NSE on $\mathbb{T}^2$

$H = L^2_{df}(\mathbb{T}^2; \mathbb{R}^2)$, divergence free, mean 0.

$\mathcal{H}_T = \{ u \in L^2_T H^1_{df};\ \partial_t u \in L^2_T H^{-1}_{df} \}$

$\|u\|_{\mathcal{H}_T} = \left(\|u\|^2_{L^2_T H^1} + \|u\|^2_{L^2_T H^{-1}} \right)^{1/2}$

- $\mathcal{H}_T \subset C_T L^2$
- Aubin-Lions compactness lemma
  $\Rightarrow \mathcal{H} \subset\subset L^2_T H^s_x(\mathbb{T}^2)$, $-1 < s < 1$
- $B(u,v) = \Pi((u\cdot\nabla)v)$
- $B(u) = B(u,u)$
- $L = \Pi\Delta$ (In Kuksin's book, $L = -\Pi\Delta$)

Rellich compactness lemma
$H^1(\mathbb{T}^2) \subset\subset H^s(\mathbb{T}^2)$, $s<1$

→ Basic properties of B

(B1) (i) $\langle B(u,v), v\rangle_{L^2_x} = 0$ $\qquad u,v,w \in H \cap C^\infty$

(ii) $\langle B(u,v), w\rangle = -\langle B(u,w), v\rangle$

proof:

(i) $(LHS) = \int_{\mathbb{T}^2} u^j \partial_j v^k v^k$

$= \int u^j \partial_j (|v|^2) \frac{1}{2}$

$= -\frac{1}{2}\int \underbrace{\partial_j u^j}_{\operatorname{div} u = 0} (|v|^2)\, dx$

$= 0$

$\Pi \leftarrow$ symmetric matrix on the Fourier side
$\int \langle \Pi(u\cdot\nabla)v, v\rangle_{\mathbb{R}^2} dx$ and use Parseval's identity
$\Pi v = v$

(ii) use (i) $\qquad \langle B(u,v+w), v+w\rangle = 0$

$0 \overset{(i)}{=} \langle B(u,v+w), v+w\rangle$

$\overset{\text{multilin. \& (i)}}{=} \langle B(u,v), w\rangle + \langle B(u,w), v\rangle$. ☐

(B2) (i) $|\langle B(u,v), w\rangle| \lesssim \|u\|_{H^{\frac{1}{2}}} \|v\|_{H^{1/2}} \|w\|_{H^1}$

(ii) $\|B(u,v)\|_{H^{-1}} \lesssim \|u\|_{H^{1/2}} \|v\|_{H^{1/2}}$

proof: (i) $\Leftrightarrow$ (ii) by duality.

(i) $|\langle B(u,v), w\rangle| \overset{(B1)}{=} |\langle B(u,w), v\rangle| \le \int \underset{L^4}{|u|} \cdot \underset{L^2}{|\nabla w|} \cdot \underset{L^4}{|v|}\, dx$

$\overset{\text{Hölder \& Sobolev}}{\lesssim} \|u\|_{H^{1/2}} \|v\|_{H^{1/2}} \|w\|_{H^1}$, $H^{1/2} \subset L^4$ $\quad d=2$

$\frac{s}{d} = \frac{1}{2}/2 = \frac{1}{2} - \frac{1}{4}$ ☐

(B3) $\| B(u,v)\|_{H^{-3}} \lesssim \|u\|_{L^2} \|v\|_{L^2}$.

proof: For $w \in H^3$,

$$|\langle B(u,v), w\rangle| = |\langle B(u,w), v\rangle| = \int \underset{L^2}{|u|} \cdot \underset{L^\infty}{|\nabla w|} \cdot \underset{L^2}{|v|}\, dx$$

$$\lesssim \|u\|_{L^2} \|v\|_{L^2} \|w\|_{H^{2+\varepsilon}} . \quad \square$$

(B4) $u,v \in \mathcal{H}_T$, $\int_0^T \langle Lu(t), v(t)\rangle dt = -\int_0^T \langle \nabla u(t), \nabla v(t)\rangle dt$

- move Π to v
- $\Pi v = v$
- integration by parts

and $\int_0^T \langle \partial_t u(t), u(t)\rangle dt = \frac{1}{2}\left(\|u(t)\|_{L^2}^2 - \|u(0)\|_{L^2}^2\right)$

(B5) $(u_1,u_2,u_3) \longmapsto \langle B(u_1(t), u_2(t)), u_3(t)\rangle$

$\mathcal{H}_T \times \mathcal{H}_T \times \mathcal{H}_T \longrightarrow L^1_T$

proof: Claim: $\mathcal{H}_T \subset L^4_T H^{1/2}$.

interpolation:
$s = \theta \cdot s_1 + (1-\theta) s_2$
$\|f\|_{H^s} \lesssim \|f\|_{H^{s_1}}^{\theta} \|f\|_{H^{s_2}}^{1-\theta}$

$$\|u\|_{L^4_T H^{1/2}}^4 = \int_0^T \underbrace{\|u(t)\|_{H^{1/2}}^4}_{\lesssim \|u(t)\|_{L^2}^2 \|u(t)\|_{H^1}^2} dt$$

$$\lesssim \|u\|^2_{L^\infty_T L^2_x} \|u\|^2_{L^2_T H^1_x}$$

$$\lesssim \|u\|^4_{\mathcal{H}_T}$$

$$\int_0^T \langle \underset{L^4_T H^{1/2}}{B(u_1(t)}, \underset{L^4_T H^{1/2}}{u_2(t))}, \underset{L^2_T H^1}{u_3(t)}\rangle dt \overset{(B2)\,\&\,\text{Hölder}}{\lesssim} \|u_1\|_{L^4_T H^{1/2}} \|u_2\|_{L^4_T H^{1/2}} \|u_3\|_{L^2_T H^1}$$

$$\lesssim \|u_1\|_{\mathcal{H}_T} \|u_2\|_{\mathcal{H}_T} \|u_3\|_{\mathcal{H}_T} . \quad \square$$

$$(NSE)\ \begin{cases} \partial_t u - Lu + B(u) = f \\ u|_{t=0} = u_0 \in L^2_{df} \end{cases}, \quad f = \Pi f \in L^2_T H^{-1}_x$$

Theorem: Given $u_0 \in H = L^2_{df}$, $\exists!$ global solution u to (NSE) with $u|_{t=0} = u_0$ and $u \in \mathcal{H}_T, \forall T>0$

$$\sup_{0\le t\le T} \left(\|u(t)\|_{L^2}^2 + \int_0^t \|u(t')\|_{H^1}^2 dt'\right) \le \|u_0\|_{L^2}^2 + \int_0^T \|f(t)\|^2_{H^{-1}} dt, \quad \forall T>0$$

proof: Fix $T>0$ and work on $[0,T]$.

① Uniqueness: Suppose $\exists$ two solutions $u,v \in \mathcal{H}_T$. Let $w = u - v$.

$\Rightarrow \partial_t w - Lw + B(w,w) + B(v,w) = 0$.

Multiply by w and integrate in x, in t.

$|\langle B(v,w),w\rangle| \overset{(B1)}{=} 0$

$$|\langle B(w,u),w\rangle| \overset{(B1)}{=} |\langle B(w,w),u\rangle| \overset{(B2)}{\le} c\|w\|_{H^{1/2}}^2 \|u\|_{H^1}$$

$$\le c \|w\|_{L^2} \|w\|_{H^1} \|u\|_{H^1}$$

$$\overset{\text{cauchy}}{\le} \frac{1}{2}\|w\|_{H^1}^2 + c\|w\|_{L^2}^2 \|u\|_{H^1}^2$$

$$\Rightarrow \frac{d}{dt}\|w\|_{L^2}^2 + 2\|w\|_{H^1}^2 = -2\langle B(w,u),w\rangle \le \|w\|_{H^1}^2 + C\|w\|_{L^2}^2\|u\|_{H^1}^2$$

$$\Rightarrow \frac{d}{dt}\|w\|_{L^2}^2 \le C\|w\|_{L^2}^2\|u\|_{H^1}^2$$

Gronwall

$$\Rightarrow \|w(t)\|_{L^2}^2 \le e^{C\int_0^t \|u(t')\|_{H^1}^2 dt'} \underbrace{\|w(0)\|_{L^2}^2}_{=0}$$

$$= 0$$

⚠ This result also gives stability of solutions.

② Existence

Step 1 : A priori bound

Suppose u is a smooth solution to (NSE).
Multiply by u and integrate.

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = \langle \partial_t u, u\rangle$$

$$= \langle Lu,u\rangle - \cancel{\langle Bu,u\rangle}^{(B1)\to 0} + \langle f,u\rangle$$

$$\le -\|u\|_{H^1}^2 + \|u\|_{H^1}\|f\|_{H^{-1}}$$

$$\le -\frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{2}\|f\|_{H^{-1}}^2$$

$$\Rightarrow \|u(t)\|_{L^2}^2 + \int_0^t \|u(t')\|_{H^1}^2 dt' \le \|u_0\|_{L^2}^2 + \int_0^t \|f(t')\|_{H^{-1}}^2 dt'$$

($\Rightarrow$ sup in $t\in[0,T]$)

Remark :

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' = \|u(0)\|_{L^2}^2 + 2\int_0^t \langle f(t'), u(t')\rangle dt'$$

$$\Rightarrow \|u\|_{L^\infty_T L^2_x} + \|u\|_{L^2_T H^1_x} \le C(u_0, f).$$

By (B2),

$$\|B(u)\|_{H^{-1}} \lesssim \|u\|_{H^{1/2}}^2 \lesssim \|u\|_{L^2}\|u\|_{H^1}.$$

$$\|\partial_t u\|_{L^2_T H^{-1}_x} \overset{(NSE)}{\le} \overset{Lu}{\|u\|_{L^2_T H^1_x}} + \|u\|_{L^\infty_T L^2_x}\|u\|_{L^2_T H^1_x} + \|f\|_{L^2_T H^{-1}_x}$$

$$\le C(u_0,f) = C(\|u_0\|_{L^2}, \|f\|_{L^2_T H^{-1}_x})$$ ← non-decreasing function.

$$\Rightarrow \|u\|_{\mathcal{H}_T} \le C(u_0,f).$$

Step 2 : Galerkin approximation

Define $P_N : L^2_{df} \longrightarrow E_N = \text{span}\{e_n : |n|\le N\}$.

Apply P_N to the equation

$$\partial_t P_N u - LP_N u + P_N B(u) = P_N f.$$

$u_N = P_N u_N$

$$(NSE_N)\begin{cases} \partial_t u_N - Lu_N + P_N B(u_N) = P_N f \\ u_N|_{t=0} = P_N u_0 \end{cases}$$

finite dimensional system of ODEs on the "Fourier" side.

Euler equation

$\partial_t u + u\cdot\nabla u + \nabla p = 0$

energy : $\frac{1}{2}\int |u|^2 dx$ is conserved if $u\in C^{1/3}$

↳ Onsager's conjecture.

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- convex integration, ∃ wild solutions
- L^∞, C^0 non-uniqueness
  given $e(t): \mathbb{R}_+ \to \mathbb{R}_+$
  ∃ u s.t. $e(t) = \frac{1}{2}\int |u(t)|^2 dx$
- Isett : C^α, α $\frac{1}{10}, \frac{1}{5}$
  ↳ $C^{1/3}$
- Buckmaster
- Vicol

By Cauchy-Lipschitz theorem, $\exists!$ local-in-time solution u_N.

$\rightarrow$ <u>blowup alternative</u>:

u_N exists on $[0,T]$

OR

$\exists T_N < T$ s.t. $\lim_{t \to T_N^-} \|u_N(t)\|_{L^2} = \infty$.

Multiply (NSE_N) by u_N and integrate

$$\Rightarrow \sup_{N \geq 1} \left(\|u_N\|_{\mathcal{H}_T} + \|u_N\|_{L^\infty_T L^2_x} \right) \leq C(u_0, f)$$

$\langle P_N B(u_N), u_N \rangle$ $=$ $\langle B(u_N), u_N \rangle$ and same computations hold.

- u_N exists on $[0,T]$
- $u_{N_j} \rightharpoonup u$ in $\mathcal{H}_T$

$\partial_t u_{N_j} \rightharpoonup \partial_t u$
$L u_{N_j} \rightharpoonup Lu$ in $L^2_T H^{-1}_x$

By Aubin-Lions, compactness Lemma, $\exists$ subsequence u_{N_j}

$$u_{N_j} \to u \quad \text{in } L^2_T H^{1/2}_x$$

(B2)
$$\Rightarrow \quad B(u_{N_j}) \to B(u) \quad \text{in } L^1_T H^{-1}_x$$

By definition of P_{N_j},

$$u_{N_j}(0) = P_{N_j} u_0 \to u_0 \quad \text{in } L^2_x.$$

$$P_{N_j} f \to f \quad \text{in } L^2_T H^{-1}_x$$

$$(NSE_{N_j}) \quad \partial_t u_{N_j} - L u_{N_j} + P_{N_j} B(u_{N_j}) = P_{N_j} f$$

problematic

$\hookrightarrow$ Apply P_m for fixed m, then $N_j \geq m$ (which holds for $j \gg 1$)

$$\partial_t P_m u_{N_j} - L P_m u_{N_j} + P_m B(u_{N_j}) = P_m f$$

$j \to \infty \quad \downarrow$ in $L^1_T H^{-1}_x$

$$(*) \quad \partial_t P_m u - L P_m u + P_m B(u) = P_m f$$

$(*)$ holds for any $m \geq 1 \Rightarrow$ Take $m \to \infty$

$$\begin{cases} \partial_t u - Lu + B(u) = f \\ u|_{t=0} = u_0 \quad \text{in } L^2 \end{cases}$$

<u>Remark</u>: We can also start with a given subsequence u_{N_j} of u_N $\Rightarrow$ show $\exists$ subseq. $u_{N_{jk}}$ of subseq. $\to u$

independent of choice of subseq. u_{N_j}

$\Rightarrow u_N$ converges to u

<u>entire sequence</u>

- <u>Energy bound</u>: By weak convergence

$$\|u\|_{L^2_T H^1_x} \leq \liminf_{j \to \infty} \|u_{N_j}\|_{L^2_T H^1}$$

& definition of P_{N_j}.

weak $*$ convergence

$$\|u(t)\|_{L^\infty_T L^2_x} \leq \liminf \|u_{N_j}\|_{L^\infty_T L^\infty}$$