II WHITE FORCED NSE

$f = \frac{d}{dt} \zeta(t,x)$

div free

$\zeta(t,x) = \sum_{n \in \mathbb{Z}^2_0} b_n \beta_n(t) e_n(x)$

ind. real BM

STOCHASTIC CONVOLUTION

$\Psi(t) = \int_0^t e^{(t-t')L} \, d\zeta(t')$, $L = \pi\Delta$

- regularity: $\zeta = \phi W$ ← div free version

$\|\phi\|_{HS(L^2; H^s)} = \|\phi(e_n)\|_{\ell^2_n H^s} = \left(\sum_n |n|^{2s} b_n^2\right)^{1/2} =: \sqrt{B_s}$ | $\phi e_n = b_n e_n$

If $\phi \in HS(L^2; H^s)$, then $\zeta \in C_t W_x^{s+1-\varepsilon, \infty}$, $\forall \varepsilon > 0$

$\zeta \in L^q_T W_x^{s+1,r}$, $T < \infty$ a.s.
no loss , $q, r < \infty$

WELL-POSEDNESS OF (SNSE)

$\partial_t u = Lu - B(u) + \partial_t \zeta$ | $du = (Lu - B(u))\,dt + d\zeta$

- Da Prato - Debussche trick ('03) ← but dates back further

$u = v + \Psi$

Recall $\partial_t \Psi = L\Psi + \partial_t \zeta$

$v = u - \Psi$
$\Rightarrow \partial_t v = Lv - B(v,v) - \underbrace{B(v,\Psi) - B(\Psi,v) - B(\Psi,\Psi)}_{\text{" forcing term"}}$ (SNSE')

- A priori estimate

$\int (SNSE') \times v$

- $\langle B(\Psi,v), v\rangle = 0$
- $\langle \underset{L^4}{B(v}, \underset{L^2}{\Psi)}, \underset{L^4}{v}\rangle \le c_1 \|v\|^2_{L^4} \|\nabla\Psi\|_{L^2} \le c_2 \|v\|_{L^2} \|v\|_{\dot H^1} \|\nabla\Psi\|_{L^2}$

$\le \frac{1}{2}\|v\|^2_{\dot H^1} + c_3 \|v\|^2_{L^2} \|\nabla\Psi\|^2_{L^2}$

- $\langle \underset{L^4}{B(\Psi}, \underset{L^4}{\Psi)}, \underset{L^2}{v}\rangle \lesssim \|v\|^2_{L^2} + \|\langle\nabla\rangle\Psi\|^4_{L^4}$

$\Rightarrow \partial_t \|v\|^2_{L^2} \le c_1(\Psi(t)) + c_2(\Psi(t)) \|v\|^2_{L^2}$

Gronwall's $\Rightarrow \sup_{t\in[0,T]} \|v(t)\|_{L^2_x} \le C(\underset{"v_0}{u_0}, \Psi, T)$

Also,
(SNSE')×v $\Rightarrow \|v\|_{L^2_T H^1_x} \le C(u_0, \Psi, T)$

- $\| B(v,\Psi)\|_{L^2_T H^{-1}_x} \lesssim \|v\|_{L^\infty_T L^2_x} \|\Psi\|_{L^2_T L^\infty_x}$

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$$\langle B(v,\Psi), w\rangle = -\langle B(v,w),\Psi\rangle \sim \int \underset{L^2}{v}\cdot \underset{L^2}{\nabla w}\cdot \underset{L^\infty}{\Psi}$$

- $\| B(\Psi,v)\| \lesssim \|v\|_{L^\infty_T L^2_x} \|\Psi\|_{L^2_T L^\infty_x}$

- $\| B(\Psi,\Psi)\|_{L^2_T H^{-1}_x} \lesssim \|\Psi\|^2_{L^4_T H^{1/2}_x}$

$$\Longrightarrow \|\partial_t v\|_{L^2_T H^{-1}_x} \lesssim \|v\|_{L^2_T H^1_x} + \|v\|_{L^\infty_T L^2_x}\|v\|_{L^2_T H^1_x} + \|\Psi\|_{L^2_T L^\infty_x}\|v\|_{L^\infty_T L^2_x} + \|\psi\|^2_{L^4_T H^{1/2}_x}$$

$$\Longrightarrow \|\partial_t u\|_{L^2_T H^{-1}_x} \le C(u_0,\Psi,T)$$

<u>GALERKIN APPROXIMATION</u>

$$\begin{cases} \partial_t v_N = L v_N - P_N B(v_N + \Psi_N), \quad \Psi_N = P_N\Psi \\ v_N|_{t=0} = P_N u_0 \end{cases}$$

→ <u>POINT</u>: same a priori estimate (unif. in N)

$\Psi_N \to \Psi$ a.s.

$\left(\Longrightarrow B(v_N + \Psi_N) \longrightarrow B(v+\Psi) \text{ in } L^1_T H^{-1}_x\right.$

result follows as before

$\Longrightarrow$ well-posed on $[0,T]$, $\forall T>0$

$\Longrightarrow$ GWP in $L^2(\mathbb{T})$ if $B_0 < \infty$ (i.e. $\phi \in HS(L^2; L^2)$)

This argument shows a control on $\|u(t)\|_{L^2} \le \|v(t)\|_{L^2} + \|\Psi(t)\|_{L^2}$.

But we can use <u>Itô's Lemma</u>, as well

Kuo's book, Da Prato - Zabczyk

$$dX^{(i)} = \sum_{j=1}^m f_{ij}\, dB_j + g_i\, dt$$

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix}_{\text{ind.}}, \quad \vec{X} = \begin{pmatrix} X^{(1)} \\ \vdots \\ X^{(n)} \end{pmatrix}, \quad f = (f_{ij})_{n\times m}, \quad g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}.$$

Then,

$$dF(t,\vec{X}_t) = \frac{\partial F}{\partial t}(t,\vec{X}_t)\,dt + \overbrace{\sum_{i=1}^m \frac{\partial F}{\partial x_i}(t,\vec{X}_t)\,dX_t^{(i)}}^{\nabla_{\vec{x}} F\cdot d\vec{X}_t}$$
$$+ \underbrace{\frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 F}{\partial x_i \partial x_j}(t,\vec{X}_t)\,dX^{(i)}dX^{(j)}}_{\frac{1}{2}\nabla^2 F(d\vec{X}_t, d\vec{X}_t)}$$

	dB_i	dt
dB_j	$\delta_{ij}\,dt$	0
dt	0	0

$$du = V_N(u)\,dt + \sum_{|n|\le N} b_n e_n \, d\beta_n$$

$$\Rightarrow dF(t, u(t)) = \partial_t F(t, u(t))\,dt + \langle \nabla_u F, V_N(u)\rangle\,dt + \langle \nabla_u F, \sum_{|n|\le N} b_n e_n \, d\beta_n\rangle$$
$$+ \frac{1}{2} \sum_{|n|\le N} \frac{\partial^2 F}{\partial \hat{u}_n} b_n^2 \cdot dt$$

with $F(t,u) = \|u\|_{L^2}^2$.

Apply Itô's lemma and take expectation on $|n|\le N$

$$\Rightarrow \frac{d}{dt}\mathbb{E}\left[\|u\|_{L^2}^2\right] = -2\,\mathbb{E}\left(\|u\|_{H^1}^2\right) + B_{0,N}$$
$$\le -2\,\mathbb{E}\left(\|u\|_{L^2}^2\right) + B_0$$

→ using Gronwall ⇒ control on $\mathbb{E}\left[\|u(t)\|_{L^2}^2\right]$ for all fixed t. But we want to control $\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|_{L^2}^2\right]$.

Key Tool: Burkholder - Davis - Gundy inequality (also called Martingale inequality)

$$\mathbb{E}\left[\sup_{0\le t\le T} |M(t)|^p\right] \sim \mathbb{E}\left[\langle M\rangle_T^{p/2}\right], \quad p>0$$

where $M(t)$ is a local martingale. quadratic variation

think of M as $\int_0^t F\,dB_{t'}$

$$\Rightarrow \mathbb{E}\left[\sup_{0\le t\le T} \|u(t)\|_{L^2_x}^2\right] \le C\left(\|u_0\|_{L^2}, B_0, T\right).$$

By applying Itô's Lemma to $F(u) = \|u\|_{\dot H^1}^2$,

$$\Rightarrow \mathbb{E}\left[\|u(t)\|_{\dot H^1}^2\right] \le \frac{1}{2} B_1 + e^{-2t}\,\mathbb{E}\left[\|u_0\|_{\dot H^1}^2\right].$$

STATIONARY MEASURE !

Semigroup: $T_t : C_b(H) \to C_b(H)$ (L^2_{if}), $\quad (T_t f)(v) = \int_H f(u) P_t(v, du) = \mathbb{E}\left[f(u(t;v))\right]$

$T_t^* : P(H) \longrightarrow P(H)$, $\quad (T_t^*\mu)(A) = \int_H P_t(v,A)\,\mu(dv) = (\Phi_t)_* \mu$

push-forward

Φ_t solution map

If $\mathcal{L}(u_0) = \mu$, then $\mathcal{L}(u(t;u_0)) = (T_t^* \mu)$

Theorem: $\exists$ invariant measure.

proof: For simplicity, assume $B_1 < \infty$. The result also holds for $B_1 = \infty$.

$u(t)$ = solution to (SNSE) with $u_0 = 0$

$\mu_t = \mathcal{L}(u(t))$

$\bar{\mu}_t = \frac{1}{t}\int_0^t \mu_{t'}\,dt'$

$B_{H^1}(r)$ = ball of radius r in H^1 (centered at 0)

$$\mu_t(H \setminus B_{H^1}(r)) = P(\|u(t)\|_{H^1} > r)$$

$$\overset{\text{chebyshev}}{\leq} \frac{C}{r^2}$$

$< \varepsilon$ by choosing $r \gg 1$ for a given $\varepsilon > 0$, uniformly in $t > 0$

$\Rightarrow \bar{\mu}_t((B_{H^1}(r))^c) < \varepsilon$

$\Rightarrow \{\bar{\mu}_t\}$ is tight $\Rightarrow$ ... rest as before. $\square$

"UNIVERSALITY" OF WHITE-NOISE FORCES

consider random kick forces $\eta_\varepsilon(t) = \sqrt{\varepsilon} \sum_{k \in \mathbb{Z}_0} \eta_k^\omega \, \delta(t - \varepsilon k)$ where $\eta_k = \sum_{n \in \mathbb{Z}_0^2} b_n g_{kn} e_n$.

Compare the dynamics with (SNSE) where $f = \frac{d}{dt}\zeta$, $\zeta = \sum_{n \in \mathbb{Z}_0^2} b_n \beta_n e_n$.

claim: "u^ε converges to u in law"

$\Longleftarrow$ Donsker's Theorem

$\{X_n\}_n$ i.i.d. mean 0, invariance σ^2

$$Z_n(t;\omega) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}(\omega) + (nt - [nt]) \frac{1}{\sigma\sqrt{n}} X_{[nt]+1}(\omega)$$

$$S_{[nt]} \;\text{"}\!=\!\text{"}\; \sum_{j=1}^{[nt]} X_j$$

$\downarrow$

BM in law

Remark: · Can also consider continuous in time forcing but not Gaussian, and still obtain "weak universality" and convergence to white forcing.

· Skorokhod's theorem can be used to upgrade convergence in law to a.s. convergence.