Lemma 1: If $u \in H \cap H^2$, then $\langle B(u), \Delta u\rangle = 0$.

($H = L^2_{df}$ mean 0)

- $B(u) = B(u,u) = \Pi((u\cdot\nabla)u)$
- Recall $\langle B(u,u), u\rangle = 0$ since $\langle B(u,v), v\rangle = 0$.

proof: $\operatorname{div} u = 0$ and $u \in H^k(\mathbb{T}^2) \Rightarrow \exists$ stream function $\psi \in H^{k+1}(\mathbb{T}^2)$, ψ unique up to constants

$u = \operatorname{curl}\psi := (-\partial_2\psi, \partial_1\psi)$

$\because \operatorname{div} u = 0 \Rightarrow \partial_1 u_1 = -\partial_2 u_2$

$$\Rightarrow u_1 = \int \partial_2 u_2\, dx_1 + c(x_2) \quad , \quad \psi = -\iint \partial_2 u_2\, dx_1 dx_2 + C$$

$$u_2 = \underbrace{\int \partial_1 u_1\, dx_2}_{= -\int \partial_2 u_2\, dx_2} + c(x_1)$$

- Since $u \in H \cap H^2$, $\exists \psi \in H^3$ s.t. $u = \operatorname{curl}\psi$.
- Also, $B(u) \in L^2_{df}$

$$\Big(\|\underbrace{B(u)}_{u\cdot\nabla u}\|_{L^2} \lesssim \|u\|_{L^\infty}\|\nabla u\|_{L^2} \overset{\text{Sobolev}}{\lesssim} \|u\|^2_{H^2}$$

Helmholtz decomposition: $H^k = H^k_{df} \oplus \nabla H^{k+1}_{\text{curl free}}$

$\Rightarrow \exists p \in H^1$ s.t. $B(u) = (u\cdot\nabla)u - \nabla p$.

- $\operatorname{curl}(u_1, u_2) = \partial_1 u_2 - \partial_2 u_1$

$\operatorname{curl}(u_1,u_2,0) = (\partial_1 u_2 - \partial_2 u_1)\hat{k}$

$$\langle B(u), \Delta u\rangle = \int ((u\cdot\nabla)u - \cancel{\nabla p}) \cdot \operatorname{curl}(\Delta\psi)\, dx$$

$$\int -\partial_1 p(-\partial_2\Delta\psi) - \partial_2 p(\partial_1\Delta\psi)\, dx \overset{\text{IBP}}{=} \int \underbrace{(\operatorname{curl}\nabla p)}_{=0}\Delta\psi\, dx$$

$$= -\int (\operatorname{curl}((u\cdot\nabla)u))\Delta\psi\, dx$$

$$= -\int ((u\cdot\nabla)\operatorname{curl} u)\Delta\psi\, dx$$

We used $\operatorname{curl}((u\cdot\nabla)u) = u\cdot\nabla \operatorname{curl} u$.

Note: $\operatorname{curl} u = \operatorname{curl}\operatorname{curl}\psi$
$= \operatorname{curl}(-\partial_2\psi, \partial_1\psi)$
$= \partial_2^2\psi + \partial_1^2\psi = \Delta\psi$

$$\Rightarrow \langle B(u), \Delta u\rangle = \int (\underbrace{(u\cdot\nabla)}_{u_1\partial_1 + u_2\partial_2}(\Delta\psi))\Delta\psi\, dx = \frac{1}{2}\int u_1\partial_1((\Delta\psi)^2) + u_2\partial_2((\Delta\psi)^2)\, dx$$

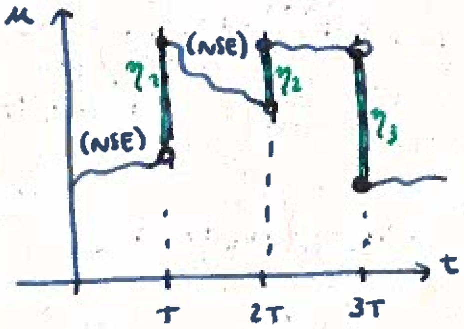
$$\overset{\text{IBP}}{=} -\frac{1}{2}\int \underbrace{(\partial_1 u_1 + \partial_2 u_2)}_{=\operatorname{div} u = 0}(\Delta\psi)^2\, dx = 0 \quad . \quad \square$$

$(NSE_f): \quad \partial_t u + B(u) = Lu + f$

① Random kick force

(kick NSE) $\quad \partial_t u + B(u) = Lu + \sum_{k=1}^{\infty} \eta_k^\omega \, \delta(t - kT)$

$\eta_k^\omega = \eta_k^\omega(x) =$ random function in x



Definition: A filtration $\{\mathcal{F}_t\}_{t \in I}$ is an increasing family of σ- algebras.

A stochastic process $\{X_t\}_{t \in I}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t \in I}$ if, for each t, X_t is $\mathcal{F}_t$ - measurable.

For a kick force under consideration,

$\mathcal{F}_t = \mathcal{F}_{(k-1)T} \quad$ for $t \in I_k := [(k-1)T, kT)$

η_k is $\mathcal{F}_k$ - measurable $\quad \Big| \; \mathcal{F}_k = \sigma\left(\begin{array}{l} \eta_j, j = 1, \dots, k \\ u_0^\omega \end{array} \right)$

Definition: A stochastic process $u(t)$, $t \geq 0$, is called a solution to (kick NSE) if it's adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and almost surely u satisfies the following

(i) $\forall k \in \mathbb{N}$, $u \in \mathcal{H}(I_k)$ is a solution to (NSE) (with $f \equiv 0$)

(ii) $u(kT+) - u(kT-) = \eta_k$, $\forall k \in \mathbb{N}$.

$\rightarrow$ on I_k, the initial condition is $u((k-1)T+) = u_0 + \sum_{j=1}^{k-1} \eta_j^\omega$.

$\rightarrow$ (kick NSE) $\iff u(t) = \underbrace{\boxed{u_0 + \sum_{j=1}^{k-1} \eta_j^\omega}}_{\text{Initial value}} + \int_0^t (B(u) + Lu)(t') \, dt'$, $\forall t \in I_k$.

Hence, from the L^2-GWP of (NSE), we obtain

Theorem: Suppose that $\eta_k \in H$, $\forall k \in \mathbb{N}$, a.s.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \dot{L}^2$ df, mean 0

Then, (kick NSE) is globally well-posed in $L^2(\mathbb{T})$.

Remark : We can take u_0 to be random ($\mathcal{F}_0$-measurable) and still GWP works.

In the following, we take $\quad \eta_K = \sum_{n \in \mathbb{Z}^2_0} b_n g_{kn} e_n$ $\quad$ $e_n \to$ O.N.B. of H. $\quad \mathbb{Z}^2_0 = \mathbb{Z}^2 \setminus \{0\}$

Assume

- $\{g_{kn}\}_{n \in \mathbb{Z}^2_0, k \in \mathbb{N}}$ = family of independent r.v.'s (identically distributed)
- $|g_{nk}(\omega)| \leq 1$, $\forall n, k, \omega \in \Omega$
- $P(|g_{kn}| \leq \varepsilon) > 0$, $\forall \varepsilon > 0$.
  - has "nice" density and $p(0) \neq 0$.
- $B_s = \sum_{n \in \mathbb{Z}^2_0} |n|^{2s} b_n^2$
- $\|\eta_K(\omega)\|^2_{L^2} = \sum b_n^2 g_{kn}^2(\omega) \leq \sum b_n^2 = B_0 < \infty$.
- $P(\|\eta_K\|_{L^2} \leq \varepsilon) > 0$, $\forall \varepsilon > 0$
  - $\geq P(|g_{kn}|^2 \leq \frac{\varepsilon}{b_n^2})$

These assumptions are not needed but they simplify the argument

Let $T = 1$ for simplicity.

$\Phi_t : u_0 \longmapsto u(t)$, solution map to (NSE) (with $f \equiv 0$)

$\Phi = \Phi_1$

Then, $\quad u(k) = \Phi(u(k-1)) + \eta_K \quad , \quad u(k) = u(k+)$

(*) $\quad u(k+t) = \Phi_t(u(k)) \quad , \; 0 \leq t < 1$.

SOME ESTIMATES

① $\|\Phi_t(u_0)\|_{L^2} \leq e^{-t} \|u_0\|_{L^2}$

proof : $(\partial_t u + B(u) = Lu) \times u \implies \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} = \langle Lu, u \rangle = -\|u\|^2_{\dot{H}^1} \leq -\|u\|^2_{L^2}$

Gronwall $\implies \|u(t)\|^2_{L^2} \leq e^{-2t}\|u(0)\|^2_{L^2}$. $\square$

①' $\|\Phi_t(u_0)\|_{H^1} \leq e^{-t}\|u_0\|_{H^1}$

proof : (NSE) $\times \Delta u$. use Lemma 1 : $\langle B(u), \Delta u \rangle = 0$

$\implies$ Gronwall. $\square$

② $0 \leq m \leq k$, $\quad \|u(k)\|_{L^2} = \|\Phi(u(k-1)) + \eta_K\|_{L^2}$

$$\overset{①}{\leq} \sqrt{B_0} + e^{-1}\|u(k-1)\|_{L^2}$$

$$\leq \sqrt{B_0} + e^{-1}(\sqrt{B_0} + e^{-1}\|u(k-2)\|_{L^2})$$

$$\leq \ldots \leq \sqrt{B_0}(1 + e^{-1} + \ldots + e^{-m}) + e^{-m}\|u(k-m)\|_{L^2}$$

$$\implies \|u(k)\|_{L^2} \leq \sqrt{B_0}\frac{e}{e-1} + e^{-m}\|u(k-m)\|_{L^2}; \quad \forall 0 \leq m \leq k, \; \forall \omega \in \Omega.$$

In particular, for $m=k$

$$\|u(k;0)\|_{L^2} \le \sqrt{B_0}\,\frac{e}{e-1}$$

$\uparrow$
$u_0=0$

Similarly, by ①' we obtain $\|u(k)\|_{H^1} \le \sqrt{B_1}\,\frac{e}{e-1} + e^{-m}\|u(k-m)\|_{H^1}$

with $u_0=0$,

(**) $\|u(k;0)\|_{H^1} \le \sqrt{B_1}\,\frac{e}{e-1}$, $\forall k\in\mathbb{Z}_{\ge 0}$, $\omega\in\Omega$.

- Now, write (*) as

$$u(k) = F_k\,(u(k-1),\omega)$$

$F_k : H\times\Omega \longrightarrow H$ meas, locally Lipschitz in $u\in H$.

$\Longrightarrow$ random dynamical system (RDS)

⚠ Every RDS defines a Markov chain ($\{u(k)\}_{k\ge 0}$ in H)

- For $u_0\in H$, $k\in\mathbb{Z}_{\ge 0}$, $A\in\mathcal{B}_H$, set $\underbrace{P_k(u_0,A)}_{\text{Transition probability}} := P(u(k;u_0)\in A)$.
  - $p_0(u_0,\cdot) = \delta_{u_0}$
  - Chapman-Kolmogorov equation:

$$P_{k+m}(u_0,A) = \int_H P_m(u,A)\,P_k(u_0,du)$$

Reference for definitions → Da Prato - "Intro to ∞-dimensional analysis", (Chap 5)

Markov semigroups

(i) on $C_b(H)$

$T_k : C_b(H) \longrightarrow C_b(H)$

$$T_k f(v) = \int_H f(z)\,P_k(v,dz) = \mathbb{E}\,(f(u(k;v)))$$

initial data

T_k can be called Kolmogorov operator

T_k^* "dual" of T_k.

(ii) on $P(H)$ = space of probability measures on H.

$T_k^* : P(H) \longrightarrow P(H)$

$$(T_k^*\mu)(A) = \int_H P_k(v,A)\,\mu(dv) = \mu\,(\{v : u(k;v)\in A\}),\quad A\in B_H$$

measure at the level of initial data

- If u_0 is random with $L(u_0)=\mu$, then $T_k^*\mu = L(u(k;u_0))$.

"Duality"

$C_b(H)$ - $(C_b(H))^*$ pairing

$\langle T_k f, \mu \rangle = \langle f, T_k^* \mu \rangle$

$$\int_H T_k f(v) \mu(dv) = \int_H f(z) (T_k^* \mu)(dz)$$

finitely additive regular bounded Borel measure. [Dunford - Schwartz]

Definition: T_k, Markov semigroup

(i) T_k is Feller if $T_k f \in C_b(H)$, $\forall f \in C_b(H)$, $\forall k \geq 0$.

(ii) T_k is strong Feller if $T_k f \in C_b(H)$, $\forall f \in L^\infty(H)$, $\forall k \geq 0$.

(iii) T_k is irreducible if $T_k \mathbb{1}_{B(x_0,r)}(x) > 0$, $\forall x, x_0 \in H$, $\forall r > 0$, "some" $k \geq 0$.

- $H \longrightarrow P(H)$ ← weak topology (for prob. measures)
  $u \longmapsto P_k(u, \cdot)$

  is continuous.

proof: $u_{0,n} \longrightarrow u_0$ in H

$$P_k(u_{0,n}, A) = P(\{\omega; u^\omega(k; u_{0,n}) \in A\}) = \int \mathbb{1}_{\{u_n^\omega(k; u_{0,n}) \in A\}}(\omega) \, dP(\omega)$$

$u_{0,n} \longrightarrow u_0$

$u_n^\omega \longrightarrow u^\omega$

Want $\longrightarrow \int \mathbb{1}_{\{u^\omega(k; u_0) \in A\}}(\omega) \, dP = P(u(k; u_0) \in A) = P_k(u_0, A)$

If A is a continuity set of measure ~~0~~ 0 ($P_k(u_0, \partial A) = 0$)

Portmanteau theorem

$$T_k f(u_{0,n}) = \int f(z) P_k(u_{0,n}, dz) \longrightarrow \int f(z) P_k(u_0, dz) = T_k f(u_0).$$

- $|T_k f| \leq \|f\|_{L^\infty} \Rightarrow T_k f \in C_b(H) \Rightarrow T_k$ is Feller.

- $T_k^* : P(H) \to P(H)$ is continuous. (Assuming T_k is Feller)

proof: Suppose $\mu_n \to \mu$.

Let $f \in C_b(H)$.

$$\int f(z) T_k^* \mu_n(dz) = \int \underbrace{T_k f}_{\text{Feller}} \, d\mu_n \Rightarrow \int T_k f \, d\mu = \int f(z) T_k^* \mu(dz)$$

$\Rightarrow T_k^* \mu_n \longrightarrow T_k^* \mu$. ☒

Definition: A probability measure $\mu \in P(H)$ is said to be invariant (or stationary) for T_k if $\int_H T_k f \, d\mu = \int_H f \, d\mu$, $\forall k \geq 0$, $\forall f \in L^\infty(H)$

(If T_k is Feller) this is equivalent to $T_k^* \mu = \mu$, $\forall k \geq 0$.

Theorem : $\exists$ an invariant measure for (kick NSE).

proof : Bogolyubov – Krylov argument.

Assume $B_1 < \infty$ $\left(B_1 = \sum_n |n|^2 b_n^2\right)$ $\qquad$ ($B_1 = \infty$ proof is longer)

Let $u(0) = 0$.

$$\mu_k = \mathcal{L}(u(k))$$

Set $\bar{\mu}_k = \frac{1}{k}\sum_{j=0}^{k-1} \mu_j$ $\qquad\qquad$ $\Big|\ T_1^* \mu_j = \mu_{j+1}$

Let $r = \sqrt{B_1}\,\frac{e}{e-1}$.

By (**), $\quad \mu_j\left(B_{H^1}(r)\right) = 1 \quad \forall j \geq 0$

$$\Longrightarrow \bar{\mu}_k\left(B_{H^1}(r)\right) = 1, \quad \forall k \geq 0$$

• $\bar{\mu}_k$ is tight

($\{\rho_k\}_k$ is tight if $\forall \varepsilon > 0$, $\exists$compact K_ε s.t. $\rho_k(K_\varepsilon) \geq 1-\varepsilon$, $\forall k$

By Prokhorov Theorem (tight $\Leftrightarrow$ weakly precompact)

$\exists \mu \in P(H)$ s.t. $\bar{\mu}_{k_m} \to \mu$.

CHECK: $T_1^*\mu = \mu$ $\qquad$ ($\Rightarrow T_k^*\mu = \mu \quad \forall k$)

$$\langle f, T_1^*\mu\rangle = \lim_{m\to\infty} \langle f, T_1^* \mu_{k_m}\rangle = \lim_{m\to\infty} \frac{1}{k_m}\sum_{j=0}^{k_m-1} \langle f, \underbrace{T_1^*\mu_j}_{=\mu_{j+1}}\rangle$$

continuity of T_1^*

$$\overset{\text{shift}}{=} \lim_{m\to\infty} \frac{1}{k_m}\sum_{j=1}^{k_m} \langle f, \mu_j\rangle$$

$$= \lim_{m\to\infty}\Big(\langle f, \bar{\mu}_{k_m}\rangle + \underbrace{\frac{1}{k_m}\big(\langle f, \mu_{k_m}\rangle - \langle f, \mu_0\rangle\big)}_{\to 0}\Big)$$

$$= \langle f, \mu\rangle, \quad \forall f \in C_b(H) \qquad \boxtimes$$