Lem. 1: $\exists$ prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $\forall R \geq 1$, $\exists N = N(R) \geq 1$ s.t. if $b_n \neq 0$, $\forall |n| \leq N$, then for any $u_1, u_2 \in B_R \subset H$, the measures $\mu_1 = P_1(u_1, \cdot)$, $\mu_2 = P_1(u_2, \cdot)$ admit a coupling (V_1, V_2), $V_j = V_j(u_1, u_2, \omega)$

(a) $V_j : B_R \times B_R \times \Omega \to H$ is measurable

(b) with $d = \|u_1 - u_2\|_{L^2}$, we have $\mathbb{P}(\|V_1 - V_2\|_{L^2} \geq \frac{1}{2} d) \leq C_0 d$.

In the last step of the proof, we controlled the H^1-norm of the difference of two solutions by the L^2-norm of the difference of the initial data (d).

Some PDE estimates: u_1, u_2 solutions to $\partial_t u_j - L u_j + B(u_j) = f_j$.

Let $w = u_1 - u_2 \Rightarrow \partial_t w - Lw + B(w, u_1) + B(u_2, w) = f_1 - f_2$. (*)

Multiply by $2w$ and $\int \cdot dx$.

$$\Rightarrow \partial_t \left(\|w\|_{L^2}^2 + \int_0^t \|w\|_{H^1}^2 dt' \right) \lesssim \|u_1\|_{H^1}^2 \left(\|w\|_{L^2}^2 + \int_0^t \|w\|_{H^1}^2 dt' \right) + 2\|f_1 - f_2\|_{H^{-1}}^2.$$

Gronwall $\Rightarrow$
$$\|w(t)\|_{L^2}^2 + \int_0^t \|w\|_{H^1}^2 dt' \leq e^{c\int_0^t \|u_1\|_{H^1}^2 dt'} \|u_1(0) - u_2(0)\|_{L^2}^2 + c\int_0^t e^{c\int_{t'}^t \|u_1\|_{H^1}^2 dt''} \|f_1(t') - f_2(t')\|_{H^{-1}}^2 dt'$$

Prop. 2.1.25 in K-S

Multiply (*) by $2t(-Lw)$ and $\int \cdot dx$.

$$\Rightarrow \partial_t \left(t \|w\|_{H^1}^2 \right) + 2t \|Lw\|_{L^2}^2 = \|w\|_{H^1}^2 + 2t \langle B(w, u_1), Lw \rangle + 2t \langle B(u_2, w), Lw \rangle + 2t \langle f_1 - f_2, Lw \rangle$$

integrate

$$\Rightarrow t \|w\|_{H^1}^2 + \int_0^t t' \|Lu\|_{L^2}^2 dt' \lesssim \int_0^t \|w\|_{H^1}^2 dt' + \int_0^t t' \|w\|_{L^2}^2 \left(\|u_1\|_{L^2}^2 \|Lu_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 \|Lu_2\|_{L^2}^2 \right) dt' + 2\int_0^t t' \|f_1 - f_2\|_{L^2}^2 dt'$$

Claim: $\int_0^t t' \|u_j\|_{L^2}^2 \|Lu_j\|_{L^2}^2 dt' \lesssim t^{-2} e^{c\int_0^t \|u_j\|_{H^1}^2 + \|f\|_{L^2}^2 dt'}$

$$\Rightarrow \|w\|_{H^1}^2 \lesssim \int_0^t \|f_1 - f_2\|_{L^2}^2 dt' + A(t) t^{-3} \left(\|u_1(0) - u_2(0)\|_{L^2}^2 + \int_0^t \|f_1 - f_2\|_{H^{-1}}^2 dt' \right)$$

CHECK!

$$A(t) = e^{c\int_0^t \|u_1\|_{H^1}^2 + \|u_2\|_{H^1}^2 + \|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2 \, dt'}$$

Def. A process $\{u(k)\}_{k\geq 0} \subset H$ defined on some probability space is called a weak solution to (kick NSE):

$u(k) = \Phi(u(k-1)) + \eta_k$. (kick NSE)

if it satisfies (kick NSE) with random kicks $\{\eta_k\}$ replaced by some other process $\{\tilde{\eta}_k\}$ with $\mathcal{L}(\tilde{\eta}_k) = \mathcal{L}(\eta_k)$.

may be different but correlated

- $\{u_j(k)\}$ weak solutions with random kicks $\{\eta_k^j\}$, $j = 1, 2$.

  Set $d(k) = \|u_1(k) - u_2(k)\|_{L^2}$

  $R(k) = \|u_1(k)\|_{L^2} + \|u_2(k)\|_{L^2}$

Recall : $\|u(k+1)\|_{L^2} \leq e^{-1}\|u(k)\|_{L^2} + \|\eta_k^j\|_{L^2}$

$\leq e^{-1}\|u(k)\|_{L^2} + \underbrace{\sqrt{B_0}}_{\text{simplification, not needed}}$

$\Rightarrow R(k+1) \leq e^{-1}R(k) + 2\sqrt{B_0}$

iterate $\Rightarrow R(l) \leq e^{-(l-k)}R(k) + \underbrace{\frac{2e}{e-1}\sqrt{B_0}}_{\leq (\frac{1}{2}-\frac{1}{e})R_0}$, choose $R_0 \geq 1$ s.t. the upper bound holds.

- If $R_0 \leq R(k)$, then

  $R(l) \leq \frac{1}{2}R(k)$, $\forall l \geq k+1$ (1)

- If $R(k) \leq R_0$, then

  $R(l) \leq (e^{-(l-k)} + \frac{1}{2} - \frac{1}{e})R_0 \leq \frac{1}{2}R_0$, $\forall l \geq k+1$

  $\Rightarrow R(l) \leq R_0$, $\forall l \geq k$. (2)

- Fix $0 \leq d_0 \leq 1$. Let u_1, u_2 be two weak solutions with the same kick $\{\eta_k^1\}$.

  Suppose $R(0) \leq R_0$. $\Rightarrow \|u_j(0)\|_{L^2} \leq R_0$

  So, if $\eta_1^1 = \eta_2^1 = \dots = \eta_T^1 = 0$, then

  $\|u_j(T)\|_{L^2} \leq e^{-T}R_0 \leq \frac{1}{4}d_0$ for $T \gg 1$

  $\left(T = \left[\log\left(\frac{4R_0}{d_0}\right)\right] + 1\right)$

By assumption $\mathbb{P}(|\eta'_k| \le \varepsilon) > 0, \forall \varepsilon > 0.$

$\Longrightarrow \|u_j(T)\|_{L^2} \le e^{-T} R_0 + \frac{e}{e-1}\varepsilon \le \frac{1}{2} d_0$ $\quad$ (if $\|\eta'_k\|_{L^2} \le \varepsilon, k=1,\dots,T$

$\Longrightarrow \mathbb{P}(\underset{\|u_1(T)-u_2(T)\|_{L^2}}{\underbrace{d(T)}} - d_0) \ge \theta > 0$ (3), for some $\theta = \theta(T) = \theta(d_0, R_0)$

- $\mathcal{P}_1(H) \subset \mathcal{P}(H)$
  - s.t. $M_1(\mu) = \int \|u\|_{L^2} \mu(du) < \infty$

Thm 2 : $\exists N = N(B_0) > 0$ s.t. if $b_n \neq 0$, $\forall |n| \le N$, then $\exists \kappa < 1$, $C \ge 1$, depending on B_0, $\{b_n, |n| \le N\}$ s.t.

$$\| T_k^* \mu_1 - T_k^* \mu_2 \|_L^* \le C(1 + M_1(\mu_1) + M_1(\mu_2)) \kappa^k$$

for any $\mu_1, \mu_2 \in \mathcal{P}_1(H)$, $\forall k \in \mathbb{N}$.

Pf. : Step 1 (coupling)

Let $\mu_j(k) = T_k^* \mu_j$, $j = 1,2$, $k \ge 0$.

1st goal : construct special coupling $(v_1(k), v_2(k))$ for $(\mu_1(k), \mu_2(k))$ $k \ge 0$.

- Take any coupling $(v_1(0), v_2(0))$ ← random initial data distributed by μ_j on $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ ← copy of $(\Omega, \mathcal{F}, \mathbb{P})$ in Lemma 1.

Apply Lemma 1 with $R = R_0$. Let $d_0 = 1 \wedge \frac{1}{16C_0}$, Impose $R_0 \ge 4d_0$.

$\Longrightarrow V_1(u_1, u_2, \omega^1)$ $\quad u_1, u_2 \in B_{R_0} \subset H$

$V_2(u_1, u_2, \omega^1)$

Set $\hat{V}_j(u_1, u_2, \omega^1) = \begin{cases} V_j & \text{if } \|u_1 - u_2\|_{L^2} \le d_0, \|u_1\|_{L^2} + \|u_2\|_{L^2} \le R_0 \\ \Phi(u_j) + \eta_j(\omega^1), & \text{otherwise} \end{cases}$

with $\mathcal{L}(\eta) = \mathcal{L}(\eta_1)$.

We now define a coupling $(v_1(1), v_2(1))$ on $\Omega^0 \times \Omega^1$ by

$$v_j^{\omega^0, \omega^1}(1) = \hat{V}_j(v_1^{\omega_0}(0), v_2^{\omega_0}(0); \omega^1)$$

with $(\Omega^j, \mathcal{F}^j, \mathbb{P}_j) = $ copy of $(\Omega, \mathcal{F}, \mathbb{P})$.

By def. of $\hat{V}_j$ (and V_j)

$$v_j^{\omega_0,\omega_1}(1) = \Phi(v_j^{\omega_0}(0)) + \eta_{1,j}^{\omega_1}, \quad j=1,2$$

where $\mathcal{L}(\eta_{1,j}) = \mathcal{L}(\eta_1)$.

$\Rightarrow$ In particular, $\mathcal{L}(v_j^{\omega^0,\omega^1}(1)) = \mu_j(1)$, $j=1,2$.

Iterate this process T times, using successively

$\hat{V}_j(\mu_1,\mu_2;\omega^2)$

$\vdots$ by Lemma 1.

$\hat{V}_j(\mu_1,\mu_2;\omega^T)$

$\Rightarrow$ Thus, we obtain $v_j(k) = \Phi(v_j(k-1)) + \eta_{k,j}$, $j=1,2$.

They are defined on

$$\Omega_T = \Omega^0 \times \Omega^1 = \Omega^0 \times \underbrace{\Omega^1 \times \ldots \times \Omega^T}_{\Omega'}$$

v_1, v_2 are weak solutions to (KickNSE). | viewed as a discrete time random dynamical system

(i) $\mathbb{P}^{\Omega'}\left(d(T) \le \frac{1}{2^T} d(0)\right) \ge 1 - C_0 d(0)(1+2^{-1}+\ldots+2^{-T+1})$

$\ge 1 - C_0 d(0)$

(ii) $\mathbb{P}^{\Omega'}(d(T) \le d_0) \ge \theta > 0$

If $R(0) \le R_0$ then,
from ③,

Step 2 (Kantorovich functional)

let $\mathrm{dist}(u,v) = \|u-v\|_{L^2} \wedge d_0$. "in coupling"

Set $F(u_1,u_2)$ by $F(u_1,u_2) = \begin{cases} d & \text{if } d \le d_0,\ R \le R_0 \\ 2d_0 & \text{if } d > d_0,\ R \le R_0 \\ R & \text{if } R > R_0 \end{cases}$

where $d = \|u_1-u_2\|_{L^2}^2$, $R = \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2$.

$\Rightarrow F(u_1,u_2) \ge \mathrm{dist}(u_1,u_2)$

Kantorovich functional: $K(\mu_1,\mu_2) = \inf\limits_{\substack{(z_1,z_2)\\ \text{coupling}}} \left(\mathbb{E}[F(z_1,z_2)] \right)$.

Set $F(k) = F(v_1(k), v_2(k))$.

(a) $\omega^\circ \in Q_1 = \{R(0) > R_0\} \Rightarrow F(0) = R(0)$

$\mathbb{E}^{\Omega'}[F(T)] \le \underbrace{\tfrac{1}{2}}_{<1} F(0)$ by (1) and $2d_0 \le \frac{1}{2}R_0 \le \frac{1}{2}R(0)$.

not needed, also holds as a deterministic estimate

(b) $\omega^\circ \in Q_2 = \{d(0) > d_0,\ R(0) \le R_0\} \Rightarrow F(0) = 2d_0$

By (2), $R(T) \le R_0$ a.s., i.e., $F(T) = \underset{\le d_0}{d(T)}$ or $2d_0$.

By (ii), with probability $\ge \theta$, we have $F(T) \le d_0$.

$\Rightarrow \mathbb{E}^{\Omega'}[F(T)] \le (1-\theta)2d_0 + \theta d_0 = (1-\frac{1}{2}\theta)\cdot 2d_0 = \underbrace{(1-\tfrac{1}{2}\theta)}_{<1}F(0)$

(c) $\omega^\circ \in Q_3 = \{d(0) \le d_0,\ R(0) \le R_0\} \Rightarrow F(0) = d(0)$

By (2), $R(T) \le R_0$ a.s.

By (1), we have

$F(T) = d(T) \le \frac{1}{2^T} d(0)$ with prob. $\ge 1 - 2C_0 d(0)$

$$\Rightarrow \mathbb{E}^{\Omega'}[F(T)] \le 2^{-T} d(0)\,(\underbrace{1-2C_0 d(0)}_{\text{drop}}) + 2d_0 \cdot 2C_0 d(0)$$

$$\le d(0)\,(\underbrace{2^{-T}}_{\le \frac{1}{2}\ (T\ge 1)} + \underbrace{4C_0 d_0}_{\frac{1}{4}}) \le \underbrace{\tfrac{3}{4}}_{<1} F(0).$$

Put (a), (b), (c) together

$$\mathbb{E}[F(T)] = \mathbb{E}^{\Omega^{(0)}}\left[\mathbb{1}_{Q_1}\mathbb{E}^{\Omega'}(F(T)) + \mathbb{1}_{Q_2}\mathbb{E}^{\Omega'}(F(T)) + \mathbb{1}_{Q_3}\mathbb{E}^{\Omega'}(F(T))\right]$$

$$\le \mathbb{E}^{\Omega^\circ}\left(\mathbb{1}_{Q_1}\tfrac{1}{2}F(0) + \mathbb{1}_{Q_2}(1-\tfrac{1}{2}\theta)F(0) + \mathbb{1}_{Q_3}\tfrac{3}{4}F(0)\right)$$

$$\le \tilde{K}\ \mathbb{E}[F(0)] \quad \text{for } \tilde{K} = (1-\tfrac{1}{2}\theta) \vee \tfrac{3}{4} < 1.$$

For $j = KT$, by iterating the argument,

$$\mathbb{E}[F(j)] \le \tilde{K}^K\, \mathbb{E}[F(0)].$$

If $j \in [1, T-1]$,

the arguments in (a) and (c) with T replaced by j hold.

But case (b) no longer holds since (3) fails for j

In this case (b), we have

$\mathbb{E}[F(j)] \le \mathbb{E}[F(0)]$

$(F(j) \le d_0 \vee 2d_0 = 2d_0 = F(0))$

$\mathbb{P}(d(T) \le d_0) > \theta > 0$.

$\Leftarrow$ we used $\|u_j(T)\| \le \frac{1}{2}d_0$ for some large T, depends on decay.

cannot say $\mathbb{P}(d(j) \le d_0) \ge \theta > 0$.

For $t = kT + j$, $0 \le j < T$, we have

$$\mathbb{E}[F(t)] \le \tilde{\kappa}^k \, \mathbb{E}[F(0)]$$
$$\le c\,\kappa^t \, \mathbb{E}[F(0)] \quad , \quad \kappa = \tilde{\kappa}^{1/T}$$
$$c = \tilde{\kappa}^{-1} > 1$$

<u>Step 3:</u>

$$\mathbb{E}[\underbrace{F(v_1(0), v_2(0))}_{\le 2d_0 + R(0)}] \le 2d_0 + \mathbb{E}[\|v_1(0)\|_{L^2} + \|v_2(0)\|_{L^2}]$$
$$\le 1 + M_1(\mu_1) + M_1(\mu_2) \quad , d_0 \le 1/2$$

$$\Longrightarrow K(\mu_1(t), \mu_2(t)) \le \mathbb{E}[F(t)]$$
$$\le c\,(1 + M_1(\mu_1) + M_1(\mu_2))\,\kappa^t \quad , \quad \kappa < 1$$

By Lemma D

$$\|\mu_1(t) - \mu_2(t)\|_L^* \le \frac{2}{d_0} \|\mu_1(t) - \mu_2(t)\|_{L, d_0}^* \leftarrow \text{dist} = \|u-v\|_{L^2} \wedge d_0$$
$$\le \frac{2}{d_0}\, c\,(1 + \cancel{M_1} M_1(\mu_1) + M_1(\mu_2))\,\kappa^t . \ \boxtimes$$

By Bogolivbov-Krylov argument, we constructed a stationary measure $\mu \in \mathcal{P}_1(H)$. In fact, $\mathbb{E}(\|u\|_{L^2}^p) < \infty$ $\forall p < \infty$.

$\uparrow \ \mathbb{E}(\|u(k)\|_{L^2}) \le \frac{e}{e-1}\sqrt{B_0}$

<u>Cor.</u>: $\exists$ stationary measure $\mu \in \mathcal{P}_1(H)$ s.t.

$$\|T_k^* \nu - \mu\|_L^* \le c\,(1 + M_1(\nu))\,\kappa^k \qquad \forall \nu \in \mathcal{P}_1(H).$$

In particular, $\underset{\overset{\|}{T_k^* \delta_u}}{P_k(u,\cdot)} \longrightarrow \mu$ as $k \to \infty$, $\forall u \in H$.

Hence, $T_k f(u) = \langle f, P_k(u,\cdot)\rangle \longrightarrow \underset{\overset{\|}{\text{const.}}}{\langle f, \mu\rangle}$, (*) $\quad \forall u \in H$, $\forall f \in C_b(H)$

<u>Cor.</u>: (kick NSE) has a unique stationary measure in $\mathcal{P}(H)$.

<u>Pf.</u>: If $\nu \in \mathcal{P}(H)$ is a stationary measure, then by (*)

$$\langle f, \nu\rangle \underset{\nu \text{ inv.}}{=} \langle f, T_k^* \nu\rangle = \langle T_k f, \nu\rangle \longrightarrow \langle \langle f,\mu\rangle \cdot \underset{\text{const. 1 function}}{1}, \nu\rangle$$
$$= \langle f, \mu\rangle \quad \forall f \in C_b(H)$$

$\Longrightarrow$ Hence, $\mu = \nu$. $\boxtimes$

On Ergodicity : T, meas-preserving map on (X,μ).

- Poincaré recurrence theorem

  If $\mu(A)>0$, $\exists\, n\in\mathbb{N}$ s.t. $\mu(T^{-n}A\cap A)>0$.

- Furstenberg multiple recurrence thm

  If $\mu(A)>0$, then $\forall k\in\mathbb{N}$, $\exists n\in\mathbb{N}$ s.t. $\mu(A\cap T^{-n}A\cap\ldots\cap T^{-kn}A)>0$.

- Von Neumann thm : $F\in L^2(\mu)$
- Birkhoff thm : $F\in L^1(\mu)$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F(T^nx)=F^*(x)\quad \text{a.e.}$$

used to prove
Szemeredi's thm
Roth thm
Van der Waerden thm

$$\int F^*d\mu=\int F d\mu$$

Def.: T is called ergodic if $TA=A$ "mod 0" (up to a set of μ-measure 0.) then $\mu(A)=0$ or 1.

TFAE: (i) T is ergodic.

(ii) if F is measurable and $F\circ T=F$, then $F\equiv$ const. a.e.

(iii) if $F\in L^2(\mu)$ and $F\circ T=F$, then $F\equiv$ const. a.e.

only invariant func. are const. func.

(iv) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F(T^nx)=\int f\,d\mu$ a.e. x.

time ave. = ensemble ave.

(v) If $\mu(A)>0$, we have $\mu\left(\bigcup_{n=0}^{\infty}T^{-n}A\right)=1$. "visits everywhere"

(vi) If $\mu(A),\mu(B)>0$, $\exists n$ s.t. $\mu(\underbrace{T^{-n}A\cap B}_{x\in B \text{ and } T^nx\in A})>0$

(vii) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(T^{-n}A\cap B)=\mu(A)\mu(B)$

Let Ω = collection of stationary prob. measures for a given Markov semigroup.

Ω is closed under convex combination, $\alpha\mu_1+(1-\alpha)\mu_2$.

Thm.: Set of invariant ergodic prob. meas. = extremal pts. of Ω.

Cor.: A unique inv. meas. is ergodic.

White Forced NSE : A similar result holds.

$k \in \mathbb{K}$ $\quad d(k) \leq d_0 \longrightarrow d(t) \leq d_0$

$R(k) \leq R_0 \qquad R(t) \leq R_0 \sqrt{t - T_*}, \quad T_* + 1 \leq t \leq K$

use "adjusted Girsanov thm"

other issues:

① random attractors

② Eulerian limit

$$\partial_t u - \nu L u + B(u) = \sqrt{\nu}\, \partial_t \zeta$$

(kinematic) viscosity
constant, not measure

send $\nu \to 0$; incompressible Euler

$\partial_t u + B(u) = 0 \qquad \leftarrow$ prove $\exists$ of inv. meas. for Euler.