28/3/2019 | TWO-DIMENSIONAL STATISTICAL HYDRODYNAMICS LECTURE 7

PRELIMINARY FROM MEASURE THEORY

X = Polish space (complete separable metric space)

Weak convergence & dual-Lipschitz distance

Recall $\mu_n \to \mu$ if $\langle f, \mu_n \rangle \to \langle f, \mu \rangle$, $\forall f \in C_b(X)$

- Lipschitz norm

$$\|f\|_L = \sup_{x \in X} |f(x)| + \mathrm{Lip}(f), \quad \text{where } \mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_X(x,y)}.$$

Set $\mathcal{V} = \{ f \in C_b(X) \mid \|f\|_L \leq 1 \}$.

Prop. A: $\mu_n \to \mu$ iff $\langle f, \mu_n \rangle \to \langle f, \mu \rangle$ $\forall f \in \mathcal{V}$.

- Dual Lipschitz distance on $\mathcal{P}(X)$ ← collection of prob. measures

$$\|\mu - \nu\|_L^* = \sup_{f \in \mathcal{V}} |\langle f, \mu \rangle - \langle f, \nu \rangle|$$

Thm. B: $\mathcal{P}(X)$ is complete w.r.t. dual Lipschitz distance.

$\{\mu_n\}$ converges to μ in this space iff $\mu_n \to \mu$.

Rmk: $\mathcal{M}(X)$ is NOT complete w.r.t. $\|\cdot\|_L^*$

← collection of measures on X.

- Let $0 < d \leq 1$. Define a new distance on X by $\tilde{d}(x_1, x_2) = d_X(x_1, x_2) \overset{\text{min}}{\wedge} d$.
- d_X and $\tilde{d}$ define the same topology in X.
- Moreover, $\|\mu - \nu\|_{L,\tilde{d}}^* \leq \|\mu - \nu\|_L^* \leq \frac{2}{d} \|\mu - \nu\|_{L,\tilde{d}}^*$

← dual lip. dist. induced by $\tilde{d}$.

$$\left(\begin{array}{l} \|f\|_{L,\tilde{d}} \geq \|f\|_L \\ \text{Let } f \in C_b \text{ s.t. } \|f\|_L \leq 1 \leq \|f\|_{L,\tilde{d}} \\ \text{For } d(x,y) > d, \quad \frac{d}{2} \frac{|f(x) - f(y)|}{d} \leq \frac{2\|f\|_{L^\infty}}{2} \leq 1 \\ \quad \Rightarrow \frac{d}{2} \|f\|_{L,\tilde{d}} \leq 1 \end{array} \right.$$

- Variational distance: $\mu, \nu \in \mathcal{P}(X)$

$$\|\mu - \nu\|_{var} = \sup_{A \in \mathcal{B}_X} |\mu(A) - \nu(A)| \qquad (\leq 1)$$

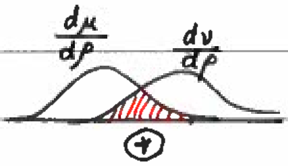
- $\mu \perp \nu$ iff $\|\mu - \nu\|_{var} = 1$

· $\|\mu-\nu\|_{\mathrm{var}} = \frac{1}{2} \sup_{\substack{f\in C_b(X)\\ \|f\|_{L^\infty}\le 1}} |\langle f,\mu\rangle - \langle f,\nu\rangle|$ ex "$\rho = \frac{\mu+\nu}{2}$

measures
"non-overlapping"
of μ & ν ⊛

$$= \frac{1}{2}\int_X \left| \frac{d\mu}{d\rho}(x) - \frac{d\nu}{d\rho}(x) \right| d\rho(x) \quad , \mu,\nu \ll \rho$$

$$= 1 - \int_X \left(\frac{d\mu}{d\rho} \wedge \frac{d\nu}{d\rho} \right) d\rho$$



$\Longrightarrow \|\mu-\nu\|_L^* \le 2\|\mu-\nu\|_{\mathrm{var}}$

Thm C: $(\mathcal{P}(X), \|\cdot\|_{\mathrm{var}})$ is complete.

- $\mu_n \to \mu$ in this space iff $\langle f,\mu_n\rangle \to \langle f,\mu\rangle$ uniformly for $f\in C_b(X)$, $\|f\|_{L^\infty} \le 1$.
- $\mathcal{P}(X) \underset{\text{closed}}{\subset} (C_b(X))^*$

Coupling

Def: Let $\mu_1,\mu_2 \in \mathcal{P}(X)$. A pair of r.v.'s ξ_1,ξ_2 defined on the same probability space is called a coupling for (μ_1,μ_2) if $\mathcal{L}(\xi_j) = \mu_j$, $j=1,2$.

·Given a coupling $\xi = (\xi_1,\xi_2)$ is a r.v. on $X\times X$ with $\mathcal{L}(\xi) = \mu$.

$$\mu_1 = \underset{\mu\circ\pi_1^{-1}}{\underset{\shortparallel}{(\pi_1)_*\mu}} \quad , \quad \mu_2 = (\pi_2)_*\mu$$

·Given a coupling (ξ_1,ξ_2) for (μ_1,μ_2)

$$\begin{aligned}\mu_1(A) - \mu_2(A) &= \mathbb{E}[\mathbb{1}_A(\xi_1) - \mathbb{1}_A(\xi_2)] \\ &= \mathbb{E}[\mathbb{1}_{\xi_1\neq\xi_2}(\mathbb{1}_A(\xi_1) - \mathbb{1}_A(\xi_2))] \\ &\le \mathbb{P}(\xi_1\neq\xi_2)\end{aligned}$$

$\Longrightarrow \|\mu_1-\mu_2\|_{\mathrm{var}} \le \mathbb{P}(\xi_1\neq\xi_2)$

Def: A coupling (ξ_1,ξ_2) is called maximal if

$\mathbb{P}(z_1 \neq z_2) = \|\mu_1-\mu_2\|_{var}$ & z_1 and z_2 conditioned on the event $N = \{z_1 \neq z_2\}$ are independent i.e. $\forall A, B \in \mathcal{B}_X$

$$\mathbb{P}(z_1 \in A, z_2 \in B \mid N) = \mathbb{P}(z_1 \in A \mid N) \times \mathbb{P}(z_2 \in B \mid N).$$

Dobrushin's lemma

Lem: Given any $\mu_1, \mu_2 \in \mathcal{P}(X)$ $\exists$ maximal coupling (z_1, z_2).

pf: $\delta = \|\mu_1-\mu_2\|_{var}$.

If $\delta = 1$, any pair (z_1, z_2) of indep. r.v.'s with $\mathcal{L}(z_j) = \mu_j$, $j = 1, 2$, is a maximal coupling for (μ_1, μ_2).

If $\delta = 0$, then $\mu_1 = \mu_2$, so any r.v. z with $\mathcal{L}(z) = \mu_1$, the pair (z, z) is a maximal coupling.

Assume $0 < \delta < 1$.

δ = non-overlapping

$1-\delta$ = overlapping

$$m = \tfrac{1}{2}(\mu_1 + \mu_2)$$

$$\rho_j = \frac{d\mu_j}{dm}, \quad \rho = \rho_1 \wedge \rho_2, \quad \hat{\rho}_j = \frac{1}{\delta}(\rho_j - \rho)$$

$$\left(\Rightarrow \rho_j = \rho + \delta \hat{\rho}_j\right)$$

CHECK: $d\hat{\mu}_j = \hat{\rho}_j \, dm$

$d\mu = \frac{1}{1-\delta} \rho \, dm$ are probability measures on X.

Let $\zeta_1, \zeta_2, \zeta, \alpha$ independent, defined on the same prob. space s.t.

$\mathcal{L}(\zeta_j) = \hat{\mu}_j$, $\mathcal{L}(\zeta) = \mu$, $\mathbb{P}(\alpha = 0) = \delta$, $\mathbb{P}(\alpha = 1) = 1-\delta$.

claim: $z_j = \alpha \zeta + (1-\alpha)\zeta_j$, $j = 1, 2$, form a maximal coupling for (μ_1, μ_2)

Given $A \in \mathcal{B}_X$,

coupling

$$\mathbb{P}(z_j \in A) = \mathbb{P}(z_j \in A, \alpha = 0) + \mathbb{P}(z_j \in A, \alpha = 1)$$

$$\overset{\text{indep.}}{=} \underline{\mathbb{P}(\alpha=0)}\,\mathbb{P}(\underset{\underset{\zeta_j}{\|}}{z_j} \in A) + \underline{\mathbb{P}(\alpha=1)}\,\mathbb{P}(\underset{\underset{\zeta}{\|}}{z_j} \in A)$$

$$= \delta \int_A \underset{\frac{1}{\delta}(\rho_j - \rho)}{\underset{\|}{\hat{\rho}_j(x)}} \, dm + (1-\delta)\frac{1}{1-\delta}\int_A \rho \, dm$$

$$= \int_A \rho_j \, dm$$

$$= \mu_j(A)$$

maximality

Moreover, $\mathbb{P}(\xi_1 \neq \xi_2) = \mathbb{P}(\xi_1 \neq \xi_2, \alpha=0) + \mathbb{P}(\overset{\zeta}{\overset{\shortparallel}{\xi_1}} \neq \cancelto{0}{\overset{\zeta}{\overset{\shortparallel}{\xi_2}}}, \alpha=1)$

$$\overset{\text{ind.}}{=} \mathbb{P}(\alpha=0)\,\mathbb{P}(\zeta_1 \neq \zeta_2)$$

$$= \delta = \|\mu_1-\mu_2\|_{var}$$

$$\begin{cases} \hat{\rho}_1(x)\,\hat{\rho}_2(x) \equiv 0 \\ \Rightarrow \mathbb{P}(\zeta_1 = \zeta_2) = \iint_{\{x_1=x_2\}} \hat{\rho}_1(x_1)\,\hat{\rho}_2(x_2)\, m(dx_1)\, m(dx_2) = 0 \\ \Rightarrow \mathbb{P}(\zeta_1 \neq \zeta_2) \end{cases}$$

CHECK: ξ_1 & ζ_2 conditional on $(\xi_1 \neq \xi_2)$ are independent.

$\xi_1 \neq \xi_2 \overset{\text{by def.}}{\Longrightarrow} \alpha = 0 \Rightarrow \xi_j = \zeta_j$

By def., ζ_1 & ζ_2 are independent. ∎

Cor: Any $\mu_1, \mu_2 \in \mathcal{P}(X)$ admits a representation

$$\mu_j = \boxed{(1-\delta)\mu} + \delta \nu_j, \quad j=1,2$$

where $\delta = \|\mu_1-\mu_2\|_{var}$,

$\nu_1, \nu_2, \mu \in \mathcal{P}(X)$,

$\nu_1 \perp \nu_2$.

called minimum of μ_1 and μ_2 denoted by $\mu_1 \wedge \mu_2$

~~Def:~~

Kantorovich functional

F measurable func. on $X \times X$ s.t.

$$F(x_1,x_2) = F(x_2,x_1) \geq \text{dist}(x_1,x_2) \quad \forall x_1, x_2 \in X$$

Define the Kantorovich functional $K = K_F$ associated with F by

$$K(\mu_1,\mu_2) = \inf \mathbb{E}[F(\xi_1,\xi_2)]$$

where infimum is taken over all couplings (ξ_1,ξ_2) for (μ_1,μ_2).

- F = Kantorovich density for the functional K_F.

Lem. D: We have $\|\mu_1-\mu_2\|_L^* \leq K_F(\mu_1,\mu_2)$.

Pf: Let (ξ_1,ξ_2) be a coupling for (μ_1,μ_2).

Take $f \in C_b(X)$, $\|f\|_L \leq 1$. Then, we have

$$\langle f, \mu_1-\mu_2\rangle = \mathbb{E}[f(\xi_1) - f(\xi_2)]$$

$$\overset{f\ \text{Lip}}{\leq} \mathbb{E}[\text{dist}(\xi_1,\xi_2)]$$

$$\leq \mathbb{E}[F(\xi_1,\xi_2)]$$

$\Rightarrow$ take sup in f

take inf in (ξ_1,ξ_2) coupling. ∎

Uniqueness of a stationary measure for kicked NSE

$u(k) = \Phi(u(k-1)) + \eta_k$

$\Phi = \Phi_1$ time 1 solution map for NSE $(f \equiv 0)$

$\eta_k = \sum_{n \in \mathbb{Z}_0^2} b_n g_{kn} e_n$

$\mathcal{L}(g_{kn}) = p_n(r)\, dr \leftarrow$ Lipschitz, $\operatorname{supp} \subset [-1,1]$, $p_n(0) \neq 0$.

We'll prove exponential mixing, i.e., distributions of any solution converges to a stationary measure exponentially fast.

$(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ underlying probability space.

$P_1(u,\cdot)$ = time 1 transition probability

$= \mathcal{L}(\Phi(u) + \eta_1)$.

Lem. 1: $\exists$ probability space $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $\forall R \geq 1$, $\exists N = N(R) \geq 1$ s.t.

if $b_n \neq 0$, $|n| \leq N$, then for any $u_1, u_2 \in B_R \subset H = \underset{\text{mean } 0}{L^2_{\text{df}}}$

the measures $\mu_1 = P_1(u_1, \cdot)$

$\mu_2 = P_2(u_2, \cdot)$

admit a coupling (V_1, V_2), $V_j = V_j(u_1, u_2, \omega)$,

(a) $V_j : B_R \times B_R \times \Omega \to H$ is measurable

(b) with $d = \|u_1 - u_2\|_{L^2}$, we have $\mathbb{P}(\|V_1 - V_2\|_{L^2} \geq \frac{1}{2} d) \leq C_0 d$

where $C_0 = C_0(R, \underset{\overset{\shortparallel}{\sum b_n^2}}{B_0}, b_n, |n| \leq N)$.

Pf: $P_N : H \to E_N = \operatorname{span}\{e_n : |n| \leq N\}$, $P_N^\perp = \mathrm{Id} - P_N$.

Look for V_1 and V_2 of the form $V_1 = \Phi(u_1) + z_2$

$V_2 = \Phi(u_2) + z_2$.

- z_1, z_2 are H-valued r.v.'s on Ω, $\mathcal{L}(z_1) = \mathcal{L}(z_2) = \eta_1$.

$\Rightarrow (V_1, V_2)$ is a coupling for (μ_1, μ_2).

We'll define z_j by specifying $P_N z_j$ for some approp. N.

$P_N^\perp z_j$

$\Omega = \Omega_1 \times \Omega_2$ for z_1, z_2.

Set $P_N^\perp z_1 = P_N^\perp z_2 = P_N^\perp \bar{\eta}_1 \leftarrow \eta_1(\omega_1, \omega_2) = \eta_1(\omega_1)$

natural extension ~~ex~~ to Ω.

$v_j = p_N \Phi(\mu_j)$

Lip. cont of Φ $\Longrightarrow \|v_1 - v_2\|_{L^2} \leq C(R)\, \underset{\|u_1-u_2\|_{L^2}}{\overset{}{\underset{\shortparallel}{d}}}$

- $b_n \neq 0$, $|n| \leq N$

$\mathcal{L}(g_{\kappa n}) = \underset{\uparrow \text{ Lipschitz}}{p_n(r)}\, dr \quad \Longrightarrow \mathcal{L}(p_N \eta_1) = \underset{\uparrow \text{ Lipschitz}}{q(x)}\, dx$, "$x \in E_N$".

Set ~~$\mu_j =$~~ $\nu_j = \mathcal{L}(\underbrace{P_N \Phi(\mu_j)}_{=v_j} + P_N \eta_1)$ | $\nu_j = (P_N)_* \mu_j$

$= q(x - v_j)\, dx$

$\|\nu_1 - \nu_2\|_{var} = \frac{1}{2} \int_{E_N} \underbrace{|q(x-v_2) - q(x-v_1)|}_{\text{Lip.} \lesssim \|v_1 - v_2\|_{L^2}} dx$

$\leq C_0\, d$

for (ν_1, ν_2)

By Dobrushin's Lemma, $\exists$ maximal coupling $(\overset{\sqcap}{u}_1, \overset{\sqcap}{u}_2)$ on a probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, $\overset{\sqcap}{u}_j = \overset{\sqcap}{u}_j(\omega_2; \mu_1, \mu_2)$.

$\Longrightarrow \mathbb{P}_2(\overset{\sqcap}{u}_1 \neq \overset{\sqcap}{u}_2) = \|\nu_1 - \nu_2\|_{var} \leq C_0\, d$

- $\overset{\sqcap}{u}_j : \Omega_2 \times B_R \times B_R \longrightarrow E_N$ is measurable (CHECK)

Let $\bar{\overset{\sqcap}{u}}_j(\omega_1, \omega_2) = \overset{\sqcap}{u}_j(\omega_2)$.

Define $p_N z_j = \bar{\overset{\sqcap}{u}}_j - P_N \Phi(\mu_j)$.

$\Longrightarrow V_j = P_N z_j + \underset{\overset{\shortparallel}{P_N^\perp \eta_1}}{P_N^\perp z_j} + \Phi(\mu_j)$

| WANT
| $P(\|v_1 - v_2\| \geq \frac{1}{2} d) \leq C_0\, d$

- If $\bar{\overset{\sqcap}{u}}_1 > \bar{\overset{\sqcap}{u}}_2 \Longrightarrow p_N V_1 = p_N V_2$

$\Longrightarrow V_1 - V_2 = p_N^\perp V_1 - p_N^\perp V_2$

$= p_N^\perp \Phi(u_1) - p_N^\perp \Phi(u_2)$

$\Longrightarrow \|V_1 - V_2\|_{L^2} = \|p_N^\perp \Phi(u_1) - p_N^\perp \Phi(u_2)\|_{L^2}$

$\leq N^{-1} \|\dots\|_{H^1} \leq N^{-1} C_1(R)\, d$

$\uparrow$ tomorrow.

$\Longrightarrow P(\|v_1 - v_2\|_{L^2} > \frac{1}{2} d) = \underset{\leq C_0 d}{\mathbb{P}(\bar{\overset{\sqcap}{u}}_1 \neq \bar{\overset{\sqcap}{u}}_2)}$

~~$\leq$ choose~~

choose $N \gg 1$ s.t. $N^{-1} C_1(R) < 1/2$.

$\square$