21/2/2019 | TWO-DIMENSIONAL STATISTICAL HYDRODYNAMICS **LECTURE 3**

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

LWP in L^3_x (large data) / LWP in $\dot H^{1/2}$ on $\mathbb{R}^3$ or $\mathbb{T}^3$.

$$\|u\|_{Y_T} = \|u\|_{L^\infty_T L^3_x}$$

$$\|u\|_{Z_T} = \sup_{t\in(0,T]} t^{1/2}\|\nabla u(t)\|_{L^3_x}$$

Lemma: $1 \le p \le q \le \infty$, $\alpha \ge 0$, K cpt in L^p.
$\exists F(t) : (0,1] \to \mathbb{R}_+$, $\lim_{t\to 0+} F(t) = 0$

$$t^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{\alpha}{2}} \|D^\alpha e^{t\Delta} f\|_{L^q} \le F(t), \quad \forall t\in(0,1], \ \forall f\in K$$

proof:

Suppose $K=\{f\}$, $\theta := \frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{\alpha}{2}$.

$$t^\theta \|D^\alpha e^{t\Delta} f\|_{L^q} \le t^\theta \|D^\alpha e^{t\Delta}(f-g)\|_{L^q} + t^\theta \|D^\alpha e^{t\Delta} g\|_{L^q}, \quad g\in\mathcal{S}(\mathbb{R}^d)$$

$\mathcal{S}(\mathbb{T}^d) = C^\infty(\mathbb{T}^d)$

$$\lesssim \|f-g\|_{L^p} + t^\theta \|D^\alpha e^{t\Delta} g\|_{L^q}$$

Given $j\ge 1$, $\exists g_j \in \mathcal{S}$ s.t.

$$t^\theta \|D^\alpha e^{t\Delta} f\|_{L^q} \le \frac{1}{2j} + t^\theta \|D^\alpha e^{t\Delta} g_j\|_{L^q}$$
$$\le \frac{1}{j}, \quad \forall 0<t\le t_j$$

$$F(t) = \inf_j (RHS)_j + t$$

General case: Given j, $K \subset \bigcup_{k=1}^{N_j} B_{\frac{1}{2j}}(g_k^j)$, $g_k^j \in \mathcal{S}$

$$t^\theta \|D^\alpha e^{t\Delta} f\|_{L^q} \le \frac{1}{2j} + t^\theta \|D^\alpha e^{t\Delta} g_k^j\|_{L^q} \quad \text{for } f\in B_{\frac{1}{2j}}(g_k^j)$$

$$(LHS) \le \max_k (RHS), \ \forall f\in K$$

Then, take infimum in j to define $F(t)$. $\boxtimes$

IDEA: Run a contraction argument in $X_T = Y_T \cap Z_T$
on $B_{R,\eta} = \{\|u\|_{Y_T} \le R, \|u\|_{Z_T} \le \eta\}$,
$\eta \ll 1$, $R = 2C_0\|u_0\|_{L^3_x}$

From Lecture 1,

$$\|\Gamma u\|_{Y_T} \le C_0\|u_0\|_{L^3_x} + \underbrace{C_1\|u\|_{Y_T}\|u\|_{Z_T}}_{\text{NOT GOOD}}$$

because $\|\Gamma u - \Gamma v\|_{Y_T} \lesssim \|u-v\|_{Y_T}\|u\|_{Z_T} + \underline{\|v\|_{Y_T}}\|u-v\|_{Z_T}$
$\leftarrow$ cannot be made small

→ Need to modify the estimate :

$$\|\Gamma u\|_{Y_T} \le C_0 \|u_0\|_{L^3_x} + c\int_0^t (t-t')^{-\frac{1}{4}} \|u\cdot\nabla u(t')\|_{L^2_x}\, dt'$$

$-\frac{3}{2}(\frac{1}{2}-\frac{1}{3})$

2nd term $\lesssim \underbrace{\int_0^t (t-t')^{-\frac{1}{4}} (t')^{-\frac{3}{4}}\, dt'}_{B(3/4,\,1/4) < \infty} \|u\|_{Y_T}^{1/2} \|\nabla u\|_{Z_T}^{3/2}$

$$\Rightarrow \|\Gamma u\|_{Y_T} \le C_0 \|u_0\|_{L^3_x} + \|u\|_{Y_T}^{1/2} \|u\|_{Z_T}^{3/2}$$
$$\|\Gamma u\|_{Z_T} \le \|e^{tL} u_0\|_{Z_T} + \|u\|_{Y_T}^{1/2} \|u\|_{Z_T}^{3/2}$$

$u \in B_{R,\eta}$, $R = 2C_0 \|u_0\|_{L^3_x}$

$$\|\Gamma u\|_{Y_T} \le \tfrac{1}{2} R + C_1 R^{1/2} \eta^{3/2} \le R$$

By Lemma, $\exists T = T(u_0) > 0$ s.t. $\|e^{tL} u_0\|_{Z_T} \le \frac{1}{2}\eta$

$$\|\Gamma u\|_{Z_T} \le \tfrac{1}{2}\eta + C_2 R^{1/2} \eta^{3/2} \le \eta$$

$\Rightarrow \boxed{\Gamma u \in B_{R,\eta}}$, $\eta = \eta(R) = \eta(\|u_0\|_{L^3_x}) \ll 1$.

→ <u>difference estimate</u> :

$$\|\Gamma u - \Gamma v\|_{X_T} \le C\,(R^{1/2} + \eta^{1/2})\,\eta^{1/2}\, \|u-v\|_{X_T}$$
$$\le \tfrac{1}{2} \|u-v\|_{X_T} \quad \text{by choosing } \eta = \eta(R) \ll 1$$

$\|u\|_{X_T} = \|u\|_{Y_T} + \|u\|_{Z_T}$

$\Rightarrow$ LWP in $L^3_x(\mathbb{R}^3)$ (or $\mathbb{T}^3$)

<u>LWP IN $\dot H^{1/2}(\mathbb{R}^3)$ (OR $H^{1/2}(\mathbb{T}^3)$)</u>

Similarly, let $\tilde Y_T = C_T \dot H^{1/2}_x$
$\tilde X_T = \tilde Y_T \cap Z_T$

$$\|\Gamma u\|_{\tilde Y_T} \le C_0 \|u_0\|_{\dot H^{1/2}} + c\underbrace{\int_0^t (t-t')^{-\frac{1}{4}} \|u\cdot\nabla u(t')\|_{L^2_x}\, dt'}$$

$$\lesssim \|u\|_{Y_T}^{1/2} \|u\|_{Z_T}^{3/2}$$

$$\overset{\text{Sobolev}}{\lesssim} \|u\|_{\tilde Y_T}^{1/2} \|u\|_{Z_T}^{3/2}$$

$\Rightarrow$ rest follows

<u>NSE WITH DETERMINISTIC FORCING</u>

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = f \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

<u>Duhamel</u> : $u(t) = \Gamma_{u_0,f}\, u(t) = e^{tL} u_0 - \int_0^t e^{(t-t')L}\, \Pi((u\cdot\nabla)u)(t')\, dt'$
$$+ \underbrace{\int_0^t e^{(t-t')L}\, \Pi f(t')\, dt'}_{= F(t)}$$

Suffices to control F in the relevant norms

ex: $\|F\|_{Y_T} \le \|f\|_{L^1_T L^3_x}$

$\|F\|_{\tilde{Y}_T} \le \|f\|_{L^1_T \dot{H}^{1/2}_x}$

$\|F\|_{Z_T}$ $\Leftarrow$ we need to be able to make "this" small by taking $T \ll 1$.

- $t^{1/2} \left\| \nabla \int_0^t e^{(t-t')L} \,\Pi f(t')\, dt' \right\|_{L^3_x}$

$$\lesssim t^{1/2} \int_0^t (t-t')^{-1/2} \|f(t')\|_{L^3_x}\, dt'$$

$$\lesssim \underbrace{t^{1/2} \int_0^t (t-t')^{-1/2} (t')^{-1/2}\, dt'}_{\lesssim 1} \sup_{t' \in (0,t)} (t')^{1/2} \|f(t')\|_{L^3_x} \quad \Rightarrow \nabla^{-1} f \in Z_T, \quad f \in \nabla Z_T$$

or

$$\lesssim \|f\|_{L^q_T L^3_x}, \text{ for some } q > 2 \quad \Rightarrow \text{requires less differentiability.}$$

Remark: can take a rougher forcing (by imposing higher integrability in time).

$$\left[\begin{array}{l} \lesssim t^{1/2} \int_0^t (t-t')^{-1+} \|D^{-1+} f(t')\|_{L^3_x}\, dt' \\ f \in L^q_T W^{-1+,3}_x, \quad q \gg 1 \end{array}\right]$$

Also, $\|F\|_{Y_T} \lesssim \|f\|_{L^q_T W^{-2+,3}_x}$, $q \gg 1$

$\|F\|_{\tilde{Y}_T} \lesssim \|f\|_{L^q_T H^{-3/2+}_x}$

BACK TO STOCHASTIC NSE

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = \phi\xi \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

ξ: space-time white noise

ϕ: smoothing operator in x

$\Rightarrow \partial_t u - Lu + \Pi((u\cdot\nabla)u) = \Pi(\phi\xi)$

($\phi = \mathrm{Id}$: Zhu-Zhu JDE

Mild Formulation

$$u(t) = e^{tL}u_0 - \int_0^t e^{(t-t')L}\, \Pi((u\cdot\nabla)u)(t')\, dt' + \underbrace{\Pi\left(\int_0^t e^{(t-t')L} \phi\, dW(t')\right)}_{\text{stochastic convolution}}$$

$\Psi = (\psi_1, \psi_2, \psi_3)$

on $\mathbb{T}^d$

$$\Psi_j = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} e^{in\cdot x} \int_0^t e^{-(t-t')|n|^2} \phi_n \, d\beta_n^j(t')$$

$$\left(\begin{array}{l} \beta_{-n}^j = \overline{\beta_n^j} \\ \phi_{-n} = \phi_n \end{array} \right.$$

Drop j in the following.

Proposition : Let $\phi \in HS(L^2; H^s)$. Then, $\psi \in C_t^{d/2-} W_x^{s+1-\alpha, r}(\mathbb{T}^d)$, a.s, $r \le \infty$.

$C_t W_x^{s+1-\varepsilon; r}(\mathbb{T}^d)$

proof : Let $t \le \tau$

$\mathbb{E}\left[\psi(t,x)\overline{\psi(\tau,y)}\right]$ space-time covariance

$$\overset{\text{ind. of BM}}{=} \mathbb{E}\left[\sum_n e^{in\cdot x} \cdots \sum_m e^{-im\cdot x} \cdots\right]$$

$$\left| \begin{array}{l} I(f) = \int_a^b f\,dB \\ \mathbb{E}[I(f)I(g)] = \int_a^b fg\,dt \end{array} \right.$$

$$\overset{\substack{n \neq m \\ \text{independent}}}{=} 2\sum_n e^{in(x-y)} |\phi_n|^2 \int_0^t e^{-(t-t')|n|^2} e^{-(\tau-t')|n|^2} dt'$$

$$= \sum_{n\neq 0} e^{in\cdot(x-y)} \frac{|\phi_n|^2}{|n|^2} \underbrace{\left(e^{(t-\tau)|n|^2} - e^{-(t+\tau)|n|^2}\right)}_{= c_n(t,\tau) \le 1}$$

Apply $\langle\nabla_x\rangle^{s+1}$, $\langle\nabla_y\rangle^{s+1}$,

$$\Rightarrow \mathbb{E}\left[\langle\nabla_x\rangle^{s+1}\psi(t,x) \, \langle\nabla_y\rangle^{s+1}\psi(\tau,y)\right] \lesssim \sum e^{in\cdot(x-y)} \langle n\rangle^{2s} |\phi_n|^2 c_n(t,\tau)$$

Set $t = \tau$, $x = y$ ← Gaussian r.v.

$$\mathbb{E}\left[|\langle\nabla\rangle^{s+1}\psi(t,x)|^p\right] \le p^{p/2}\, \mathbb{E}\left[|\langle\nabla\rangle^{s+1}\psi(t,x)|^2\right]^{p/2}$$

$$\le p^{p/2} \|\phi\|^p_{HS(L^2_j H^s)} \quad \forall p < \infty$$

- ψ sum of Gaussian r.v. $\Rightarrow$ ψ Gaussian r.v.
- $\|g\|_{L^p(\Omega)} \le p^{1/2} \|g\|_{L^2(\Omega)}$ for any g Gaussian
- Hilbert-Schmidt norm

$\|\phi\|_{HS(X;Y)} = \left(\sum_n \|\phi e_n\|_Y^2\right)^{1/2}$

$\{e_n\}$ ONB in X

$\to X = L^2$, $\phi(e_n) = \phi_n e^{in\cdot x}$

Let $r < \infty$. Then, $\forall r \le p < \infty$

$$\left\| \|\psi(t)\|_{W_x^{s+1,r}} \right\|_{L^p(\Omega)}$$

$$\overset{\text{Minkowski}}{\le} \left\| \|\langle\nabla\rangle^{s+1}\psi(t,x)\|_{L^p(\Omega)} \right\|_{L^r_x(\mathbb{T}^d)}$$

$$\lesssim p^{1/2} \|\phi\|_{HS(L^2_j H^s)}$$

For $r = \infty$, use Sobolev in x

$$\|\psi(t)\|_{W_x^{s+1-\varepsilon,\infty}} \lesssim \|\psi(t)\|_{W_x^{s+1,r}}, \quad r < \infty$$

Fix $t > 0$, $\psi(t) \in W_x^{s+1,r}$ a.s. $r < \infty$

$\psi(t) \in W_x^{s+1-\varepsilon,\infty}$ a.s.

Issue : Set of probability 1 depends on t.

Given $h \in \mathbb{R}$, s.t. $t+h > 0$,

$$\delta_h \psi(t,x) = \psi(t+h,x) - \psi(t,x)$$

$$\mathbb{E}\left[\delta_h\psi(t,x)\,\delta_h\psi(t,y)\right] = \mathbb{E}\left[\psi(t+h,x)\psi(t+h,y)\right] - \mathbb{E}\left[\psi(t+h,x)\psi(t,y)\right]$$
$$- \mathbb{E}\left[\psi(t,x)\psi(t+h,y)\right] + \mathbb{E}\left[\psi(t,x)\psi(t,y)\right]$$

$$= \sum e^{in\cdot(x-y)} \frac{|\phi_n|^2}{|n|^2} \left(e_n(t+h,t+h) - e_n(t+h,t) - e_n(t,t+h) + e_n(t,t) \right)$$

$$|e_n(t+h,t+h) - e_n(t+h,t)| = \left| 1 - e^{-2(t+h)|n|^2} - e^{-h|n|^2} + e^{-(2t+h)|n|^2} \right|$$

$$= \left| (1-e^{-h|n|^2})(\underbrace{1+e^{-(2t+h)|n|^2}}_{\lesssim 1}) \right|$$

mean value theorem $\rightarrow$

$$\lesssim |h|^\alpha |n|^{2\alpha}, \quad \forall \alpha \in [0,1]$$

$$\| \langle \nabla \rangle^{s+1-\alpha} \delta_h \psi(t,x) \|_{L^p(\Omega)} \le p^{1/2} \| \langle \nabla \rangle^{s+1-\alpha} \delta_h \psi(t,x) \|_{L^2(\Omega)} \qquad \forall p < \infty$$

$$\lesssim p^{1/2} |h|^{\alpha/2} \|\phi\|_{HS(L^2;H^s)}$$

$$\| \| \delta_h \psi(t) \|_{W_x^{s+1-\alpha,r}} \|_{L^p(\Omega)} \lesssim p^{1/2} |h|^{\alpha/2} \|\phi\|_{HS(L^2;H^s)}. \qquad \forall r \le p < \infty.$$

Kolmogorov continuity criterion: $|h|^{p\frac{\alpha}{2}}$

$$\frac{\theta}{p} = \frac{p\frac{\alpha}{2}-1}{p} = \frac{\alpha}{2} - \frac{1}{p}$$

$\mathbb{E}[|X_t - X_s|^p] \lesssim |t-s|^{1+\theta}$

$(\frac{\theta}{p}-)$ - Hölder

$$\Rightarrow \psi \in C_t^{\frac{\alpha}{2}-} W_x^{s+1-\alpha,r}, \quad \text{a.s. } r < \infty \quad (\text{and } r = \infty). \ \square$$

SNSE on $\mathbb{T}^3$

- LWP in $H^{1/2}$

$\alpha = \varepsilon$

$\psi \in \tilde{Y}_T = L_T^\infty H_x^{1/2} \Leftarrow s+1 > 1/2$, $s > -1/2$

$\psi \in Z_T$

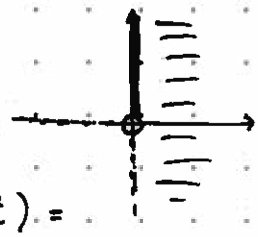
need $\|\nabla \psi(t)\|_{L_x^3} < \infty$ a.s. $\Leftarrow s+1 > 1$, $s > 0$

- LWP in L_x^3

$\psi \in Y_T \Leftarrow s+1 > 0$, $s > -1$

In the remaining part of the course, we focus on $\mathbb{T}^2$.

- $H = L^2(\mathbb{T}^2; \mathbb{R}^2)$ divergence free, mean 0.
- $\mathbb{Z}_+^2 = \{ (n_1, n_2) : n_1 > 0 \text{ OR } n_1 = 0 \text{ and } n_2 > 0 \}$



- ONB on H

$$e_n = \begin{cases} c_n \, n^\perp \sin(n\cdot x), & n \in \mathbb{Z}_+^2 \\ c_n \, n^\perp \cos(n\cdot x), & n \in -\mathbb{Z}_+^2 \end{cases}, \qquad \mathbb{Z}_+^2 \cup (-\mathbb{Z}_+^2) = \mathbb{Z}^2 \setminus \{0\}$$

where $c_n = \frac{1}{\sqrt{2}\pi |n|}$, $n = (n_1, n_2)$, $n^\perp = (-n_2, n_1)$

- $\mathcal{H}_T = \{ u \in L_T^2 H^1, \ \partial_t u \in L_T^2 H^{-1} \}$

$$\|u\|_{\mathcal{H}_T} = \left(\|u\|_{L_T^2 H^1}^2 + \|\partial_t u\|_{L_T^2 H^{-1}}^2 \right)^{1/2}$$

- $t \mapsto \langle \partial_t u(t), u(t) \rangle_{L_x^2} \in L_T^1$ if $u \in \mathcal{H}_T$

• $\int_0^T \langle \partial_t u, u \rangle_{L^2_x}\, dt \le \|\partial_t u\|_{L^2_T H^{-1}_x} \|u\|_{L^2_T H^1_x}$ $\Rightarrow$ $\|u(t)\|^2_{L^2}$ is (abs) conti
i.e. $\mathcal{H}_T \subset C_T L^2_x$.

Aubin-Lions lemma (compactness): $\mathcal{H} \subset\subset L^2_T H^s_x$, $s<1$. (proof from Constantin-Foias)

Separable reflexive Banach spaces $X_1 \subset\subset X_0 \subset X_{-1}$

Proposition: $\{u_n\}$ bdd in $L^{p_1}_T X_1$
$\{\partial_t u_n\}$ bdd in $L^{p_2}_T X_{-1}$, $1 < p_1, p_2 < \infty$.

$\Rightarrow$ $\exists$ subsequence u_{n_j} of u_n convergent in $L^{p_1}_T X_0$.

Lemma: $\forall \varepsilon > 0$ $\exists c_\varepsilon > 0$ s.t. $\forall x \in X_1$, $\|x\|_0 \le \varepsilon \|x\|_1 + c_\varepsilon \|x\|_{-1}$.

proof: Suppose NOT, $\exists \{x_n\} \subset X_1$ s.t. $\|x_n\|_0 \ge \varepsilon \|x_n\|_1 + n \|x_n\|_{-1}$.
Let $y_n = x_n / \|x_n\|_1$.
Then,
(*) $\|y_n\|_0 \ge \varepsilon + n \|y_n\|_{-1}$, $\|y_n\|_1 = 1$
X_1 is separable and reflexive $\Rightarrow$ $B_1 \subset X_1$ is weakly compact.
$\Rightarrow$ $\exists$ subsequence, denoted by y_n, s.t. $y_n \rightharpoonup y$ in X_1
$\overset{X_1 \subset\subset X_0}{\Longrightarrow}$ $y_n \to y$ in X_0
By (*), $\|y_n\|_{-1} \le \frac{1}{n} \|y_n\|_0 \le \frac{c}{n} \to 0$

$\Rightarrow$ $y = 0$, but by (*) $\|y\|_0 \ge \varepsilon$, contradiction. ☐

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proof of Prop: W.L.O.G., assume $\exists$ subsequence $u_n \rightharpoonup 0$ in $L^{p_1}_T X_1$. ($v_n = u_n - u$)
WTS: $u_n \to 0$ in $L^{p_1}_T X_0$ (strongly)
claim: By Lemma, it suffices to show $u_n \to 0$ in $L^{p_1}_T X_{-1}$.
If so,
$$\int_0^T \|u_n\|^{p_1}_{X_0}\, dt \le \varepsilon \sup_n \int_0^T \|u_n(t)\|_1^{p_1}\, dt + c_{\varepsilon, p_1} \int_0^T \|u_n(t)\|_{-1}^{p_1}\, dt$$
$$\le \varepsilon C + o(1) \quad \text{as } n \to \infty$$
$\Rightarrow$ $u_n \to 0$ in $L^{p_1}_T X_0$

: I, bdd interval in $\mathbb{R}$
$L \in X_1'$, $\mathbb{1}_I(t) L \in (L^{p_1}_T X_1)'$
$$\underbrace{\langle u_n, \mathbb{1}_I L \rangle}_{\downarrow\, 0} = \int_I \langle u_n(t), L \rangle\, dt = \left\langle \int_I u_n(t)\, dt, L \right\rangle$$
because $u_n \rightharpoonup 0$ in $L^{p_1}_T X_1$

$\Rightarrow$ $\int_I u_n(t)\, dt$ weakly convergent in X_1 (to 0)

up to subseq. $\overset{}{\underset{X_1 \subset\subset X_0}{\Longrightarrow}}$ $\int_I u_n(t)\, dt \to 0$ in X_0 $\Rightarrow$ in X_{-1} (**)

Fix $t \in [0,T]$ and write $u_n(t) - u_n(t_1) = \int_{t_1}^t \frac{du_n}{ds}\, ds$

Average in t_1 over $[t-\varepsilon, t]$

$$u_n(t) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} u_n(t_1)\,dt_1 + \underbrace{\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} (s-t+\varepsilon)\frac{du_n}{ds}(s)\,ds}_{\|\cdot\|_{-1} \underset{\text{Hölder}}{\lesssim} \varepsilon^{1/p_2'} \left\|\frac{du_n}{ds}\right\|_{L_T^{p_2} X_{-1}} \lesssim \varepsilon^{1/p_2'}}$$

$\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\int_{t_1}^{t} \dots \, ds\, dt_1$

Fubini

↖ implicit constant C

Given $\varepsilon_0 > 0$, choose $\varepsilon > 0$ small s.t. $C\varepsilon^{1/p_2'} < \frac{\varepsilon_0}{2}$

$$\Rightarrow \|u_n(t)\|_{X_{-1}} \leq \frac{\varepsilon_0}{2} + \frac{1}{\varepsilon}\underbrace{\left\|\int_{t-\varepsilon}^{t} u_n(t_1)\,dt_1\right\|_{X_{-1}}}_{\downarrow (**) \; 0}$$

$$\Rightarrow \|u_n(t)\|_{X_{-1}} \to 0$$

on the other hand, we have

$$\|u_n(t_1) - u_n(t_2)\|_{X_{-1}} \leq C|t_1 - t_2|^{1/p_2'}, \; \forall n \geq 1 \qquad \text{(using FTC)}$$

Fix $\varepsilon > 0$.

$$\sup_{t\in[0,T]} \|u_n(t)\|_{X_{-1}} \leq \max_{j=1,\dots,\lceil \frac{T}{\varepsilon}\rceil} \|u_n(j\varepsilon)\|_{X_{-1}} + C\varepsilon^{1/p_2'}$$

$$\leq \varepsilon_0 \qquad \forall n \geq N(\varepsilon_0)$$

$$\Longrightarrow \sup_{t\in[0,T]} \|u_n(t)\|_{X_{-1}} \longrightarrow 0$$

$$\Longrightarrow u_n \to 0 \text{ in } L_T^{p_1} X_{-1}. \quad \square$$