{aligned}$$

From (1.3),

$$\begin{aligned}
\text{curl } u &= \text{curl curl } \psi \\
&= \text{curl } (-\partial_2\psi, \partial_2\psi) \\
&= \partial_2^2\psi + \partial_1^2\psi \\
&= \Delta\psi.
\end{aligned}$$

Hence using integration by parts,

$$\begin{aligned}
\langle B(u), \Delta u\rangle &= -\int (u \cdot \nabla) \text{ curl } u \cdot \Delta\psi \, dx \\
&= -\frac{1}{2}\int u_1\partial_1(\Delta\psi)^2) + u_2\partial_2(\Delta\psi)^2) \, dx \\
&= \frac{1}{2}\int (\partial_1 u_1 + \partial_2 u_2)(\Delta\psi)^2 \, dx \\
&= 0.
\end{aligned}$$

□

Random kick forcing. We study the two-dimensional kick forced Naiver-Stokes equation (Kick NSE),

$$\partial_t u + B(u) = Lu + \sum_{k=1}^{\infty} \eta_k^\omega \delta(t - kT). \tag{1.6}$$

Here $\eta_k^\omega = \eta_k^\omega(x)$ is a random function in x.

Before we can go deep into the study of Kick NSE, we need to introduce some probabilistic notions.

Definition 1.2. *A filtration $\{\mathcal{F}_t\}_{t\in I}$ is an increasing family of $\sigma-$algebras. A stochastic process $\{X_t\}_{t\in I}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t\in I}$ if for each t, X_t is $\mathcal{F}_t-$measurable.*

For Kick NSE, we work with the filtration

$$\mathcal{F}_t = \mathcal{F}_{(k-1)T} = \mathcal{F}_{k-1} \quad \text{for } t \in I_k := [(k-1)T, kT)$$

and we choose $\mathcal{F}_k = \sigma(\eta_j, j = 1, 2, .., k)$. Note that η_k is $\mathcal{F}_k$-measurable.

Definition 1.3. *A stochastic process $u(t)$, $t \geq 0$ is called a solution to Kick NSE if it's adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and almost surely statisfies the following*

(1) *For all $k \in \mathbb{N}$, $u \in \mathcal{H}(I_k)$ is a solution to NSE (with $f = 0$).*
(2) *For all $k \in \mathbb{N}$, $u(kT+) - u(kT-) = \eta_k$.*

On I_k the initial condition for Kick NSE is

$$u((k-1)T+) = u_0 + \sum_{j=1}^{k-1} \eta_j^\omega. \tag{1.7}$$

So solving Kick NSE is equivalent to solving

$$u(t) = u_0 + \sum_{j=1}^{k-1} \eta_j^\omega + \int_0^t (B(u) + Lu)(t')dt', \text{ for all } t \in I_k.$$

Hence from the L^2-GWP of NSE we have the following global existence result for Kick NSE.

Theorem 1.4. *Suppose that $\eta_k \in H := L^2_{df}$ almost surely for all $k \in \mathbb{N}$. Then Kick NSE is globally well-posed in $\mathcal{H} = \{u \in L^2_T H^1_{df} : \partial_t u \in L^2_T H^{-1}_{df}\}$.*

Remark 1.5. *We can take u_0 to be random ($\mathcal{F}_0$-measurable) and GWP still works.*

In the following we assume the kick forces are of the form

$$\eta_k = \sum_{n\in\mathbb{Z}^2\setminus\{0\}} b_n g_{kn} e_n$$

where $\{e_n\}_{n\in\mathbb{Z}^2}$ is an orthonormal basis for $H := L^2_{df}$. We set $B_s = \sum_{n\in\mathbb{Z}^2\setminus\{0\}} |n|^{2s} b_n^2$. We further place the following assumptions on $\{g_{kn}\}_{n\in\mathbb{Z}^2\setminus\{0\},k\in\mathbb{N}}$

(1) $\{g_{kn}\}_{n\in\mathbb{Z}^2\setminus\{0\},k\in\mathbb{N}}$ is a family of independent identically distributed random variables.
(2) $|g_{nk}(\omega)| \leq 1$ for all n, k and for all $\omega \in \Omega$.
(3) $P(|g_{nk}| \leq \varepsilon) > 0$ for all $\varepsilon > 0$ (so $|g_{kn}|$ has a nice density).
(4) $\|\eta_k(\omega)\|^2_{L^2} = \sum_{n\in\mathbb{Z}^2\setminus\{0\}} b_n^2 g_{kn}^2(\omega) \leq \sum_{n\in\mathbb{Z}^2\setminus\{0\}} b_n^2 = B_0 < \infty$.

These assumptions can be considerably weakened, however they simplify the coming argument.

Combining the third and fourth assumptions with the pigeon hole principle gives

$$P(\|\eta_k\|_{L^2} \leq \varepsilon) > 0 \quad \text{for all } \varepsilon > 0. \tag{1.8}$$

From now on we will assume $T = 1$ for simplicity. We let

$$\Phi_t : u_0 \mapsto u(t)$$

denote the solution map to NSE (with $f = 0$) and we set $\Phi = \Phi_1$. Then,

$$\begin{aligned} u(k) &= \Phi(u(k-1)) + \eta_k, \qquad \text{where } u(k) = u(k+) \\ u(k+t) &= \Phi_t(u(k)) \qquad \text{for } 0 \le t < 1. \end{aligned} \tag{1.9}$$

Energy estimates. We now establish some estimates needed to prove the existence of an invariant measure.

Proposition 1.6. *The following estimates hold*

(1) $\|\Phi_t(u_0)\|_{L^2} \le e^{-t}\|u_0\|_{L^2}$.
(2) $\|\Phi_t(u_0)\|_{H^1} \le e^{-t}\|u_0\|_{H^1}$.

Proof. For the first estimate multiplying both sides of the equation $\partial_t u + B(u) = Lu$ by u, integrating and noting that $\langle B(u), u\rangle = 0$ gives,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = \langle Lu, u\rangle = -\|u\|_{\dot{H}^1}^2 \le -\|u\|_{L^2}^2.$$

Where in the last inequality we used the fact that u has mean 0. An application of Gronwall's inequality then gives

$$\|u(t)\|_{L^2}^2 \le e^{-2t}\|u(0\|_{L^2}^2.$$

For the second estimate multiplying both sides of the equation $\partial_t u + B(u) = Lu$ by Δu, integrating and using Lemma 1.1 gives,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\dot{H}^1}^2 = \langle Lu, \Delta u\rangle = -\|u\|_{\dot{H}^2}^2 \le -\|u\|_{\dot{H}^1}^2.$$

An application of Gronwall's inequality then gives

$$\|u(t)\|_{\dot{H}^1}^2 \le e^{-2t}\|u(0\|_{\dot{H}^1}.$$

The desired inequality follows from the fact that $u(t)$ has mean 0. □

We use the notation $u(k; u_0)$ to denote the solution to Kick NSE with initial data u_0.

Proposition 1.7. *The followings inequalities hold*

(1) $\|u(k;0)\|_{L^2} \le \sqrt{B_0}\frac{e}{e-1}$.
(2) $\|u(k;0)\|_{H^1} \le \sqrt{B_1}\frac{e}{e-1}$.

Proof. For $0 \le m \le k$, using Proposition 1.6,

$$\begin{aligned} \|u(k)\|_{L^2} &= \|\Phi(u(k-1)) + \eta_k\|_{L^2} \\ &\le \sqrt{B_0} + e^{-1}\|u(k-1)\|_{L^2} \\ &\le \sqrt{B_0} + e^{-1}(\sqrt{B_0} + e^{-1}\|u(k-2)\|_{L^2}) \\ &\le \cdots \\ &\le \sqrt{B_0}(1 + e^{-1} + \cdots + e^{-m}) + e^{-m}\|u(k-m)\|_{L^2}. \end{aligned}$$

Hence,

$$\|u(k)\|_{L^2} \le \sqrt{B_0}\frac{e}{e-1} + e^{-m}\|u(k-m)\|_{L^2}.$$

In particular, for $k = m$, $\|u(k;0)\|_{L^2} \leq \sqrt{B_0}\frac{e}{e-1}$. The proof of the second inequality is similar. □

We can write (1.9) as a random dynamical system (RDS). That is we can write (1.9) in the form

$$u(k) = F_k(u(k-1), \omega)$$

where $F_k : H \times \Omega \to H$ is a locally Lipschitz (from LWP theory) measurable function in $u \in H$.

Remark 1.8. *Every RDS defines a Markov chain ($\{u(k)\}_{k\geq 0}$ in H) in a canonical way.*

Definition 1.9. *For $u_0 \in H$, $k \in \mathbb{Z}_{\geq 0}$, $A \in \mathcal{B}_H$ we define the transition probability*

$$P_k(u_0, A) = P(u(k; u_0) \in A).$$

Note that δ_{u_0} by $\delta_{u_0}(A) = P_0(u_0, A)$. We will later use the following, well known, result in probability theory.

Theorem 1.10. *(Chapman-Kolmogorov equation) The following holds*

$$P_{k+m}(u_0, A) = \int_H P_m(u, A) P_k(u_0, du).$$

For a proof of this result and for more information concerning transition probabilities, see [2, Chapter 5].

Markov semigroups.

Definition 1.11. *We let $C_b(H)$ denote the set of continuous and bounded Borel functions on H. We define the Kolmogorov operator $T_k : C_b(H) \to C_b(H)$ by*

$$T_k f(v) = \int_H f(z) P_k(v, dz) = \mathbb{E}\left[f(u(k; v))\right].$$

It is not immediately obvious from the above definition that $T_k f(v) \in C_b(H)$. We show this later.

Definition 1.12. *We let $P(H)$ denote the set of all probability measures on H. We define the dual Kolmogorov operator $T_k^* : P(H) \to P(H)$ by*

$$(T_k^* \mu)(A) = \int_H P_k(v, A) \mu(dv) = \mu(\{v : u(k; v) \in A\}).$$

Remark 1.13. *If u_0 is random with $L(u_0) = \mu$, then $T_k^* \mu = L(u(k; u_0))$.*

From the definitions one can show

$$\langle T_k f, \mu \rangle = \int_H T_k f(v) \mu(dv) = \int_H f(z) (T_k^* \mu((dz).$$

Hence, calling T_k^* the dual operator of T_k is justified.

Definition 1.14. *Let T_k be a Markov semi-group. Then,*

1. *T_k is Feller if $T_k \in C_b(H)$ for all $f \in C_b(H)$, for all $k \geq 0$.*
2. *T_k is Strong Feller if $T_k \in C_b(H)$ for all $f \in L^\infty(H)$, for all $k > 0$.*
3. *T_k is irreducible if $T_k \mathbf{1}_{B(x_0,r)}(x) > 0$, for all $x, x_0 \in H$ for all $r > 0$, for any $k \geq 0$.*

Proposition 1.15. *Equipping $P(H)$ with the weak topology, the mapping $H \to P(H)$ defined by $u \mapsto P_k(u, \cdot)$ is continuous.*

Proof. Let $u_{0,n} \to u_0$ be a sequence in H. By the Portmanteau Theorem, see [1], it suffices to show that if A is a continuity set of measure $P_k(u_0, \cdot)$, that is $P_k(u_0, \partial A) = 0$ then $P_k(u_{0,n}, A) \to P_k(u_0, A)$. From LWP theory and the fact that A is a continuity set we have,

$$\begin{aligned} P_k(u_{0,n}, A) &= P(\{\omega : u(k; u_{0,n}) \in A\}) \\ &= \int \mathbf{1}_{u(k;u_{0,n})\in A}(\omega) dP(\omega) \\ &\to \int \not\models_{u(k;u_0)\in A}(\omega) dP(\omega) \\ &= P_k(u_0, A). \end{aligned}$$

□

We now show that the semi-group associated to Kick NSE is Feller.

Proposition 1.16. *$T_k : C_b(H) \to C_b(H)$ is Feller.*

Proof. Suppose $u_0, n \to u_0$ be a sequence in H. By the Portmanteau Theorem,

$$T_k f(u_{0,n}) = \int f(z) P_k(u_{0,n}, dz) \to \int f(z) P_k(u_0, dz) = T_k f(u_0)$$

and so $T_k f$ is continuous. Further,

$$|T_k f(v)| = \left| \int_H f(z) P_k(v, dz) \right| \leq \|f\|_{L^\infty} \tag{1.10}$$

and so $T_k f$ is bounded. □

Proposition 1.17. *$T_k^* : P(H) \to P(H)$ is continuous.*

Proof. Suppose $\mu_n \to \mu$. Let $f \in C_b(H)$. Then from the fact that T_k is Feller and the Portmanteau Theorem we have,

$$\int f(z)(T_k^* \mu_n)(dz) = \int T_k f d\mu_n \to \int T_k f d\mu = \int f(z)(T_k^* \mu)(dz).$$

□

Definition 1.18. *A probability measure $\mu \in P(H)$ is said to be invariant (or stationary) for T_k if*

$$\int_H t_k f d\mu = \int f d\mu$$

for all $k \geq 0$, for all $f \in L^\infty(H)$.

If T_k is Feller then the above definition is equivalent to $T_k^* \mu = \mu$ for all $k \geq 0$.

With all the necessary probabilistic machinery laid out, we are now in a position to prove the main theorem of this section.

Theorem 1.19. *There exists and invariant measure for Kick NSE.*

Proof. The proof of this theorem uses the well known Bogolyubov-Krylov argument, see [4].

For simplicity we assume $B_1 < \infty$. The theorem can also be proven if $B_1 = \infty$ but the proof is much longer.

Let $u(0) = 0$. Set $\mu_k = L(u(k))$ and

$$\overline{\mu}_k = \frac{1}{k}\sum_{j=0}^{k-1}\mu_j.$$

Let $r = \sqrt{B_1}\frac{e}{e-1}$. By Proposition 1.7 $\mu_j(B_{H^1}(r) = 1$ for all $j \geq 0$ and hence $\overline{mu}_k(B_{H^1}(r)) = 1$ for all $k \geq 0$.

Hence $\{\overline{\mu}_k\}_{k\geq 0}$ is tight and so by Prokhorov's Theorem, see [2, Theorem 6.7], $\{\overline{\mu}_k\}_{k\geq 0}$ is weakly precompact. Hence there exists $\mu \in P(H)$ such that $\overline{\mu}_{k_n} \rightharpoonup \mu$. We claim that μ is an invariant measure for Kick NSE. To do this is suffices to show $T_1\mu = \mu$ as then it would follow that $T_k\mu = \mu$ for all $k \geq 0$. For $f \in C_b(H)$ we have,

$$\begin{aligned}
\langle f, T_1^*, \mu\rangle &= \lim_{m\to\infty}\langle f, T_1^*\mu\rangle \\
&= \lim_{m\to\infty}\frac{1}{km}\sum_{j=0}^{km-1}\langle f, T_1^*\mu_j\rangle \\
&= \lim_{m\to\infty}\frac{1}{km}\sum_{j=1}^{km}\langle f, \mu_j\rangle \\
&= \lim_{m\to\infty}\langle f, \overline{\mu}_{k_m}\rangle + \lim_{m\to\infty}\frac{1}{km}(\langle f, \mu_{k_m}\rangle - \langle f, \mu_0\rangle) \\
&= \langle f, \mu\rangle
\end{aligned}$$

where in the first equality we used Proposition 1.17, the third we shifted the sum and used the fact that $T_1^*\mu_j = \mu_{j+1}$. □

2. Lecture 6: Existence of an invariant measure for the white-forced two dimensional Naiver-Stokes equation

In the previous section we proved that there exists an invariant measure for Kick NSE. In this section we will go through a similar procedure and prove the existence of a invariant measure for the White Forced NSE (SNSE).

Formally SNSE is NSE with forcing

$$f = \frac{d}{dt}\zeta(t,x)$$

where,

$$\zeta(t,x) = \sum_{n\in\mathbb{Z}_0^2} b_n\beta_n(t)e_n(x).$$

Recall $\{e_n\}_{n\in\mathbb{Z}_0^2}$ is a basis for L^2_{df}.

Stochastic Convolution. The stochastic convolution

$$\Psi(t) = \int_0^t e^{t-t')L d\zeta(t')}, \quad L = \Pi\Delta$$

is important in the study of White-Forced NSE. Note that

$$\zeta = \phi dW \tag{2.1}$$

where dW is a Wiener process on L^2_{df}. In the following we assume ϕ is a Hilbert-Schmidt operator, $\phi \in HS(L^2, H^s)$. The Hilbert-Schmidt norm is given by,

$$\|\phi\|_{HS(L^2;H^s)} := \left(\sum_{n\in\mathbb{Z}_0^2} |n|^{2s} b_n^2 \right)^{\frac{1}{2}} =: B_s.$$

It is easy to prove the following regularity estimate for ζ.

Proposition 2.1. *If $\phi \in HS(L^2; H^S)$ then for $\varepsilon > 0$, $q, r < \infty$ and $T < \infty$,*

$$\zeta \in C_t W_x^{s+1-\varepsilon,\infty} \text{ a.s.} \quad \text{and} \quad \zeta \in L_t^q W_x^{s+1,r} \text{ a.s.} \tag{2.2}$$

Well-posedness of SNSE. We aim to solve

$$\partial_t u = Lu - B(u) + \partial_t \zeta$$

or in Ito formulation

$$du = (Lu - B(u))dt + d\zeta.$$

To solve SNSE, we use the well known Da Prato-Debussche trick, see [3]. That is we make the ansatz

$$u = v + \Psi.$$

As

$$\partial_t \Psi = Lv + \partial_t \zeta$$

it follows that v solves the equation

$$\partial_t v = Lv - B(v,v) - B(v,\Psi) - B(\Psi,v) - B(\Psi,\Psi) \tag{2.3}$$

which we call (SNSE'). We have the following estimate on the nonlinear piece of SNSE',

$$\begin{aligned}
\langle B(\Psi,v), v\rangle &= 0 \\
\langle B(v,\Psi), v\rangle &\le C\|v\|_{L^4}^2 \|\nabla\Psi\|_{L^2} \\
&\le C\|v\|_{\dot H^1}\|v\|_{L^2}\|\nabla\Psi\|_{L^2} \\
&\le \frac{1}{2}\|v\|_{\dot H^1} + C\|v\|_{L^2}^2\|\nabla\Psi\|_{L^2}^2 \\
\langle B(\Psi,\Psi), v\rangle &\lesssim \|v\|_{L^2}^2 + \|\langle\nabla\rangle\Psi\|_{L^4}^4
\end{aligned}$$

Multiplying SNSE', (2.3), by v and integrating,

$$\frac{1}{2}\partial_t\|v\|_{L^2}^2 + \|\nabla v\|_{H^1}^2 = -\langle B(v,v),v\rangle - \langle B(\Psi,v),v\rangle - \langle B(v,\Psi),v\rangle - \langle B(\Psi,\Psi),v\rangle. \tag{2.4}$$

Combining this with the estimates for the nonlinear terms above gives

$$\frac{1}{2}\partial_t\|v\|_{L^2}^2 \le C_1(\Psi(t)) + C_2(\Psi(t))\|v\|_{L^2}^2.$$

Multiplying this by an integrating factor gives,

$$\partial_t \left(e^{-\int_0^t C_2(\Psi(t'))dt'} \|v(t)\|_{L^2}^2 \right) \leq C_1(\Psi(t))$$

and after integrating we get,

$$\sup_{t\in[0,T]} \|v(t)\|_{L^2_x} \leq C(u_0, \Psi, T).$$

Putting this estimate back into (2.4) gives

$$\|v\|_{L^2_T, H^1_x} \lesssim C(u_0, \Psi, T).$$

Also

$$\|B(v,\Psi)\|_{L^2_T H^{-1}_x} \lesssim \|v\|_{L^\infty_T L^2_x} \|\Psi\|_{L^2_T L^\infty_x}.$$

Indeed this follows by duality. Testing by $w \in \dot{H}^1$,

$$\|B(v,\Psi)\|_{L^2_T H^{-1}_x} \sim \langle B(v,\Psi), w\rangle = -\langle B(v,w),\Psi\rangle \sim \int v\nabla w \Psi \lesssim \|v\|_{L^2}\|\nabla w\|_{L^2}\|\Psi\|_{L^\infty}.$$

Similarly we also have

$$\begin{aligned} \|B(\Psi,v)\|_{L^2_T H^{-1}_x} &\lesssim \|v\|_{L^\infty_T L^2_x}\|\Psi\|_{L^2_T L^\infty_x} \\ \|B(\Psi,\Psi)\|_{L^2_T H^{-1}_x} &\lesssim \|\Psi\|^2_{L^4_T H^{\frac{1}{2}}_x}. \end{aligned}$$

Hence taking the $L^2_T H^{-1}_x$-norm of SNSE', (2.3), we get,

$$\|\partial_t v\|_{L^2_T H^{-1}_x} \lesssim \|v\|_{L^2_T H^1_x} + \|v\|_{L^\infty_T L^2_x}\|v\|_{L^2_T H^1_x} + \|\Psi\|_{L^2_T L^\infty_x}\|v\|_{L^\infty_T L^2_x} + \|\Psi\|_{L^4_T H^{\frac{1}{2}}_x}$$

and so

$$\|\partial_t u\|_{L^2_T H^{-1}_x} \leq C(u-0, \Psi, T).$$

Galerkin approximation. We consider the following truncated version of SNSE', (2.3),

$$\begin{aligned} \partial v_N &= Lv_N - P_N B(v_N + \Psi_N) \\ v_N|_{t=0} &= P_N u_0 \end{aligned} \tag{2.5}$$

where $\Psi_N = P_N \Psi$. We have

$$\Psi_N \to \Psi \text{ a.s.} \quad \text{and so} \quad B(v_N+\Psi_N) \to B(v+\Psi) \text{ in } L^1_T H^{-1}_x \text{ a.s.}$$

This truncated equation, (2.5), is well-posed on $[0,T]$ for all $T>0$ and hence is globally well-posed in $L^2(\mathbb{T}^2)$ if $B_0 < \infty$, that is $\psi \in HS(L^2; L^2)$.

The point of considering this approximated equation is that (2.5) is a finite dimensional system of SDEs and so we can use standard results in stochastic calculus like Itô's Lemma. Further it is easy to show that (2.5) satisfies the same a priori estimates as (2.3), uniformly in N.

Itô's Lemma. We have shown a control on

$$\|u(t)\|_{L^2} \leq \|v(t)\|_{L^2} + \|\Psi(t)\|_{L^2}.$$

We can use Itô's Lemma to get another estiamte.

Lemma 2.2. *(Itô's Lemma) Suppose*

$$dX^{(i)} = \sum_{j=1}^{m} f_{ij} dB_j + g_i dt$$

where $B = (B_1, \dots, B_m)$ *are independent Brownian motions,* $\vec{X} = (X^{(1)}, \dots, X^{(n)})$*,* $f = (f_{ij})_{n\times m}$*,* $g = (g_1, dots, g_n)$*. Then,*

$$dF(t, \vec{X}_t) = \frac{\partial F}{\partial t}(t, \vec{X}_t)dt + \sum_{i=1}^{m} \frac{\partial F}{\partial x_i}(t, \vec{X}_t)dX^{(i)} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j}(t, \vec{X}_t)dX^{(i)}dX^{(j)}$$

where $dB_i dB_j = \delta_{ij}dt$*,* $dB_i dt = 0$*,* $dtdB_i = 0$ *and* $dtdt = 0$*.*

For a proof of Itô's Lemma see [4] or [7].

Writing (2.5) as

$$du = V_N(u)dt + \sum_{|n|\leq N} b_n e_n d\beta_n$$

and using Itô's Lemma with $F = \|u(t)\|^2_{L^2}$ we have,

$$dF(t, u(t)) = \partial_t F(t, u(t))dt + \langle \nabla_u F, v_N(u)\rangle dt + \langle \nabla_u F, \sum_{|n|\leq N} b_n e_n d\beta_n\rangle + \frac{1}{2}\sum_{|n\leq N} \frac{\partial^2 F}{\partial \widehat{u}_n^2} b_n^2 dt.$$

Taking an expectation we get,

$$\frac{d}{dt}\mathbb{E}\left\|u(t)\right\|^2_{L^2} = -\mathbb{E}\left\|u(t)\right\|^2_{H^1} + B_{0,N} \leq -\mathbb{E}\left\|u(t)\right\|^2_{L^2} + B_0.$$

Gronwall's inequality then gives a bound on $\mathbb{E}, \|u(t)\|^2_{L^2}$. But we actually want a bound on $\mathbb{E} \sup_{t\in[0,T]} \|u(t)\|^2_{L^2}$.

To get this bound we use the Burkhölder-Davis-Grundy inequality: for $p > 0$ and local martingale $M(t)$ we have

$$\mathbb{E} \sup_{t\in[0,T]} |M(t)|^p \sim \mathbb{E}\langle M\rangle_T^{\frac{p}{2}}.$$

In our setting this gives the desired bound

$$\mathbb{E} \sup_{t\in[0,T]} \|u(t)\|^2_{L^2} \leq C(\|u_0\|_{L^2}, B_0, T).$$

By applying Itô's Lemmas to $F(u) = \|u\|^2_{H^1}$ we get

$$\mathbb{E}\|u(t)\|^2_{\dot{H}^1} \leq \frac{1}{2}B_1 + e^{-2t}\mathbb{E}\|u_0\|^2_{\dot{H}^1}.$$

Stationary measure. We consider the semi-group $T_t : C_b(H) \to C_b(H)$ defined by

$$(T_t f)(v) = \int_H f(u) P_t(v, du) = \mathbb{E}[f(u(t; v))].$$

Here $H = L^2_{df}$, $P_t(u_0, A)$ is the transition probability at time t, defined in a manner similar to the previous lecture and $u(t, v)$ denotes the solution to SNSE at time t with inital data v.

We also consider the dual of T_t, $T_t^* : P(H) \to P(H)$ defined by

$$(T^* \mu)(A) = \int_H P_t(v, A) \mu(dv) = (\Phi_t)_* \mu$$

where here Φ_t is the solution map to SNSE. The dual operator is defined in such a way so that if $\mu = L(u_0)$ then $T_t^* \mu = L(u(t))$.

We are now in a position to prove the main theorem of this section.

Theorem 2.3. *There exists an invariant measure for SNSE.*

Proof. We assume $B_1 < \infty$. The result id still true if $B_1 = \infty$ but our assumption simplifies the proof. Set $\mu_t = L(u(t))$ and

$$\overline{\mu}_t = \frac{1}{t} \int_0^t \mu_{t'} dt'.$$

We define $B_{H^1}(r)$ to be the ball of radius r centered at 0 in H^1. Then using Chebyshev,

$$\begin{aligned} \mu_t(H \backslash B_{H^1}(r))0 &= \mathbb{P}(\|u(t)\|_{H^1} > r) \\ &\leq \frac{C}{r^2} \\ &< \varepsilon \end{aligned}$$

where we choose r sufficiently large in the last inequality. This shows that

$$\overline{\mu}_t((B_{H^1}(r))^c) < \varepsilon.$$

This shows the family of measures $\overline{\mu}_t$ is tight. The rest of the proof is identical to the proof of the existence of an invariant measure for Kick NSE in the previous section. $\square$

Universality of White Noise Forces. Consider the random kick forces

$$\eta_\varepsilon(t) = \sqrt{\varepsilon} \sum_{k \in \mathbb{Z}_0} \eta_k \delta_(t - \varepsilon k)$$

where

$$\eta_k = \sum_{k \in \mathbb{Z}_0^2} b_n g_{kn} e_n.$$

We compare the dynamics of Kick NSE with the above kick forcing to SNSE with forcing

$$f = \frac{d}{dt} \zeta$$

where

$$\zeta = \sum_{k \in \mathbb{Z}_0^2} b_n \beta_n e_n.$$

We have the following result.

Theorem 2.4. *The solution u^ε to the above kick forced NSE equation converges to u, the solution to SNSE in law.*

The proof of the above result follows from Donsker's Theorem which states that if $\{X_n\}_n$ is a i.i.d mean zero variance σ^2 sequence of random variables and

$$Z_n(t;\omega) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}(\omega)_(nt - [nt]) \frac{1}{\sigma\sqrt{n}} X_{[nt]+1}(\omega)$$

then Z_n converges in law to a Brownian motion.

Remark 2.5. *We can also consider continuous in time forcing but not Gaussian forcing and still obtain "weak universality" and convergence to white forcing. Skorokhod's theorem can be used to upgrade this convergence to a.s. convergence.*

References

[1] Billingsley, P. *Convergence of probability measures*, John Wiley & Sons, Inc., New York, 1999.
[2] Da Prato, G. *An introduction to infinite-dimensional analysis*, Universitext. Springer-Verlag, Berlin, 2006.
[3] Da Prato, G; Debussche, A. *Strong solutions to the stochastic quantization equations*, Ann. Probab. 31 (2003), no. 4, 1900–1916.
[4] Da Prato, G.; Zabczyk, J. *Ergodicity for infinite-dimensional systems*, London Mathematical Society Lecture Note Series, 229. Cambridge University Press, Cambridge, 1996.
[5] Kuksin, S. B. *Randomly forced nonlinear PDEs and statistical hydrodynamics in 2 space dimensions*, Zurich Lectures in Advanced Mathematics. European Mathematical Society, Zürich, 2006.
[6] Kuksin, .; Shirikyan, A. *Mathematics of two-dimensional turbulence*, Cambridge Tracts in Mathematics, 194. Cambridge University Press, Cambridge, 2012.
[7] Kuo, Hui-Hsiung, *Introduction to stochastic integration*, Universitext. Springer, New York, 2006.

UNIQUENESS OF THE INVARIANT MEASURE FOR THE KICKED NSE ON $\mathbb{T}^2$

TYPED BY JUSTIN FORLANO

ABSTRACT. These notes are based on lectures 7 and 8.

CONTENTS

1. PRELIMINARY FROM MEASURE THEORY

The main references for this section are [1, Chapter 5] and [4, Chapter 2].

In the following, let (X, d_X) denote an arbitrary Polish space (= complete and separable metric space). We denote by $C_b(X)$ the space of continuous and bounded functions from X to $\mathbb{R}$ and $\mathcal{P}(X)$ the set of probability measures on X.

1.1. Metrizing weak convergence of probability measures.

Recall that we say a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ converges *weakly* to some $\mu \in \mathcal{P}(X)$, denoted $\mu_n \rightharpoonup \mu$, if

$$\langle \mu_n, f\rangle \longrightarrow \langle \mu, f\rangle \tag{1.1}$$

as $n \to \infty$, for every $f \in C_b(X)$. In fact, we may interpret the duality pairing through integration so that $\mu_n \rightharpoonup \mu$ if and only if

$$\int_X f(x)d\mu_n(x) \longrightarrow \int_X f(x)d\mu(x)$$

for every $f \in C_b(X)$. For $f \in C_b(X)$, we define the *Lipschitz norm* of f by

$$\|f\|_{\mathrm{Lip}} := \sup_{x\in X} |f(x)| + \mathrm{Lip}(f), \tag{1.2}$$

where

$$\mathrm{Lip}(f) := \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(x)-f(y)|}{d_X(x,y)}. \tag{1.3}$$

Note that replacing the denominator above by $d_X(x,y)^\alpha$, for some $\alpha \in (0,1)$, yields the α-Hölder seminorm of f

Let $\mathcal{U} := \{f \in C_b(X) : \|f\|_{\mathrm{Lip}} \le 1\}$. The following proposition shows that for establishing the weak convergence of probability measures, it suffices to establish the convergence in (1.1) along the subset of functions in $\mathcal{U}$.

Proposition 1.1. *Let $\mu_n, \mu \in \mathcal{P}(X)$. Then $\mu_n \rightharpoonup \mu$ if and only if*

$$\langle \mu_n, f\rangle \longrightarrow \langle \mu, f\rangle \tag{1.4}$$

for every $f \in \mathcal{U}$.

For the proof see, [1, Proposition 5.1].

We now endow $\mathcal{P}(X)$ with the following *dual-Lipschitz distance*:

$$\|\mu-\nu\|^*_{\mathrm{Lip}} := \sup_{f\in\mathcal{U}} |\langle \mu, f\rangle - \langle \nu, f\rangle|. \tag{1.5}$$

It turns out $\mathcal{P}(X)$ endowed with the dual-Lipschitz distance is a good space for analysis (since $(\mathcal{P}(X), \|\cdot\|^*_{\mathrm{Lip}})$ is a complete metric space) and furthermore the dual-Lipshitz distance metrizes the notion of weak convergence of probability measures. More precisely, we have:

Theorem 1.2. *The metric space $\mathcal{P}(X)$ endowed with the dual-Lipschitz distance is complete. Moreover, $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ converges to $\mu \in \mathcal{P}(X)$ with respect to the dual-Lipschitz distance if and only if $\mu_n \rightharpoonup \mu$.*

A small point: $\mathcal{P}(X)$ is convex but is not in general a vector space.

Remark 1.3. The dual-Lipschitz distance also makes the space of measures $\mathcal{M}(X)$ into a metric space, however it is not complete.

Let $0 < d \le 1$. We define a new distance on X by

$$\widetilde{d}(x_1,x_2) := d_X(x_1,x_2) \wedge d. \tag{1.6}$$

Notice that d_X and $\widetilde{d}$ define the same topology on X since balls with respect to d_X are also balls with respect to $\widetilde{d}$. Moreover, the dual-Lipshcitz norms with respect to d_X and $\widetilde{d}$ are equivalent; that is for any $\mu, \nu \in \mathcal{P}(X)$, we have

$$\|\mu-\nu\|^*_{\mathrm{Lip},\widetilde{d}} \le \|\mu-\nu\|^*_{\mathrm{Lip},d_X} \le \frac{2}{d}\|\mu-\nu\|^*_{\mathrm{Lip},\widetilde{d}}. \tag{1.7}$$

To see (1.7), we set

$$\mathcal{U}_{d_X} := \{f \in C_b(X) : \|f\|_{\mathrm{Lip},d_X} \le 1\},$$
$$\mathcal{U}_{\widetilde{d}} := \{f \in C_b(X) : \|f\|_{\mathrm{Lip},\widetilde{d}} \le 1\},$$

and since $\widetilde{d}(x_1,x_2) \leq d_X(x_1,x_2)$ for any $x_1,x_2 \in X$, we have $\mathcal{U}_{\widetilde{d}} \subset \mathcal{U}_{d_X}$ which implies the first inequality in (1.7). For the second inequality, we notice that if $f \in \mathcal{U}_{d_X}$, then $\frac{d}{2}f \in \mathcal{U}_{\widetilde{d}}$.

1.2. **Variational distance.** Given $\mu,\nu \in \mathcal{P}(X)$, we define

$$\|\mu-\nu\|_{\text{var}} = \sup_{A\in\mathcal{B}(X)} |\mu(A)-\nu(A)|, \tag{1.8}$$

where $\mathcal{B}(X)$ is the set of Borel sets in (X,d). This is the *variational distance* between the probability measures μ and ν. Notice that $\|\mu-\nu\|_{\text{var}} \leq 1$ for any $\mu,\nu \in \mathcal{P}(X)$. The variational distance is a measure of the 'non-overlapping' of two measures and this heuristic is motivated by the following two properties:

(i) μ and ν are singular if and only if $\|\mu-\nu\|_{\text{var}} = 1$.

(ii) Suppose there exist $\rho \in \mathcal{P}(X)$ such that $\mu,\nu \ll \rho$. Then we have

$$\begin{aligned}\|\mu-\nu\|_{\text{var}} &= \frac{1}{2}\int_X \left|\frac{d\mu}{d\rho}(x) - \frac{d\nu}{d\rho}(x)\right| d\rho(x) \\ &= 1 - \int_X \frac{d\mu}{d\rho}(x) \wedge \frac{d\nu}{d\rho}(x) d\rho(x).\end{aligned} \tag{1.9}$$

Thus, the variation distance is roughly

$$1 - (\text{total overlap}) = (\text{total non-overlap}).$$

Remark 1.4. In fact, the measure $\rho := \frac{1}{2}(\mu+\nu) \in \mathcal{P}(X)$ and satisfies $\mu,\nu \ll \rho$ so (1.9) 'always holds.'

The equality in (1.9) follows from the general equivalent characterisation

$$\|\mu-\nu\|_{\text{var}} = \frac{1}{2} \sup_{\substack{f\in C_b(X)\\ \|f\|_{L^\infty}\leq 1}} \left| \int f(x)d\mu(x) - \int f(x)d\nu(x) \right|. \tag{1.10}$$

As the class of functions in the above supremum contains the class $\mathcal{U}$, we find

$$\|\mu-\nu\|^*_{\text{Lip}} \leq 2\|\mu-\nu\|_{\text{var}}. \tag{1.11}$$

Consequently, we have:

Theorem 1.5. *The following hold:*

(i) *The space $(\mathcal{P}(X), \|\cdot\|_{\text{var}})$ is complete.*

(ii) *$\mu_n \to \mu$ in the variation distance if and only if*

$$\langle f, \mu_n\rangle \to \langle f, \mu\rangle$$

uniformly in $f \in C_b(X)$ such that $\|f\|_{L^\infty} \leq 1$.

(iii) *$\mathcal{P}(X)$ with the variational distance embeds continuously into $C_b(X)^*$.*

Compare (ii) to Proposition 1.1. The result of (iii) follows from (1.10) with (1.9) and Remark 1.4.

1.3. **Coupling.** We denote the law of a random variable ξ by $\mathcal{L}(\xi)$.

Definition 1.6. *Given $\mu_1, \mu_2 \in \mathcal{P}(X)$, a pair of random variables (ξ_1, ξ_2) defined on the same probability space is called a **coupling** for (μ_1, μ_2) if $\mathcal{L}(\xi_j) = \mu_j$ for $j = 1, 2$.*

Given a coupling (ξ_1, ξ_2), the random variable $\xi = (\xi_1, \xi_2)$ on $X \times X$ has $\mathcal{L}(\xi) = \mu$ (a measure on $\mathcal{B}(X \times X)$) and marginals

$$\mu_1 = (\Pi_1)_{\#}\mu = \mu \circ \Pi_1^{-1} \qquad \text{and} \qquad \mu_2 = (\Pi_2)_{\#}\mu = \mu_2 \circ \Pi_2^{-1},$$

where $\Pi_j(\xi) = \xi_j$, for $j = 1, 2$ are the coordinate projections. In this sense, a coupling has "doubled" the number of variables in going from X to $X \times X$.

Given a coupling (ξ_1, ξ_2) for (μ_1, μ_2), for any $A \in \mathcal{B}(X)$ we have

$$\begin{aligned} \mu_1(A) - \mu_2(A) &= \mathbb{E}[\mathbf{1}_A(\xi_1) - \mathbf{1}_A(\xi_2)] \\ &= \mathbb{E}[\mathbf{1}_{\{\xi_1 \neq \xi_2\}}(\mathbf{1}_A(\xi_1) - \mathbf{1}_A(\xi_2))] \\ &\leq \mathbb{P}(\xi_1 \neq \xi_2), \end{aligned}$$

which implies

$$\|\mu_1 - \mu_2\|_{\text{var}} \leq \mathbb{P}(\xi_1 \neq \xi_2). \tag{1.12}$$

Definition 1.7. A coupling (ξ_1, ξ_2) is called **maximal** if:

(i) $\|\mu_1 - \mu_2\|_{\text{var}} = \mathbb{P}(\xi_1 \neq \xi_2)$, that is equality holds in (1.12).

(ii) ξ_1 and ξ_2 conditioned on the event $\mathcal{N} = \{\xi_1 \neq \xi_2\}$ are independent, that is, for all $A, B \in \mathcal{B}(X)$,

$$\mathbb{P}(\xi_1 \in A, \, \xi_2 \in B \,|\, \mathcal{N}) = \mathbb{P}(\xi_1 \in A \,|\, \mathcal{N})\mathbb{P}(\xi_2 \in B \,|\, \mathcal{N}).$$

It is natural to ask whether any pair of probability measures has a maximal coupling. This turns out to be the case.

Lemma 1.8 (Dobrushin's Lemma)**.** *Given any $\mu_1, \mu_2 \in \mathcal{P}(X)$, there exists a maximal coupling (ξ_1, ξ_2).*

Proof. Put $\delta = \|\mu_1 - \mu_2\|_{\text{var}}$. If $\delta = 1$, any pair (ξ_1, ξ_2) of independent random variables with $\mathcal{L}(\xi_j) = \mu_j$, $j = 1, 2$, is a maximal coupling for (μ_1, μ_2) (use (1.12)). On the other hand, if $\delta = 0$, then $\mu_1 = \mu_2$ so any random variable ξ with $\mathcal{L}(\xi) = \mu_1$ gives rise to a maximal coupling (ξ, ξ). We now assume $0 < \delta < 1$ and begin to set up some definitions and notations. With

$$m := \frac{1}{2}(\mu_1 + \mu_2),$$

we have $\mu_1, \mu_2 \ll m$ and we write

$$\rho_j := \frac{d\mu_j}{dm},$$

for $j = 1, 2$, $\rho := \rho_1 \wedge \rho_2$ and $\widehat{\rho}_j = \frac{1}{\delta}(\rho_j - \rho)$. In particular, we have $\rho_j = \rho + \delta\widehat{\rho}_j$. By Remark 1.4 and (1.9), the measures

$$d\widehat{\mu}_j := \widehat{\rho}_j dm \qquad \text{and} \qquad d\mu := \frac{\rho}{1 - \delta} dm$$

are probability measures on X. Let ζ_1, ζ_2, ζ and α be independent random variables on the same probability space[1] such that

$$\begin{aligned} \mathcal{L}(\zeta_j) &= \widehat{\mu}_j, \qquad \mathcal{L}(\zeta) = \mu, \\ \mathbb{P}(\alpha = 0) &= \delta, \qquad \mathbb{P}(\alpha = 1) = 1 - \delta. \end{aligned} \tag{1.13}$$

With all the setup in place, we now claim that the random variables (ξ_1, ξ_2), defined by

$$\xi_j := \alpha\zeta + (1-\alpha)\zeta_j \quad \text{for } j = 1, 2, \tag{1.14}$$

are a maximal coupling for (μ_1, μ_2).

We first verify (ξ_1, ξ_2) are a coupling. Given $A \in \mathcal{B}(X)$ and $j = 1, 2$ fixed, we have

$$\begin{aligned} \mathbb{P}(\xi_j \in A) &= \mathbb{P}(\xi_j \in A,\, \alpha = 0) + \mathbb{P}(\xi_j \in A,\, \alpha = 1) \\ &= \mathbb{P}(\alpha = 0)\mathbb{P}(\xi_j \in A) + \mathbb{P}(\alpha = 1)\mathbb{P}(\xi_j \in A) \\ &= \delta \int_A \widehat{\rho}_j(x) dm(x) + (1-\delta) \int_A \frac{\rho(x)}{1-\delta} dm(x) \\ &= \int_A \rho_j(x) dm(x) = \rho_j(A). \end{aligned}$$

Thus, $\mathcal{L}(\xi_j) = \rho_j$ for each $j = 1, 2$. We now verify (i) in Definition 1.7. By the independence of α with ζ_1 and ζ_2, we have[2]

$$\begin{aligned} \mathbb{P}(\xi_1 \neq \xi_2) &= \mathbb{P}(\xi_1 \neq \xi_2,\, \alpha = 0) + \mathbb{P}(\xi_1 \neq \xi_2,\, \alpha = 1) \\ &= \mathbb{P}(\alpha = 0)\mathbb{P}(\xi_1 \neq \xi_2) \\ &= \mathbb{P}(\alpha = 0) = \delta. \end{aligned}$$

Finally, since $\mathbb{P}(\zeta_1 \neq \zeta_2) = 1$ and $\{\xi_1 \neq \xi_2\} = \{\zeta_1 \neq \zeta_2\} \cap \{\alpha = 0\}$, we have

$$\begin{aligned} \mathbb{P}(\xi_1 \in A, \xi_2 \in B \,|\, \{\xi_1 \neq \xi_2\}) &= \mathbb{P}(\zeta_1 \in A, \zeta_2 \in B, \alpha = 0) \\ &= \mathbb{P}(\zeta_1 \in A, \alpha = 0)\mathbb{P}(\zeta_2 \in B, \alpha = 0) \\ &= \mathbb{P}(\xi_1 \in A \,|\, \{\xi_1 \neq \xi_2\})\mathbb{P}(\xi_2 \in B \,|\, \{\xi_1 \neq \xi_2\}) \end{aligned}$$

for any $A, B \in \mathcal{B}(X)$. Thus, (ξ_1, ξ_2) are a maximal coupling for (μ_1, μ_2). □

The constructive proof of Dobrushin's Lemma immediately implies the following corollary.

Corollary 1.9. *Any $\mu_1, \mu_2 \in \mathcal{P}(X)$ admits a representation*

$$\mu_j = (1-\delta)\mu + \delta\nu_j \quad \textit{for } j = 1, 2,$$

where $\delta := \|\mu_1 - \mu_2\|_{\mathrm{var}}$, $\mu, \nu_1, \nu_2 \in \mathcal{P}(X)$ and $\nu_1 \perp \nu_2$.

The measure $(1-\delta)\mu$ is sometimes referred to as the *minimum of μ_1 and μ_2* and is denoted $\mu_1 \wedge \mu_2$.

---

[1] A priori, such random variables exist on different probability spaces. What we write here is their natural extensions to the product space formed from these probability spaces.

[2] We use here that $\mathbb{P}(\zeta_1 \neq \zeta_2) = 1$. To concretely see this, note that by definition of $\widehat{\rho}_j$, we have $\widehat{\rho}_1(x)\widehat{\rho}_2(x) = 0$ for a.e. $x \in X$. Then,

$$\mathbb{P}(\zeta_1 = \zeta_2) = \iint\limits_{\{x_1 = x_2\}} \widehat{\rho}_1(x_1)\widehat{\rho}_2(x_2) dm(x_1) dm(x_2) = 0.$$

1.4. **Kantorovich functional.** Let $F : X \times X \to \mathbb{R}$ be a measurable function such that

$$F(x_1, x_2) = F(x_2, x_1) \geq \operatorname{dist}(x_1, x_2) \quad \text{for all } x_1, x_2 \in X. \tag{1.15}$$

We define the *Kantorovich functional* $K = K_F : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$, associated with F, by

$$K(\mu_1, \mu_2) = \inf_{\substack{\text{all couplings} \\ (\xi_1,\xi_2) \text{ for } (\mu_1,\mu_2)}} \mathbb{E}[F(\xi_1, \xi_2)]. \tag{1.16}$$

The function F is known as the Kantorovich density for the functional K_F. We will make use of the following inequality.

Lemma 1.10. *For any $\mu_1, \mu_2 \in \mathcal{P}(X)$ and any measurable $F : X \times X \to \mathbb{R}$ satisfying* (2.42)*, we have*

$$\|\mu_1 - \mu_2\|^*_{\text{Lip}} \leq K_F(\mu_1, \mu_2).$$

Proof. Let (ξ_1, ξ_2) be a coupling for (μ_1, μ_2) and let $f \in \mathcal{U}$. Then,

$$\begin{aligned} \langle f, \mu_1 \rangle - \langle f, \mu_2 \rangle &= \mathbb{E}[f(\xi_1) - f(\xi_2)] \\ &\leq \mathbb{E}[\text{Lip}(f) \operatorname{dist}(\xi_1, \xi_2)] \\ &\leq \mathbb{E}[F(\xi_1, \xi_2)]. \end{aligned}$$

We now take a supremum over $f \in \mathcal{U}$ followed by an infimum over all such couplings. □

2. Uniqueness of the invariant measure for (KickNSE)

2.1. **The kicked NSE.** For the sake of reference, we recall our formulation of the (KickNSE). The (KickNSE) is the PDE

$$\partial_t u - Lu + B(u) = \eta^\omega(t), \tag{2.1}$$

where $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}^2$, $L = \Pi\Delta$, $B(u) = \Pi(u \cdot \nabla)u$ and $\eta^\omega(t)$ is a random stochastic process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\eta(t) = \eta^\omega(t) = \sum_{k \in \mathbb{Z}} \eta^\omega_k \delta(t - k), \quad \omega \in \Omega,$$

which provides 'kicks' of a random strength η^ω_k at each time $t = k \in \mathbb{Z}$. We make the following assumptions on the kicks η^ω_k: for each $k \in \mathbb{Z}$, we have

$$\eta^\omega_k = \sum_{n \in \mathbb{Z}^2_0} b_n g_{kn}(\omega) e_n, \tag{2.2}$$

where

- $\{e_n\}_{n \in \mathbb{Z}^2_0}$ are an orthonormal basis of $H := L^2_{\text{df},0}(\mathbb{T}^2 \to \mathbb{R}^2)$, the space of L^2 vector fields which are divergence free and have zero mean,
- $b_n \geq 0$ and $B_0 := \sum_{n \in \mathbb{Z}^2_0} b_n^2 < +\infty$, and
- $\{g_{kn}(\omega)\}_{k \in \mathbb{Z}, n \in \mathbb{Z}^2_0}$ are independent random variables satisfying

  $$|g_{kn}(\omega)| \leq 1 \quad \text{for all } n \in \mathbb{Z}^2_0,\, k \in \mathbb{Z},\, \omega \in \Omega \tag{2.3}$$

  and

  $$\mathcal{L}(g_{kn}) = p_n(r)dr, \quad n \in \mathbb{Z}^2_0,\, k \in \mathbb{Z}, \tag{2.4}$$

  where the p_n are Lipschitz functions supported in $[-1, 1]$ with $p_n(0) \neq 0$.

These assumptions have two important consequences:

$$\sup_{k\in\mathbb{Z}} \|\eta_k(\omega)\|_{L^2}^2 = \sup_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}_0^2} b_n^2 g_{kn}^2(\omega) \le B_0, \text{ and} \tag{2.5}$$

$$\mathbb{P}(\|\eta_k\|_{L^2} \le \varepsilon) > 0, \quad \text{for any } \varepsilon > 0. \tag{2.6}$$

We saw in the previous lectures that applying the deterministic global well-posedness theory for the forced NSE on $\mathbb{T}^2$ we obtain a unique, global-in-time solution $u \in C_T H$, ω-a.s., for any $T > 0$, to (2.1) under the standing assumptions. Denoting by $\Phi = \Phi_1$ the time $t = 1$ nonlinear solution map of NSE with *no* forcing, we may write

$$u(k) = \Phi(u(k-1)) + \eta_k, \quad k \in \mathbb{Z}. \tag{2.7}$$

This formulation of (2.1) then gave rise to a formulation in terms of a random dynamical system and hence there was a naturally associated Markov chain with transition probabilities

$$p_k(u_0, A) := \mathbb{P}(u(k; u_0) \in A), \tag{2.8}$$

for $u_0 \in H$, $k \in \mathbb{Z}_{\ge 0}$ and $A \in \mathcal{B}(H)$, and hence also an associated Markov semigroup T_k. We then showed there exists a measure $\mu \in \mathcal{P}(H)$ invariant under the flow of (2.1); that is, $T_k^* \mu = \mu$ for every $k \in \mathbb{Z}_{\ge 0}$.

Goal: Prove μ is the *unique* invariant measure under the flow of (KickNSE).

In fact, we will actually prove a stronger statement about (2.1): it is *exponentially mixing.* Essentially, this says that given any initial distribution, the law of the resulting solution converges exponentially fast to the invariant measure. The uniqueness of the invariant measure follows readily from exponential mixing and it is the proof of the latter result that requires the bulk of the forthcoming work.

2.2. **The main lemma.** Recall that the time-one transition probability can be written as

$$p_1(u, \cdot) = \mathcal{L}(\Phi(u) + \eta_1). \tag{2.9}$$

Lemma 2.1. *There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $R \ge 1$, there exists $N = N(R) \ge 1$ such that if $b_n \ne 0$ for $|n| \le N$, then for any $u_1, u_2 \in B_R \subset H$, the measures*

$$\mu_1 = p_1(u_1, \cdot) \quad \textit{and} \quad \mu_2 = p_1(u_2, \cdot)$$

admit a coupling (V_1, V_2), where $V_j = V_j(u_1, u_2; \omega)$ such that

(i) *$V_j : B_R \times B_R \times \Omega \longmapsto H$ is measurable and*
(ii) *with $d := \|u_1 - u_2\|_{L^2}$, we have*

$$\mathbb{P}\left(\|V_1 - V_2\|_{L^2} \ge \frac{1}{2} d \right) \le C_0 d, \tag{2.10}$$

where $C_0 = C_0(R, B_0, \{b_n\}_{|n|\le N})$.

The inequality (2.10) is nontrivial when $C_0 d \le 1$.

Proof. Let

$$\mathbf{P}_N : H \longmapsto E_N = \operatorname{span}\{e_n : |n| \le N\}$$

and $\mathbf{P}_N^\perp := \mathrm{Id} - \mathbf{P}_N$. We search for V_1 and V_2 of the form:

$$V_1 = \Phi(u_1) + \xi_1,$$
$$V_2 = \Phi(u_2) + \xi_2,$$

where the random variables $\xi_1, \xi_2 \in H$ and satisfy $\mathcal{L}(\xi_1) = \mathcal{L}(\xi_2) = \eta_1$. Then, noting (2.9), (V_1, V_2) will be a coupling for (μ_1, μ_2). Our goal is to define the random variables ξ_1 and ξ_2. We do this by specifying $\mathbf{P}_N\xi_j$ and $\mathbf{P}_N^\perp\xi_j$ for some appropriate $N \ge 1$.

We will define ξ_1 and ξ_2 on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \Omega_1 \times \Omega_2$ for $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ to be defined later. With $\overline{\eta}_1$ the natural extension of η_1 to Ω; that is, $\overline{\eta}_1(\omega_1, \omega_2) = \eta_1(\omega_1)$, we set

$$\mathbf{P}_N^\perp\xi_1 = \mathbf{P}_N^\perp\xi_2 = \mathbf{P}_N^\perp\overline{\eta}_1. \tag{2.11}$$

We now move onto defining $\mathbf{P}_N\xi_j$ for $j = 1, 2$. Setting $v_j = \mathbf{P}_N\Phi(u_j)$, for $j = 1, 2$, the Lipschitz continuity of Φ implies

$$\|v_1 - v_2\|_{L^2} \le C(R)\|u_1 - u_2\|_{L^2} = C(R)d. \tag{2.12}$$

We have

$$\mathcal{L}(\mathbf{P}_N\eta_1) = q(x)dx,$$

where $x \in \mathbb{R}^{\dim E_N}$ (we have identified E_N with $\mathbb{R}^{\dim E_N}$). Here, $q(x) = \prod_{|n| \le N} b_n^{-1} p_n(b_n^{-1}x_n)$ and we have used the assumptions $b_n \neq 0$ for $|n| \le N$ and (2.4). Notice that q is Lipschitz. Then

$$\nu_j := (\mathbf{P}_N)_\# \mu_j = \mathcal{L}(v_j + \mathbf{P}_N\eta_1) = q(x - v_j)dx$$

and it follows from (1.9) and (2.12) that

$$\|\nu_1 - \nu_2\|_{\mathrm{var}} = \frac{1}{2}\int_{E_N} |q(x - v_1) - q(x - v_2)|dx \le C_0 d. \tag{2.13}$$

By Dobrushin's Lemma, there exists a maximal coupling (Ξ_1, Ξ_2) for (ν_1, ν_2) on a probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, where $\Xi_j = \Xi_j(u_1, u_2; \omega_2)$. Recalling Definition 1.7 and using (2.13), we have

$$\mathbb{P}_2(\Xi_1 \neq \Xi_2) = \|\nu_1 - \nu_2\|_{\mathrm{var}} \le C_0 d. \tag{2.14}$$

We now check that $\Xi_j : B_R \times B_R \times \Omega_2 \longmapsto E_N$ is measurable. For $u \in B_R$, let $v(u) = \mathbf{P}_N\Phi(u)$ and we have $\nu(u) = (\mathbf{P}_N)_\# p_1(u, \cdot) = q(x - v(u))dx$. Here, $\nu(u)$ is a measure on E_N and $q_v(x) := q(x - v)$ is measurable with respect to $(x, v) \in E_N^2$. Then, the map $\rho : B_R \times B_R \longmapsto \mathbb{R}$, defined by

$$\rho(u_1, u_2) = \|\nu(u_1) - \nu(u_2)\|_{\mathrm{var}},$$

is measurable. Putting $m = \frac{1}{2}(\nu(u_1) + \nu(u_2))$, we construct as in the proof of Lemma 1.8, the following measures on E_N:

$$\widehat{\mu}_j = \widehat{p}_j(u_1, u_2, x)dx, \quad \mu_0 = p_0(u_1, u_2, x)dx,$$

for $j=1,2$, where $\widehat{p}_j$ and p_0 are measurable with respect to (u_1,u_2,x). We then obtain probability spaces $\{(\widetilde{\Omega}_j,\widetilde{\mathcal{F}}_j,\widetilde{\mathbb{P}}_j)\}_{j=0}^2$, with corresponding random variables $\{\xi_j(u_1,u_2)\}_{j=0}^2$ such that

$$\mathcal{L}(\xi_j(u_1,u_2))=\mu_j \quad \text{for } j=0,1,2.$$

Furthermore, by [2, Lemma 4.3] (see also [4, Theorem 1.2.28]), $\xi_j(u_1,u_2,\widetilde{\omega}_j)$ is measurable with respect to $(u_1,u_2,\widetilde{\omega}_j)$. On the probability space $([0,1],\mathcal{B}([0,1]),\mathrm{Leb})$, we set

$$\alpha_{\rho(u_1,u_2)}=\mathbf{1}_{[0,1-\rho(u_1,u_2)]}(s).$$

Then, on the probability space

$$(\Omega_2,\mathcal{F}_2,\mathbb{P}_2)=\big(\times_{j=0}^2\widetilde{\Omega}_j\times[0,1],\times_{j=0}^2\widetilde{\mathcal{F}}_j\times\mathcal{B}([0,1]),\otimes_{j=1}^2\widetilde{\mathbb{P}}_j\times\mathrm{Leb}\big)$$

with the naturally extended version of the (now independent) random variables ξ_j and α to this probability space, we set

$$\Xi_j(u_1,u_2;\omega_2)=\alpha\xi_0+(1-\alpha)\xi_j \quad \text{for } j=1,2,$$

as in (1.14). Now $\Xi_j(u_1,u_2;\omega_2)$ is a maximal coupling for $(\nu(u_1),\nu(u_2))$ and is measurable with respect to (u_1,u_2,ω_2).

Let $\overline{\Xi}_j(\omega_1,\omega_2):=\Xi_j(\omega_2)$ for $j=1,2$ and we set

$$\mathbf{P}_N\xi_j=\overline{\Xi}_j-\mathbf{P}_N\Phi(u_j) \quad \text{for } j=1,2. \tag{2.15}$$

Recalling (2.11), we put

$$V_j=\mathbf{P}_N\xi_j+\mathbf{P}_N^{\perp}\eta_1+\Phi(u_j) \quad \text{for } j=1,2. \tag{2.16}$$

Then, (V_1,V_2) is a coupling for (μ_1,μ_2) and are measurable with respect to (u_1,u_2,ω). It remains to verify (2.10). From (2.16) and (2.11), we have

$$V_1-V_2=\overline{\Xi}_1-\overline{\Xi}_2+\mathbf{P}_N^{\perp}[\Phi(u_1)-\Phi(u_2)].$$

Therefore,

$$\mathbb{P}\bigg(\|V_1-V_2\|_{L^2}\ge\frac{1}{2}d\bigg)\le\mathbb{P}\bigg(\|\mathbf{P}_N^{\perp}[\Phi(u_1)-\Phi(u_2)]\|_{L^2}\ge\frac{1}{2}d\bigg)+\mathbb{P}(\overline{\Xi}_1\ne\overline{\Xi}_2).$$

By Proposition 2.2, we have[3]

$$\|\mathbf{P}_N^{\perp}[\Phi(u_1)-\Phi(u_2)]\|_{L^2}\le N^{-1}\|\Phi(u_1)-\Phi(u_2)\|_{H^1}\le N^{-1}C_1(R)d.$$

Thus, given $R>0$, we choose $N\gg1$ such that $N^{-1}C_1(R)<\frac{1}{2}$. Then (2.14) implies (2.10), which completes the proof. □

2.3. **Some PDE estimates.** In this subsection, we derive a useful smoothing property of the NSE flow on $\mathbb{T}^2$ with forcing:

$$\partial_t u-Lu+B(u)=f. \tag{2.17}$$

Proposition 2.2. [4, Proposition 2.1.25] *Let $\Phi_t(u_0;f)$ denote the solution of (2.17) at time t with initial data u_0 and forcing f. Then, the following hold:*

[3]We are using the smoothing properties of the NSE flow; see Subsection 2.3.

(i) *There exists* $C>0$ *such that for any* $u_{0,1}, u_{0,2} \in H$ *and* $f_1, f_2 \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \dot{H}^{-1})$, *we have*

$$\|\Phi_t(u_{0,1}; f_1) - \Phi_t(u_{0,2}; f_2)\|^2_{L^2} \leq \exp\left(C\int_0^t \|\Phi_s(u_{0,1}; f_1)\|^2_{\dot{H}^1} ds\right)\|u_{0,1} - u_{0,2}\|^2_{L^2} + \int_0^t \exp\left(C\int_s^t \|\Phi_{t'}(u_{0,1}; f_1)\|^2_{\dot{H}^1} dt'\right)\|f_1(s) - f_2(s)\|^2_{\dot{H}^{-1}} ds.$$

(ii) *There exists* $C>0$ *such that* $0 < t \leq 1$ *and for any* $u_{0,1}, u_{0,2} \in H$ *and* $f_1, f_2 \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \dot{H})$, *we have*

$$\|\Phi_t(u_{0,1}; f_1) - \Phi_t(u_{0,2}; f_2)\|^2_{\dot{H}^1} \leq C\int_0^t \|f_1 - f_2\|^2_{L^2} ds + A(t)t^{-3}\left[\|u_{0,1} - u_{0,2}\|^2_{L^2} + \int_0^t \|f_1 - f_2\|^2_{\dot{H}^{-1}} ds\right],$$

where

$$A(t) := \exp\left(C\int_0^t \|\Phi_s(u_{0,1}; f_1)\|^2_{\dot{H}^1} + \|\Phi_s(u_{0,2}; f_2)\|^2_{\dot{H}^1} + \|f_1\|^2_{L^2} + \|f_2\|^2_{L^2} ds\right).$$

Proof. The following computations can be justified by first considering suitable Galerkin approximations; we omit these technicalities. Let $u_j = u_j(t) = \Phi_t(u_{0,j}; f_j)$, for $j = 1, 2$, and set $w := u_1 - u_2$ which solves

$$\partial_t w - Lw + B(w, u_1) + B(u_2, w) = f_1 - f_2. \tag{2.18}$$

(i) We take the inner product of (2.18) with $2w$ and use

$$\langle B(u, v), v\rangle = 0,$$

to obtain

$$\partial_t(\|w\|^2_{L^2}) + 2\|w\|^2_{\dot{H}^1} = -2\langle B(w, u_1), w\rangle + 2\langle f_1 - f_2, w\rangle.$$

Using[4]

$$|\langle B(w, u_1), w\rangle| \leq C\|u\|^2_{\dot{H}^1}\|w\|^2_{\dot{H}^{\frac{1}{2}}} \leq C\|u\|^2_{\dot{H}^1}\|w\|_{L^2}\|w\|_{\dot{H}^1} \leq \frac{1}{4}\|w\|^2_{\dot{H}^1} + C\|u_1\|^2_{L^2}\|w\|^2_{L^2},$$

$$|\langle f_1 - f_2, w\rangle| \leq \|f_1 - f_2\|_{\dot{H}^{-1}}\|w\|_{\dot{H}^1} \leq \frac{1}{4}\|w\|^2_{\dot{H}^1} + \|f_1 - f_2\|^2_{\dot{H}^{-1}},$$

we have

$$\partial_t\left(\|w\|^2_{L^2} + \int_0^t \|w\|^2_{\dot{H}^1} dt'\right) \leq 2C\|u_1\|^2_{\dot{H}^1}\left(\|w\|^2_{L^2} + \int_0^t \|w\|^2_{\dot{H}^1} dt'\right) + 2\|f_1 - f_2\|^2_{\dot{H}^{-1}}. \tag{2.19}$$

By Gronwall's inequality, we get

$$\|w(t)\|^2_{L^2} + \int_0^t \|w(t')\|^2_{\dot{H}^1} dt' \leq \exp\left(2C\int_0^t \|u_1\|^2_{\dot{H}^1} dt'\right)\|u_{0,1} - u_{0,2}\|^2_{L^2} + 2\int_0^t \exp\left(2C\int_{t'}^t \|u_1(s)\|^2_{\dot{H}^1} ds\right)\|f_1(t') - f_2(t')\|^2_{\dot{H}^{-1}} dt' \tag{2.20}$$

This now implies (i).

[4]The constants change from line to line.

(ii) We take the inner product with $2t(-Lw)$ and have

$$\begin{aligned}\partial_t(t\|w\|^2_{\dot H^1}) + 2t\|Lw\|^2_{L^2} =& \|w\|^2_{\dot H^1} + 2t\langle B(w,u_1), Lw\rangle \\ &+ 2t\langle B(u_2,w), Lw\rangle - 2t\langle f_1 - f_2, Lw\rangle.\end{aligned} \tag{2.21}$$

Using the inequalities

$$\|w\|_{L^\infty} \le C\|w\|^{\frac12}_{L^2}\|Lw\|^{\frac12}_{L^2}, \quad \text{and} \quad \|w\|_{\dot H^1} \le \|w\|^{\frac12}_{L^2}\|Lw\|^{\frac12}_{L^2}$$

we estimate

$$\begin{aligned}|\langle B(w,u_1), Lw\rangle| &\le C\|w\|_{L^\infty}\|u_1\|^{\frac12}_{\dot H^1}\|Lw\|_{L^2} \\ &\le C\|w\|^{\frac12}_{L^2}\|Lw\|^{\frac32}_{L^2}\|u_1\|^{\frac12}_{L^2}\|Lu_1\|^{\frac12}_{L^2} \\ &\le \frac18\|Lw\|^2_{L^2} + C\|w\|^2_{L^2}\|u_1\|^2_{L^2}\|Lu_1\|^2_{L^2}\end{aligned}$$

and

$$\begin{aligned}|\langle B(u_2,w), Lw\rangle| &\le C\|u_2\|_{L^\infty}\|w\|_{\dot H^1}\|Lw\|_{L^2} \\ &\le C\|u_2\|^{\frac12}_{L^2}\|Lu_2\|^{\frac12}_{L^2}\|w\|^{\frac12}_{L^2}\|Lw\|^{\frac32}_{L^2} \\ &\le \frac18\|Lw\|^2_{L^2} + C\|w\|^2_{L^2}\|u_2\|^2_{L^2}\|Lu_2\|^2_{L^2}.\end{aligned}$$

Using these with

$$|\langle f_1 - f_2, Lw\rangle| \le \frac14\|Lw\|^2_{L^2} + \|f_1 - f_2\|^2_{L^2},$$

we integrate in time (2.21) which yields the estimate

$$\begin{aligned}t\|w\|^2_{\dot H^1} + \int_0^t t'\|Lw\|^2_{L^2}dt' \le& \int_0^t \|w\|^2_{\dot H^1}dt' + C\int_0^t t'\|w\|^2_{L^2}(\|u_1\|_{L^2}\|Lu_1\|^2_{L^2} \\ &+ \|u_2\|^2_{L^2}\|Lu_2\|^2_{L^2})dt' + 2\int_0^t t'\|f_1 - f_2\|^2_{L^2}dt'.\end{aligned}$$

Noting $0 < t \le 1$, we use (2.20) to bound the first term on the right hand side and estimate $\|w\|^2_{L^2}$ in the second term. Then, in order to obtain (ii), we are left to show

$$\int_0^t t'\|u_j\|^2_{L^2}\|Lu_j\|^2_{L^2}dt' \le Ct^{-2}\exp\left(C\int_0^t \|u_j\|^2_{\dot H^1} + \|f_j\|^2_{L^2}\,dt'\right) \tag{2.22}$$

for $j = 1, 2$ and $0 < t \le 1$. We require the following estimates on solutions to (2.17): There exists a constant $\alpha > 0$ such that for any $u_0 \in H$ and $f \in L^2_{\text{loc}}(\mathbb{R}_+; \dot H^{-1})$, we have

$$\|\Phi_t(u_0)\|^2_{L^2} \le e^{-\alpha t}\|u_0\|^2_{L^2} + \int_0^t e^{-\alpha(t-s)}\|f(s)\|^2_{\dot H^{-1}}ds, \tag{2.23}$$

$$t\|\Phi_t(u_0)\|^2_{\dot H^1} + \int_0^t s\|\Phi_s(u_0)\|^2_{L^2}ds \le \|u_0\|^2_{L^2} + \int_0^t s\|f(s)\|^2_{L^2}ds + \int_0^t \|f(s)\|^2_{\dot H^{-1}}ds. \tag{2.24}$$

The proof of (2.23) follows from the energy bound in Lecture 4, Poincare's inequality and Gronwall's inequality; see [4, Proposition 2.1.21 (i)] for more. The proof of (2.24) is a direct computation from differentiating in time the quantity $t\langle L\Phi_t(u_0), \Phi_t(u_0)\rangle$; see [4, Theorem 2.1.18]. For $0 \le t \le 1$, (2.24) implies

$$\int_0^t s\|u_j(s)\|_{L^2}^2 ds \le \|u_{0,j}\|_{L^2}^2 + 2\int_0^t \|f_j\|_{L^2}^2 ds. \tag{2.25}$$

Now, taking the inner product of (2.17) with $2u_j$ and integrating in time gives

$$\|u_j(t)\|_{L^2}^2 + 2\int_0^t \|u_j\|_{\dot{H}^1}^2 ds = \|u_{0,j}\|_{L^2}^2 + 2\int_0^t \langle u_j, f_j\rangle ds.$$

This implies

$$\|u_{0,j}\|_{L^2}^2 \le \|u_j(t)\|_{L^2}^2 + 3\int_0^t \|u_j\|_{\dot{H}^1}^2 ds + \int_0^t \|f_j\|_{\dot{H}^{-1}}^2 ds$$

and hence by (2.23), we find

$$(1 - e^{-\alpha t})\|u_{0,j}\|_{L^2}^2 \le C\int_0^t \|u_j\|_{\dot{H}^1}^2 + \|f_j\|_{\dot{H}^{-1}}^2\, ds. \tag{2.26}$$

Using (2.23), (2.25) and (2.26), we obtain

$$\begin{aligned}\int_0^t s\|u_j\|_{L^2}^2\|Lu_j\|_{L^2}^2 ds &\le C\bigg(\|u_{0,j}\|_{L^2}^2 + 2\int_0^t \|f_j\|_{L^2}^2 ds\bigg)^2 \\ &\le C\bigg(\frac{1}{1-e^{-\alpha t}}\int_0^t \|u_j\|_{\dot{H}^1}^2 + \|f_j\|_{\dot{H}^{-1}}^2 ds + 2\int_0^t \|f_j\|_{L^2}^2\, ds\bigg)^2.\end{aligned}$$

Since $1 - e^{-\alpha t} = \alpha t(1 + \mathcal{O}(\alpha t))$ as $t \to 0$, the inequality above implies (2.22) and thus completes the proof. □

2.4. **The main theorem.**

Definition 2.3 (Weak Solutions)**.** *A process $\{u(k)\}_{k\ge 0} \subset H$ defined on some probability space is called a **weak solution** to the kicked NSE* (2.7) *if it satisfies* (2.7) *with the random kicks $\{\eta_k\}$ replaced by some other process $\{\widehat{\eta}_k\}$ which satisfies*

$$\mathcal{L}(\widehat{\eta}_k) = \mathcal{L}(\eta_k) \quad k \ge 0.$$

We now state some useful estimates related to weak solutions. Let $\{u_j(k)\}_{k\ge0}$, $j = 1, 2$, be two weak solutions with random kick forces $\{\eta_k^j\}_{k\ge0}$, respectively. We define

$$d(k) := \|u_1(k) - u_2(k)\|_{L^2}, \tag{2.27}$$

$$R(k) := \|u_1(k)\|_{L^2} + \|u_2(k)\|_{L^2}. \tag{2.28}$$

In Lecture 5, we proved the estimate[5]

$$\|u(k+1)\|_{L^2} \le e^{-1}\|u(k)\|_{L^2} + \|\eta_k\|_{L^2} \le e^{-1}\|u(k)\|_{L^2} + \sqrt{B_0}. \tag{2.29}$$

Therefore, we have

$$R(k+1) \le e^{-1}R(k) + 2\sqrt{B_0}$$

[5]In the second inequality, we used the uniform (in ω) assumption (2.3). For the case of white noise forcing, the second inequality would hold with high probability (i.e. no longer uniformly in ω).

for each $k \geq 0$, which implies

$$R(\ell) \leq e^{-(\ell-k)} R(k) + \frac{2e}{e-1}\sqrt{B_0} \leq e^{-(\ell-k)} R(k) + \left(\frac{1}{2} - \frac{1}{e}\right) R_0,$$

by choosing $R_0 \geq 1$. Thus, we have

$$R(\ell) \leq \frac{1}{2} R(k) \quad \text{for all } \ell \geq k+1, \text{ if } R(k) \geq R_0 \tag{2.30}$$

and

$$R(\ell) \leq \frac{1}{2} R_0 \quad \text{for all } \ell \geq k+1, \text{ if } R(k) \leq R_0. \tag{2.31}$$

Fix $0 < d_0 \leq 1$ and let u_1, u_2 be two weak solutions with the same kick force $\{\eta'_k\}$. Suppose $R(0) \leq R_0$ which implies $\|u_j(0)\|_{L^2} \leq R_0$, $j = 1, 2$. Note that (2.30) and (2.31) continue to hold for just one norm $\|u_j(k)\|_{L^2}$. If $\eta'_1 = \eta'_2 = \cdots = \eta'_T = 0$ (no kicks), then

$$\|u_j(T)\|_{L^2} \leq e^{-T} R_0 \quad \text{for } T \gg 1$$

and hence

$$\|u_j(T)\|_{L^2} \leq \frac{1}{4} d_0 \tag{2.32}$$

for $j = 1, 2$ if

$$T = \left[\log\left(\frac{4R_0}{d_0}\right)\right] + 1.$$

If the kicks do not all vanish up to time T, then (2.6) implies

$$\|u_j(T)\|_{L^2} \leq e^{-T} R_0 + \frac{e}{e-1}\varepsilon \leq \frac{1}{2} d_0,$$

on the set where $\|\eta'_k\|_{L^2} \leq \varepsilon$ for all $k = 1, \ldots, T$. Therefore,

$$\mathbb{P}(d(T) \geq d_0) \geq \theta > 0 \tag{2.33}$$

for some $\theta = \theta(T) = \theta(d_0, R_0)$.

From now, we consider measures in the space $\mathcal{P}_1(H) \subset \mathcal{P}(H)$, which are those measures $\mu \in \mathcal{P}(H)$ with finite 'first moment':

$$M_1(\mu) := \int_H \|u\|_{L^2} d\mu(u) < +\infty. \tag{2.34}$$

Theorem 2.4. *There exists $N = N(B_0) > 0$ such that if $b_n \neq 0$ for all $|n| \leq N$, then there exists $\kappa < 1$, $C \geq 1$, depending on B_0 and $\{b_n : |n| \leq N\}$ such that*

$$\|T_k^* \mu_1 - T_k^* \mu_2\|_{\mathrm{Lip}}^* \leq C(1 + M_1(\mu_1) + M_1(\mu_2))\kappa^k \tag{2.35}$$

for any $\mu_1, \mu_2 \in \mathcal{P}_1(H)$ and for all times $k \in \mathbb{N}$.

Proof. Let $\mu_j(k) := T_k^* \mu_j$ for $j = 1, 2$ and $k \geq 0$.[6] We want to estimate $\|\mu_1(k) - \mu_2(k)\|_{\mathrm{Lip}}^*$. From Lemma 1.10, we reduce this to constructing an appropriate Kantorovich functional, which we then bound from above by a 'special' coupling; see (1.16). We divide the implementation of this idea into three main steps.

[6]This is "the law of the solution at time k."

Step 1: (Coupling) Our first goal is to construct a 'special' coupling $(U_1(k), U_2(k))$ for $(\mu_1(k), \mu_2(k))$, $k \geq 0$. Take any coupling $(U_1(0), U_2(0))$ on $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$.[7] We now apply Lemma 2.1 with $R = R_0$. Let $d_0 = 1 \wedge \frac{1}{16C_0}$, where C_0 is the constant coming from (2.10) and impose $R_0 \geq 4d_0$. We obtain coupling maps $V_1(u_1, u_2; \omega^1)$ and $V_2(u_1, u_2; \omega^1)$ which are defined for $u_1, u_2 \in B_{R_0} \subset H$. For $j = 1, 2$, we set

$$\widehat{V}_j(u_1, u_2; \omega^1) = \begin{cases} V_j(u_1, u_2; \omega^1) & \text{if } \|u_1 - u_2\|_{L^2} \leq d_0, \ \|u_1\|_{L^2} + \|u_2\|_{L^2} \leq R_0, \\ \Phi(u_j) + \eta(\omega^1) & \text{otherwise, where } \mathcal{L}(\eta) = \mathcal{L}(\eta_1). \end{cases} \tag{2.36}$$

We now define a coupling $(U_1(1), U_2(1))$ on the product space $\Omega^0 \times \Omega^1$ (with the product σ-algebra and product measure) by

$$U_j^{\omega^0, \omega^1}(1) = \widehat{V}_j(U_1^{\omega^0}(0), U_2^{\omega^0}(0); \omega^1), \qquad j = 1, 2. \tag{2.37}$$

By definition of $\widehat{V}_j$ and V_j (see (2.16)), we have

$$U_j^{\omega^0, \omega^1}(1) = \Phi(U_j^{\omega^0}(0)) + \eta_{1,j}^{\omega^1}, \qquad j = 1, 2,$$

where $\mathcal{L}(\eta_{1,j}) = \mathcal{L}(\eta_1)$. In particular, $\mathcal{L}(U_j^{\omega^0, \omega^1}(1)) = \mu_j(1)$ for $j = 1, 2$, that is, $U_j(1)$ is 'distributed correctly.' We now iterate this process T times using successively Lemma 2.1 and obtain maps

$$\widehat{V}_j(u_1, u_2; \omega^2), \ldots, \widehat{V}_j(u_1, u_2; \omega^T)$$

and hence, for $k = 1, 2, \ldots, T$, maps

$$U_j(k) = \Phi(U_j(k-1)) + \eta_{k,j}, \qquad j = 1, 2, \qquad \mathcal{L}(\eta_{k,j}) = \mathcal{L}(\eta_1), \tag{2.38}$$

which are all defined on the same probability space

$$\Omega_T := \Omega^0 \times \Omega' := \Omega^0 \times \Omega^1 \times \cdots \times \Omega^T.$$

The equality (2.38) says $\{U_1(k)\}_{k=0}^T$ and $\{U_2(k)\}_{k=0}^T$ are weak solutions to (KickNSE) in the sense of Definition 2.3. We set $d(k) := \|U_1(k) - U_2(k)\|_{L^2}$ and $R(k) := \|U_1(k)\|_{L^2} + \|U_2(k)\|_{L^2}$ and we consider two cases depending on whether $d(0) \leq d_0$ or otherwise. [8]

Case 1: ('In coupling') Suppose $d(0) \leq d_0$ and $R(0) \leq R_0$. Then, we have

$$\begin{aligned} \mathbb{P}^{\Omega'}\big(d(T) \leq 2^{-T} d_0\big) \geq \mathbb{P}^{\Omega'}\big(d(T) \leq 2^{-T} d(0)\big) &\geq 1 - C_0 d(0)(1 + 2^{-1} + \cdots + 2^{-T+1}) \\ &\geq 1 - 2C_0 d(0) \\ &\geq 1 - 2C_0 d_0. \end{aligned} \tag{2.39}$$

The second inequality in (2.39) above follows from

$$\mathbb{P}^{\Omega'}\big(d(T) \geq 2^{-T} d(0)\big) \leq C_0 d(0)(1 + 2^{-1} + \cdots + 2^{-T+1}). \tag{2.40}$$

We will prove (2.40) by induction on T. The base case $T = 1$ follows immediately from (2.36), (2.37) and (2.10). Now suppose that (2.40) is satisfied for every $1 \leq T \leq T_0 - 1$. Our aim is to verify (2.40) for $T = T_0$. In view of (2.31), we have $R(T) \leq R_0$ for every

[7] Just random initial data distributed according to $\mu_j(0) = \mu_j$.

[8] The broad picture is that when $d(0) \leq d_0$ (re. Case 1), we have a 'good' probability ($\geq 1 - 2C_0 d_0$) that we follow the couplings V_j. We are 'knocked out of coupling' at any later time with probability $\leq 2C_0 d_0$ or if we did not start 'in coupling,' that is, if $d(0) > d_0$. In the latter case (re. Case 2), we may return to coupling with probability θ at which point we then follow Case 1.

$T \le T_0 - 1$. Writing $\Omega'_T := \Omega^1 \times \cdots \Omega^T$, from (2.36), (2.38) and (2.10) and the inductive hypothesis, we have

$$\begin{aligned}
&\mathbb{P}^{\Omega'_{T_0}}\big(\|U_1(T_0) - U_2(T_0)\|_{L^2} \ge 2^{-1}(2^{-(T_0-1)}d(0))\big) \\
&\le \mathbb{P}^{\Omega'_{T_0}}\big(\|V_1(U_1(T_0-1), U_2(T_0-1); \omega^{T_0}) - V_2(U_1(T_0-1), U_2(T_0-1); \omega^{T_0})\|_{L^2} \ge 2^{-1}d(T_0-1) \\
&\qquad\qquad \text{and } d(T_0-1) \le 2^{-(T_0-1)}d(0)\big) + \mathbb{P}^{\Omega'_{T_0}}\big(d(T_0-1) \ge 2^{-(T_0-1)}d(0)\big) \\
&\le C_0 d(0) 2^{-(T_0-1)} + C_0 d(0)(1 + 2^{-1} + \cdots + 2^{-T_0+2})
\end{aligned}$$

which completes the inductive step.

Case 2: ('Not in coupling') In this case, we suppose $d(0) > d_0$ and $R(0) \le R_0$. Then (2.33) implies

$$\mathbb{P}^{\Omega'}(d(T) \le d_0) \ge \theta > 0. \tag{2.41}$$

Step 2: (Kantorovich functional)

Let $\mathrm{dist}(u, v) = \|u - v\|_{L^2} \wedge d_0$ and we set $d = \|u_1 - u_2\|_{L^2}$ and $R = \|u_1\|_{L^2} + \|u_2\|_{L^2}$. We define

$$F(u_1, u_2) = \begin{cases} d & \text{if } d \le d_0, \ \ R \le R_0, \\ 2d_0 & \text{if } d > d_0, \ \ R \le R_0, \\ R & \text{if } R > R_0. \end{cases} \tag{2.42}$$

From a case-by-case analysis[9], we have

$$F(u_1, u_2) \ge \mathrm{dist}(u_1, u_2) \tag{2.43}$$

and hence it follows from this and (2.42) that

$$K(\mu_1, \mu_2) = \inf_{\substack{\text{all couplings} \\ (\xi_1, \xi_2) \text{ for } (\mu_1, \mu_2)}} \mathbb{E}[F(\xi_1, \xi_2)] \tag{2.44}$$

is a Kantorovich functional. Given any coupling $(U_1(0), U_2(0))$ for (μ_1, μ_2), we apply the construction in Step 1 to obtain couplings $(U_1(k), U_2(k))$ of $(\mu_1(k), \mu_2(k))$ for each $k = 1, \dots, T$. We set $F(k) = F(U_1(k), U_2(k))$ and we will estimate $\mathbb{E}[F(k)]$ by $\mathbb{E}[F(0)]$. We partition Ω^0 into three sets and estimate each contribution separately.

Case (a) $\omega^0 \in Q_1 := \{R(0) > R_0\}$

In this case, $F(0) = R(0)$. From (2.30), (2.42) and $2d_0 \le \frac{1}{2}R_0 < \frac{1}{2}R(0)$, we have

$$F(T) \le \frac{1}{2}R(0)$$

and thus

$$\mathbb{E}^{\Omega'}[F(T)] \le \frac{1}{2}F(0). \tag{2.45}$$

Case (b) $\omega^0 \in Q_2 := \{d(0) > d_0, \ R(0) \le R_0\}$

[9] Recall we chose $R_0 \ge 4d_0$.

Here we have $F(0) = 2d_0$. By (2.31), we have $R(T) \leq \frac{1}{2}R_0$ almost surely. Now from (2.41), $F(T) \leq d_0$ with probability $\geq \theta$. Therefore,

$$\mathbb{E}^{\Omega'}[F(T)] \leq (1-\theta)2d_0 + \theta d_0 = \left(1 - \frac{1}{2}\theta\right)2d_0 \leq \left(1 - \frac{1}{2}\theta\right)F(0). \tag{2.46}$$

Case (c) $\omega^0 \in Q_3 := \{d(0) \leq d_0,\ R(0) \leq R_0\}$
We have $F(0) = d(0)$ and again by (2.31), we have $R(T) \leq R_0$ almost surely. From (2.39), we have

$$F(T) = d(T) \leq \frac{1}{2^T}d(0) \quad \text{with probability} \quad \geq 1 - 2C_0 d(0).$$

Thus,

$$\mathbb{E}^{\Omega'}[F(T)] \leq 2^{-T}d(0) + 2d_0\mathbb{P}^{\Omega'}(d(T) > 2^{-T}d(0)) \leq d(0)(2^{-T} + 4C_0 d_0) \leq \frac{3}{4}F(0) \tag{2.47}$$

because $F(0) = d(0)$.

Putting (2.45), (2.46) and (2.47) together, we have

$$\mathbb{E}[F(T)] = \mathbb{E}^{\Omega^0}\left[\sum_{j=1}^{3}\mathbf{1}_{Q_j}\mathbb{E}^{\Omega'}[F(T)]\right] \leq \tilde{\kappa}\,\mathbb{E}^{\Omega^0}[F(0)],$$

where $\tilde{\kappa} = (1 - \frac{1}{2}\theta) \vee \frac{3}{4} < 1$. For $j = kT$, we can iterate the above argument to obtain

$$\mathbb{E}[F(j)] \leq \tilde{\kappa}^k\,\mathbb{E}^{\Omega^0}[F(0)]. \tag{2.48}$$

If $j \in [1, T-1]$, the arguments in Cases (a) and (c) with T replaced by j hold too. However, Case (b) no longer holds true since we used the exponential decay in time (of $\|u_j(T)\|_{L^2}$) to ensure T was large enough so that $\mathbb{P}(d(T) \leq d_0) > \theta > 0$. We cannot say $\mathbb{P}(d(j) \leq d_0) > \theta > 0$. We do have $F(j) \leq d_0 \vee 2d_0 = 2d_0 = F(0)$, which implies

$$\mathbb{E}[F(j)] \leq \mathbb{E}^{\Omega^0}[F(0)].$$

For $t = kT + j$, where $0 \leq j < T$, we have

$$\mathbb{E}[F(t)] \leq \tilde{\kappa}^k\mathbb{E}^{\Omega^0}[F(0)] \leq C\kappa^t\mathbb{E}^{\Omega^0}[F(0)].$$

for some $C > 1$ and $\kappa = \tilde{\kappa}^{\frac{1}{T}}$.

Step 3:
It remains now to compute $\mathbb{E}^{\Omega^0}[F(0)]$. By the definition (2.42), we have

$$\mathbb{E}^{\Omega^0}[F(0)] \leq 2d_0 + \mathbb{E}^{\Omega^0}[\|U_1(0)\|_{L^2} + \|U_2(0)\|_{L^2}] \leq 1 + M_1(\mu_1) + M_1(\mu_2).$$

Since $(U_1(t), U_2(t))$ is a particular coupling, we have

$$K(\mu_1(t), \mu_2(t)) \leq \mathbb{E}[F(t)] \leq C(1 + M_1(\mu_1) + M_1(\mu_2))\kappa^t,$$

for some $\kappa < 1$. Finally, by (1.7) and Lemma 1.10, we obtain

$$\begin{aligned}\|\mu_1(t) - \mu_2(t)\|^*_{\text{Lip}} &\leq \frac{2}{d_0}\|\mu_1(t) - \mu_2(t)\|^*_{\text{Lip,dist}} \\ &\leq \frac{2}{d_0}C(1 + M_1(\mu_1) + M_1(\mu_2))\kappa^t,\end{aligned}$$

which completes the proof of Theorem 2.4.

□

2.5. **Corollaries of the main theorem.** In lecture 5, we saw that (2.7) has an invariant measure $\mu \in \mathcal{P}(H)$. In fact, by iterating (2.29) (assuming zero initial data) we obtain

$$\mathbb{E}[\|u(k)\|_{L^2}] \leq \frac{e}{e-1}\sqrt{B_0}$$

and hence $\mu \in \mathcal{P}_1(H)$. As T_k is Feller, this implies $T_k^*\mu = \mu$ and thus Theorem 2.4 yields the following corollary.

Corollary 2.5. *There exists an invariant measure $\mu \in \mathcal{P}_1(H)$ such that*

$$\|T_k^*\nu - \mu\|_{\mathrm{Lip}}^* \leq C(1 + M_1(\nu))\kappa^k$$

for any $\nu \in \mathcal{P}_1(H)$.

Given any $u \in H$, $\delta_u \in \mathcal{P}_1(H)$ and $T_k^*\delta_u = p_k(u, \cdot)$. Therefore, Corollary 2.5 implies

$$p_k(u, \cdot) \rightharpoonup \mu \quad \text{for any } u \in H,$$

as $k \to \infty$. Then, for any $f \in C_b(H)$ and for any $u \in H$, we have

$$T_k f(u) = \langle T_k f, \delta_u\rangle = \langle f, T_k^*\delta_u\rangle = \langle f, p_k(u, \cdot)\rangle \longrightarrow \langle f, \mu\rangle. \tag{2.49}$$

Corollary 2.6. *The* (KickNSE) *has a unique invariant measure in $\mathcal{P}(H)$.*

Proof. Let $\nu \in \mathcal{P}(H)$ be another invariant measure. Then by (2.49), we have

$$\begin{aligned}\langle f, \nu\rangle = \langle f, T_k^*\nu\rangle &= \langle T_k f, \nu\rangle \\ &\longrightarrow \langle\langle f, \mu\rangle 1, \nu\rangle \\ &= \langle f, \mu\rangle.\end{aligned}$$

That is, $\langle f, \nu\rangle = \langle f, \mu\rangle$ for any $f \in C_b(H)$ and hence $\nu = \mu$. □

3. On ergodicity

In the previous section, we went to great lengths to prove that the invariant measure we constructed by the (comparatively simple) Bogolyubov-Krylov method is in fact the unique invariant measure for (KickNSE) (Corollary 2.6). It turns out that this uniqueness result now implies rather strong and previously inaccessible qualitative information on the long time behaviour of (KickNSE). To see a framework of these implications, we state them in a general fashion.

Given a measure space $(X, \mathcal{F}, \mu)$, we consider a measure preserving map $T : X \to X$; that is, for every $A \in \mathcal{F}$, we have $(T_\#\mu)(A) = \mu(A)$. In particular, the measure μ is invariant under the map T. Invariance alone yields some interesting consequences:

- **Poincare recurrence theorem:** Given $\mu(A) > 0$, there exists $n \in \mathbb{N}$ such that
$$\mu(T^{-n}A \cap A) > 0.$$
- **Furstenberg recurrence theorem:**[10] If $\mu(A) > 0$, then for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that
$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A) > 0.$$

[10]This result was used to prove some famous results in number theory such as Szemeredi's theorem, Roth's theorem, and the van der Waerden theorem.

- **Von Neumann theorem:** Let $\mathcal{I}$ denote the σ-algebra of sets which are invariant under T. Then, for $F \in L^2(X, \mathcal{F}, \mu)$, we have
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} F(T^n x) = \mathbb{E}[f \,|\, \mathcal{I}](x)$$
for μ-almost every $x \in X$.
- **Birkhoff's theorem:** The statement is as in Von Neumann's theorem above except the assumption $F \in L^2(X, \mathcal{F}, \mu)$ is replaced by $F \in L^1(X, \mathcal{F}, \mu)$.

Definition 3.1. A measure preserving map T is said to be **ergodic** if whenever $T^{-1}A = A$, we have $\mu(A) = 0$ or 1.

In other words, the map T is ergodic if the only invariant sets are trivial.

There are many equivalent ways to define ergodicity and we state a few of these below.

Theorem 3.2. *The following are equivalent:*

(i) *T is ergodic with respect to μ*

(ii) *If F is measurable and $F \circ T = F$, then F is constant a.e.*

(iii) *If $F \in L^2(\mu)$ and $F \circ T = F$, then F is constant a.e.*

(iii) *If $F \in L^1(\mu)$, we have*
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(T^n x) = \int F d\mu \qquad \mu - \text{a.e. } x$$

(v) *If $\mu(A) > 0$, we have*
$$\mu\bigg(\bigcup_{n=0}^{\infty} T^{-n} A\bigg) = 1.$$

(vi) *If $\mu(A)$, $\mu(B) > 0$, then there exists n such that*
$$\mu(T^{-n}A \cap B) > 0.$$

(vii) *We have*
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

We denote by Λ the collection of invariant probability measures for a given Markov semigroup. It is easy to see that Λ is convex. It turns out the ergodic measures hold a special place in Λ.

Theorem 3.3. *The set of extremal points of Λ is equal to the set of ergodic probability measures.*

Immediately following this theorem, we have:

Corollary 3.4. *A unique invariant measure is ergodic.*

We proved in Corollary 2.6 that μ is the unique invariant measure under T_k. Hence, μ is ergodic for T_k and thus the seven other statements in Theorem 3.2 hold for the dynamics of (KickNSE).

A uniqueness result also holds in the case of the white forced NSE where, for instance, we make the following modifications to the argument in Subsection 2.4:

KickNSE	White NSE
$d(k) \leq d_0$	$d(t) \leq d_0$
$R(k) \leq R_0$	$R(t) \leq R_0\sqrt{t - T_*}, \qquad T_* + 1 \leq t \leq k$
	use an 'adjusted' Girsanov theorem

See [1, page 57] for a more complete list of the differences.

4. Further issues to study

- **Random attractors:** see [4, Section 4.2].
- **The Eulerian limit:** consider the equation

$$\partial_t u - \nu L u + B(u) = \sqrt{\nu}\partial_t \zeta, \tag{4.1}$$

where $\nu > 0$ is the kinematic viscosity. For each $\nu > 0$ we have a unique invariant measure μ_ν for the flow of the corresponding ν-equation (4.1). The idea is to send $\nu \to 0$ and thereby construct an invariant measure for the incompressible Euler equation ($\nu = 0$ in (4.1)). See for instance [1, Section 10], [2, Section 5]. This idea has been applied for Schrödinger-Heat type equations [3] and for constructing invariant measures supported on smooth functions for the (purely dispersive) Benjamin-Ono equation [5].

References

[1] S. B. Kuksin, *Randomly forced nonlinear PDEs and statistical hydrodynamics in 2 space dimensions*, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2006. x+93 pp. ISBN: 3-03719-021-3.

[2] S. B. Kuksin, A. Shirikyan, *A coupling approach to randomly forced nonlinear PDE's. I*, Commun. Math. Phys. 221, 351–366 (2001).

[3] S. B. Kuksin, A. Shirikyan, *Randomly forced CGL equation: stationary measures and the inviscid limit*, Journal of Physics A: Mathematical and General 37 (2004), no. 12, 3805–3822.

[4] S. B. Kuksin, A. Shirikyan, *Mathematics of two-dimensional turbulence*, Cambridge Tracts in Mathematics, 194. Cambridge University Press, Cambridge, 2012. xvi+320 pp

[5] M. Sy, *Invariant measure and long time behavior of regular solutions of the Benjamin–Ono equation*, Anal. PDE Volume 11, Number 8 (2018), 1841–1879.