14/2/2019 | TWO-DIMENSIONAL STATISTICAL HYDRODYNAMICS LECTURE 2

(NSE) $\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u - \nabla p = f \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$

Π = Leray projection : $H^s \to H^s_{df}$ ← divergence free [Projection onto space of divergence free functions

$L = \Pi\Delta$

$$\Rightarrow \begin{cases} \partial_t u - Lu + \Pi((u\cdot\nabla)u) = \Pi f \\ u|_{t=0} = u_0 \end{cases} \quad , \quad \Pi(\nabla p) = \nabla p + \nabla(-\Delta)^{-1}\nabla\cdot\nabla p = \nabla p - \nabla p = 0 .$$

$$\Rightarrow u(t) = \Gamma_{u_0} u(t) = e^{tL}u_0 - \int_0^t e^{(t-t')L}\Pi((u\cdot\nabla)u)(t')\,dt' + \int_0^t e^{(t-t')L}\Pi(f)(t')\,dt'$$

NEXT WEEK: JCMB 5328

<u>Linear estimates</u> : $1<p\le q<\infty$, $\alpha\ge 0$, $\|D^\alpha e^{tL} f\|_{L^q} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\alpha}{2}} \|f\|_{L^p}$ on $\mathbb{R}^d$ (on $\mathbb{T}^d$ for $0<t\le 1$)

$X = C_t([0,\infty); L^3_x) \cap L^\infty_t((0,\infty); W^{1,3}_x)$ ← df →

⚠ Always consider divergence free spaces.

- $\|u\|_{Y_T} = \|u\|_{L^\infty_T L^3_x}$
- $\|u\|_{Z_T} = \sup_{t\in(0,T)} t^{1/2}\|\nabla u(t)\|_{L^3_x}$.

Last week,

$\|\Gamma u\|_Y \le c_0\|u_0\|_{L^3_x} + c_1\|u\|_Y\|u\|_Z$

$\|\Gamma u\|_Z \lesssim \|u_0\|_{L^3_x} + \|u\|_Y^{1/2}\|u\|_Z^{3/2}$

<u>Similarly</u> : $\|\Gamma u - \Gamma v\|_Y \lesssim \|u-v\|_Y\|u\|_Z + \|v\|_Y\|u-v\|_Z$

$\|\Gamma u - \Gamma v\|_Z \lesssim \|u-v\|_Y^{1/2}\|u-v\|_Z^{1/2}\|u\|_Z + \|v\|_Y^{1/2}\|v\|_Z^{1/2}\|u-v\|_Z$

Let $\|u_0\|_{L^3_x} \ll 1$, and consider a closed ball of radius η , $B_\eta \subset X$, for $\eta = 10c_0\|u_0\|_{L^3_x} \ll 1$

$$\Rightarrow \|\Gamma u\|_X \le 2c_0\|u_0\|_{L^3_x} + c_1\|u\|_X^2 < \eta$$

$$\|\Gamma u - \Gamma v\|_X \le \underbrace{c_2(\underbrace{\|u\|_X + \|v\|_X}_{\le 2\eta})}_{\le \frac{1}{2}}\|u-v\|_X$$

Banach Fixed Point Thm

$\Rightarrow \Gamma u = u$ in $B_\eta \subset X$ $\Rightarrow$ small data global-well-posedness in $L^3_x(\mathbb{R}^3)$

- uniqueness in $L^3_x(\mathbb{R}^3)$ follows from a continuity argument.
- to show uniform continuity, use same estimates to get $\|u-v\|_X \lesssim \|u_0-v_0\|_{L^3_x}$.

HOW TO RECOVER PRESSURE p?

$$\partial_t u - \Delta u + (u\cdot\nabla)u - \nabla p = 0$$

$$\Rightarrow \nabla p = \partial_t u - \Delta u + (u\cdot\nabla)u =: G(t)$$

$$\Pi(G(t)) = \partial_t u - Lu - \Pi((u\cdot\nabla)u) = 0$$

$$\Rightarrow G(t) = \text{curl free} = \nabla\phi$$

$$\Rightarrow p = \phi + \text{const}.$$

LWP IN $L^3_x(\mathbb{R}^3)$? NEXT WEEK

$$\|\Gamma u(t)\|_{L^3_x} \leq c\|u_0\|_{L^3_x} + \int_0^t (t-t')^{-\frac{1}{2}} (u(t'))(\nabla u(t'))\,dt'$$

We cannot expect improvement from Y_T, Z_T norms by considering a small time interval since both take L^∞_t.

SMALL DATA GLOBAL WELL-POSEDNESS IN $\dot{H}^{1/2}_x(\mathbb{R}^3)$?

$$\|\Gamma u(t)\|_{\dot{H}^{1/2}} \leq c\|u_0\|_{\dot{H}^{1/2}_x} + c\int_0^t (t-t')^{-\frac{1}{2}} \underbrace{\|(u\cdot\nabla)u(t')\|_{L^{3/2}_x}}_{\leq \underbrace{\|u(t')\|_{L^3_x}}_{\lesssim \|u(t')\|_{\dot{H}^{1/2}_x}} \|\nabla u(t')\|_{L^3_x}} dt'$$

$$\leq c\|u_0\|_{\dot{H}^{1/2}_x} + c\int_0^t (t-t')^{-\frac{1}{2}}(t')^{-\frac{1}{2}}\,dt' \; \|u\|_{L^\infty_t \dot{H}^{1/2}} \|u\|_Z$$

Sobolev inequality
$\frac{s}{d} = \frac{1}{p} - \frac{1}{q}$
$\|f\|_{L^q_x} \lesssim \|f\|_{\dot{W}^{s,p}_x}$
on $\mathbb{R}^d$
(or on $\mathbb{T}^d$ with $\int f\,dx = 0$)

Similarly,

$$\|\nabla\Gamma(t)\|_{L^3_x} \leq c\,t^{-1/2}\,\||\nabla|^{-1/2}u_0\|_{L^2_x} + \underbrace{\|u\|^{1/2}_{L^\infty_t L^3_x}}_{\lesssim \|u\|^{1/2}_{L^\infty_t \dot{H}^{1/2}_x}} \|u\|_Z^{3/2}, \quad -\tfrac{3}{2}\left(\tfrac{1}{2}-\tfrac{1}{3}\right) - \tfrac{1/2}{2} = -1/2$$

$\Rightarrow$ Run a contraction mapping argument in $\tilde{X} = C_t\dot{H}^{1/2}_x \cap Z$.

- forced NSE $\quad \partial_t u - \Delta u + (u\cdot\nabla)u - \nabla p = f$, $\quad f$ deterministic, given.

- stochastic NSE (on $\mathbb{T}^d$) $\quad \partial_t u - \Delta u + (u\cdot\nabla)u - \nabla p = \zeta$, $\quad \zeta$ = white in time (or rough force in time), smooth in x, stochastic forcing.

interested in long time behaviour of solutions.

- white in time:

$\zeta = \phi\xi$, $\quad \xi$ = space-time white noise
ϕ = smoothing operator in x, Hilbert-Schmidt from L^2_x to H^s.

- $W(t) = L^2$- cylindrical Wiener process $= (W^1(t), \dots, W^d(t))$.

$$W^j(t,x) = \sum_{n \in \mathbb{Z}^d} \beta_n^j \, e^{in\cdot x} \quad , \quad \{\beta_n^j\}_{\substack{n\in\mathbb{Z}^d \\ j\in\{1,\dots,d\}}}$$ family of independent complex valued Brownian motions.

$$\beta_n^j = \underbrace{Re(\beta_n^j)} + i\, \underbrace{Im(\beta_n^j)}$$

independent real-valued BM's

BASIC STOCHASTIC ANALYSIS

- $(\Omega, \mathcal{F}, P)$ probability space
- A Brownian motion (BM) B on $\mathbb{R}_+$ $(t \geq 0)$ is a stochastic process s.t.
  - (i) $B(0) = 0$, a.s.,
  - (ii) $B(t) - B(s) \sim \mathcal{N}(0, t-s)$, $t > s$
    mean variance
  - (iii) independent increment on disjoint time intervals: $B(t_1) - B(s_1)$, $B(t_2) - B(s_2)$ are independent, $t_2 > s_2 > t_1 > s_1$.

- mean: $\mathbb{E}[f] = \int f\,dP$
- variance: $\mathbb{E}[(f - \mathbb{E}(f))^2] = var(f)$

Properties:

- $$\mathbb{E}\left(|B(t) - B(s)|^{2k}\right) = \underbrace{\frac{(2k)!}{2^k (k!)}}_{=(2k-1)!!} (t-s)^k$$

- $\mathbb{E}\left(|B(t) - B(s)|^p\right) \sim_p |t-s|^{p/2}$, with implicit constant $c_p \lesssim p^{p/2}$.

Kolmogorov continuity criterion: $\{X_t\}$ stochastic process with values in a metric space S. Suppose there exists $p \geq 1, \alpha > 0$ s.t. $\mathbb{E}(d(X_t, X_s)^p) \lesssim |t-s|^{1+\alpha}$ $\forall t, s$.
Then,
$$P\left(\sup_{t \neq s} \frac{d(X_t, X_s)}{|t-s|^{\frac{\alpha}{p} - \gamma}} \geq \lambda\right) \leq \frac{c}{\lambda^p}, \quad \forall\, 0 < \gamma < \frac{\alpha}{p}.$$

$\Rightarrow X_t$ is a.s. $(\frac{\alpha}{p} - \varepsilon)$ - Hölder continuous. In particular, a.s. continuous.

$\boxed{\text{proof}}$ Follows from Borel-Cantelli Lemma

Since $\mathbb{E}(|B(t) - B(s)|^p) \lesssim |t-s|^{1 + (\frac{p}{2} - 1)}$ for all finite p, using Kolmogorov's continuity criterion, we get $\frac{\alpha}{p} = \frac{1}{2} - \frac{1}{p} \to \frac{1}{2}$ as $p \to \infty$,

$$\boxed{\text{BM is a.s. } (\tfrac{1}{2}-) \text{ - Hölder continuous}}$$

- $j \in \mathbb{Z}$, $Q_j(f) = \int \varphi\left(\frac{|\xi|}{2^j}\right) \hat{f}(\xi) e^{i\xi\cdot x} d\xi$, $\sum_{j\in\mathbb{Z}} \varphi\left(\frac{|\xi|}{2^j}\right) \equiv 1$.

nice bump function, C_c^∞ supported on $[1/2, 2]$

$P_j = Q_j$, $j \geq 1$

$P_0 = \sum_{j\leq 0} Q_j$ = projection onto $\{|\xi| \leq 1\}$

- Besov space

$\|f\|_{B^s_{p,q}} = \| 2^{js} \|P_j(f)\|_{L^p_x} \|_{\ell^q_j(\mathbb{Z}_{\geq 0})}$

$\to p = q = 2$, $B^s_{2,2} = H^s$

$\|f\|_{\dot{B}^s_{p,q}} = \| 2^{js} \|Q_j(f)\|_{L^p_x} \|_{\ell^q_j(\mathbb{Z})}$

- <u>FACT</u> : $0 < s < 1$, $\dot{C}^s = \dot{B}^s_{\infty,\infty}$, $\sup_{t_1\neq t_2} \frac{f(t_2) - f(t_1)}{(t_2-t_1)^s}$

$C^s = \dot{C}^s \cap L^\infty = B^s_{\infty,\infty}$.

$\to$ <u>Locally in time</u> : BM $\in B^{1/2-}_{\infty,\infty} \supset W_t^{\frac{1}{2}-,p}$, $1 \leq p \leq \infty$.

BM $\in B^{1/2-}_{p,q}$, $1 \leq p \leq q \leq \infty$.

- <u>covariance</u> $\mathbb{E}[B(t)B(s)] = t \wedge s$ ← min

$t > s$, $\mathbb{E}[(B(t) - B(s))B(s)] + \mathbb{E}[B^2(s)] = s$

independent

- <u>Wiener integral</u> : $I(f) = \int_a^b f(t)\,dB(t)$, $f \in L^2([a,b])$ deterministic.

<u>Step 1</u> : Step function $f(t) = \sum_{j=1}^n a_{j-1} \mathbb{1}_{[t_{j-1}, t_j]}(t)$

deterministic.

Define $I(f) = \sum_{j=1}^n a_{j-1} (B(t_j) - B(t_j - t_{j-1}))$ (left endpoint of Riemann sum)

Then,

① $\mathbb{E}(I(f)) = 0$

② $\mathbb{E}[(I(f))^2] = \sum_{j=1}^n \sum_{k=1}^n a_{j-1} a_{k-1} \mathbb{E}[(B(t_j) - B(t_j - t_{j-1}))(B(t_k) - B(t_{k-1}))]$

$t_0 < t_1 < \dots < t_n$

$= \sum_{j=1}^n a_{j-1}^2 (t_j - t_{j-1})$

$= \|f\|^2_{L^2([a,b])}$

<u>Step 2</u>: general $f \in L^2([a,b])$

Approximate f by step functions f_n in $L^2([a,b])$ and define

$I(f) = \lim_{n\to\infty} I(f_n)$.

$\Rightarrow$ ① & ② hold

$I : L^2([a,b]) \longrightarrow L^2(\Omega)$, isometry (onto the image)

Remark : If B is complex-valued,

$$\mathbb{E}[|I(f)|^2] = 2\|f\|^2_{L^2([a,b])}$$

Back to SNSE :

- mild formulation: $u(t) = e^{tL}u_0 - \int_0^t e^{(t-t')L}\,\Pi\,((u\cdot\nabla)u)(t')\,dt'$

$$+ \underbrace{\Pi\int_0^t e^{(t-t')L}\,\phi\, dW(t')}_{=}\,,\quad \xi = \phi\,\xi$$
$$= \phi\, dW$$

j-th comp.

$$\Pi\left(\sum_{n\in\mathbb{Z}^d} e^{in\cdot x}\underbrace{\int_0^t e^{-(t-t')|n|^2}\phi_n\, d\beta_n^j(t')}_{\text{Wiener integral}}\right)$$

$(\phi f)^\wedge(n) = \phi_n\,\hat{f}(n)$

spatially homog.
(translation inv.)