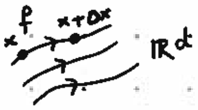
7/2/2019 | TWO-DIMENSIONAL STATISTICAL HYDRODYNAMICS LECTURE 1

$\vec{u}$ = velocity of fluid



Want to take derivative of f

Euler coord. ① $\frac{\partial f}{\partial t}$ = usual time derivative

Lagrange coord. ② $\lim_{\Delta t \to 0} \frac{f(t+\Delta t, x+\vec{u}\,\Delta t) - f(t,x)}{\Delta t} = \frac{Df}{Dt}$

$= \partial_t f + \vec{u}\cdot\nabla f$

material derivative
advective
hydrodynamic
Lagrangian
Stokes

if we want to understand the change of f with respect to the flow, we must follow the particles x

$\dot{x} = \vec{u}$, $\frac{f(t+\Delta t) - f(t)}{\Delta t}$ (with $\overset{x+\Delta x}{}$ over $t+\Delta t$ and $\overset{x}{}$ over t)

INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

$u = (u_1, u_2, u_3) : \mathbb{R}_t \times \mathbb{R}^3 \to \mathbb{R}^3$, velocity field

$p : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = -\nabla p + \Delta u + f, & t>0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

div u = 0 ⤷ incompressibility condition [volume does not change]

→ 4 equations, 4 unknowns, $\partial_t u_j + \sum_{k=1}^{3} u_k \partial_k u_j = -\partial_j p + \Delta u_j + f_j$.

$\sum_{j=1}^{3} \partial_j u_j = 0$.

Helmholtz decomposition : $u = \underbrace{\nabla\times A}_{\text{div. free}} + \underbrace{\nabla\phi}_{\text{curl free}}$

T. Tao 254A website

→ valid decomposition in L^2, $H^s(\mathbb{R})$

→ on the torus, we also have a harmonic term, which we can get rid of by considering u with mean zero. [Hodge decomposition, Tao's website]

→ for other L^p, requires more care.

$u_0 = \overset{\text{div. free}}{v_0} + \nabla w_0$ ⟹ Take divergence $-\Delta w_0 = -\operatorname{div} u_0$

$w_0 = -\nabla(-\Delta)^{-1}\nabla\cdot u_0$

$\Rightarrow v_0 = u_0 - \nabla w_0$

$= (\mathrm{Id} + \nabla(-\Delta)^{-1}\nabla\cdot)\, u_0$

$= \Pi u_0$

↳ Leray projection

$\Rightarrow v_{0j} = \sum_{k=1}^{3} (\delta_{jk} + \partial_j(-\Delta)^{-1}\partial_k)\, u_{0k}$

$= \mathcal{F}^{-1}\left(\sum_{k=1}^{3} \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2}\right) \hat{u}_{0k}(\xi) \right)$

- **Riesz Transform** : $\frac{i\xi_j}{|\xi|}$ ← higher dimensional analogue of Hilbert transform

$Hf(x) = \text{p.v.}\int \frac{f(x-y)}{y}\,dy \quad : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R}), \quad 1<p<\infty$

on $\mathbb{T}$: $\widehat{Hf}(n) = i\,\mathrm{sgn}(n)\,\hat{f}(n)$.

- $R_j f(x) = \int \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\,dy$, $R_j : L^p \to L^p$, $1 < p < \infty$.
- $\sum_{j=1}^{d} R_j^2 = -\mathrm{Id}$ (like $H^2 = -\mathrm{Id}$).

We want to apply the Leray projection to the equation and study only u, not p.
Assume u is divergence free,

$$(NSE) \xRightarrow{\Pi} \partial_t u + \Pi((u\cdot\nabla)u) = Lu + \Pi f \qquad \text{with } L = \Pi\Delta$$

$$\Pi(\nabla p) = 0$$

with $\mathrm{div}\,u = 0 \Rightarrow \Pi u = u$

$u|_{t=0} = u_0$.

DUHAMEL FORMULATION (= MILD FORMULATION)

$$u(t) = e^{tL}u_0 - \int_0^t e^{(t-t')L}\,\Pi((u\cdot\nabla)u)(t')\,dt'$$
$$+ \int_0^t e^{(t-t')L}\,\Pi f(t')\,dt'$$

→ Need to know the mapping property of e^{tL} !

heat Proposition [$L^p - L^q$ estimate] : $1 \le p \le q \le \infty$, $\|e^{t\Delta}f\|_{L^q_x} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p_x}$

$$\|D^\alpha(e^{t\Delta}f)\|_{L^q_x} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}}\|f\|_{L^p_x}, \quad t>0, \alpha \ge 0.$$

proof:

$$e^{t\Delta}f(x) = \int \overset{\text{Gaussian}}{K_t}(x-y)f(y)\,dy \Rightarrow \|e^{t\Delta}f\|_{L^q_x} \underset{\text{Young's}}{\lesssim} \|K_t\|_{L^r_x}\|f\|_{L^p_x}, \quad \frac{1}{q}+1 = \frac{1}{r}+\frac{1}{p}$$

$$\hat{K}_t(\xi) = e^{-t|\xi|^2}$$
$$= \hat{K}(t^{1/2}\xi), \quad \text{for } K = K_1$$

$$K_t(x) = \frac{1}{t^{d/2}}K\left(\frac{x}{t^{1/2}}\right) \Rightarrow \|K_t\|_{L^r_x} = t^{-d/2}\left\|K\left(\frac{x}{t^{1/2}}\right)\right\|_{L^r_x} = t^{-d/2}\,t^{d/2\cdot 1/r}\,C_K$$
$$\sim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$$

Assume $D = \sqrt{-\Delta}$, otherwise the proof follows by scaling as before.

$$D^\alpha(e^{t\Delta}f) = D^\alpha(K_t * f) = (D^\alpha K_t) * f$$
$$\widehat{D^\alpha K_t}(\xi) = |\xi|^\alpha e^{-t|\xi|^2} = t^{-\frac{\alpha}{2}}\underbrace{(t^{1/2}|\xi|)^\alpha e^{-t|\xi|^2}}_{\hat{G}_t}$$

Let $G = G_1 \in \mathcal{S}(\mathbb{R}^d)$, $G_t = t^{-d/2}G\left(\frac{x}{t^{1/2}}\right)$

$$\|D^\alpha K_t\|_{L^r_x} = t^{-\frac{\alpha}{2}}\|G_t(x)\|_{L^r_x} = t^{-\frac{\alpha}{2}}\,t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \quad .\,\emptyset$$

- What happens on $\mathbb{T}^d$? $e^{t\Delta}f = R_t * f$, $\hat{R}_t(n) = e^{-t|n|^2}$

→ Cannot use scaling argument, thus we use Poisson summation formula:

$|f(x)| + |\hat{f}(x)| \le C(1+|x|)^{-d-\varepsilon}$,

$$\sum_{n\in\mathbb{Z}^d}\hat{f}(n)e^{inx} = \sum_{n\in\mathbb{Z}^d} f(n+x).$$

proof: $F(x) = \sum_{n\in\mathbb{Z}^d} f(x+n)$ ← periodic function

POISSON
SUMMATION
FORMULA

$$\overset{\shortparallel}{\sum_{n\in\mathbb{Z}^d}} \hat{F}(n)\, e^{in\cdot x}$$

$$\hat{F}(n) = \int_{\mathbb{T}^d} F(x)\, e^{-in\cdot x}\, dx = \int_{\mathbb{T}^d} \sum_m f(x+m)\, e^{-in\cdot x}\, dx$$

$$\begin{matrix} x\in\mathbb{T}^d \\ y = x+m \\ \in m+\mathbb{T}^d \end{matrix} \nearrow \quad = \sum_m \int_{\mathbb{T}^d} f(y)\, e^{-in\cdot(y-m)}\, dy$$

$$= \int_{\mathbb{R}^d} f(x)\, e^{-in\cdot x}\, dx - \hat{f}(n)\,. \quad \square$$

Want to study $\|R_t\|_{L^r_x(\mathbb{T}^d)}$, $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$, $\hat{R}_t(n) = \hat{K}_t(n) = e^{-t|n|^2}$

$$\|R_t\|_{L^r_x(\mathbb{T}^d)} = \Big\| \sum_n \hat{K}_t(n)\, e^{in\cdot x} \Big\|_{L^r_x} \underset{\substack{\uparrow \\ \text{PSF}}}{=} \Big\| \sum_n K_t(x+n) \Big\|_{L^r_x}$$

$$\text{Hölder} \lesssim \Big\| \underbrace{\Big(\sum_n \langle n\rangle^{-\beta r'}\Big)^{1/r'}}_{<\infty} \big\| \langle n\rangle^{\beta} K_t(x+n) \big\|_{\ell^r_n} \Big\|_{L^r_x(\mathbb{T}^d)} \quad \beta r' > d$$

$$\lesssim \| \langle x\rangle^{\beta} K_t(x) \|_{L^r_x(\mathbb{R}^d)} =: ④$$

$$\begin{cases} K_t(x) = \frac{1}{t^{d/2}}\, K\left(\frac{x}{t^{1/2}}\right) \quad , 0\le t\le 1 \,, \quad \langle x\rangle \lesssim \left\langle \frac{x}{t^{1/2}} \right\rangle \\ \tilde{K} = \langle\cdot\rangle^{\beta} K(\cdot) \end{cases}$$

$$④ \le \frac{1}{t^{d/2}} \Big\| \left\langle \frac{x}{t^{1/2}} \right\rangle^{\beta} K\left(\frac{x}{t^{1/2}}\right) \Big\|_{L^r_x(\mathbb{R}^d)} = \tilde{C}_K\, t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

⚠ A similar computation holds for $\|D^{\alpha} R_t\|_{L^r_x(\mathbb{T}^d)}$. $\square$

⚠ The estimate on the torus is only valid for $0\le t\le 1$.

- $\underline{t \gg 1}$, $e^{-t|\xi|^2}$ has exponential decay , but weaker for $|\xi| \ll 1$
- but on the torus we cannot expect decay unless we have mean zero .

$$P_{\neq 0}\, e^{t\Delta} \longrightarrow \| P_{\neq 0} R_t \|_{L^r_x} = \Big\| \sum_{n\neq 0} \underbrace{e^{-t|n|^2}}_{\le e^{-\frac{t}{2}} e^{-\frac{|n|^2}{2}},\ t>1} e^{-in\cdot x} \Big\|_{L^r_x(\mathbb{T}^d)} \le e^{-\frac{t}{2}} C \,.\, , \quad t>1 .$$

Corollary: $\| D^{\alpha} e^{tL} f \|_{L^q_x} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right) - \frac{\alpha}{2}} \|f\|_{L^p_x}$, $1 < p \le q < \infty$
(or $1 < p < q = \infty$)

SCALING (u,p) solution . $\Rightarrow$ $\begin{cases} u^{\lambda}(t,x) = \lambda\, u(\lambda^2 t, \lambda x) \\ p^{\lambda}(t,x) = \lambda^2\, p(\lambda^2 t, \lambda x) \end{cases}$, $\lambda > 0$, $(u^{\lambda}, p^{\lambda})$ solution .

$$\|u^{\lambda}\|_{L^q_t L^r_x(\mathbb{R}_+ \times \mathbb{R}^d)} = \lambda^{1-\frac{d}{r}-\frac{2}{q}} \|u\|_{L^q_t L^r_x}$$

→ scaling invariant constant : $\frac{2}{q} + \frac{d}{r} = 1$. (for u) $[\text{e.g., } L^{\infty}_t L^d_x]$

→ For $\mathcal{P}^{\lambda}$, scaling invariant constant : $\frac{2}{q} + \frac{d}{r} = 2$ [e.g. $L^{\infty}_t L^{d/2}_x$]

→ <u>d=3</u> : $\dot{H}^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$

$\|f\|_{\dot{H}^s} = \left(\int |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$, $f|_{|x|=\infty} = 0$ or modulo polynomials] to guarantee that space is well-defined.

$\|f\|_{W^{s,p}} = \| \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{f}) \|_{L^p_x}$

$\int |u(T)|^2 dx + \int_0^T \int |\nabla u(t)|^2 \, dx dt = \int |u_0|^2 dx$ ← conserved quantity

⚠ Quantity is too weak to control the $L^3_x(\mathbb{R}^3)$-norm , making it difficult to show GWP.

Theorem : (i) $\exists \delta > 0$ s.t. if $\|u_0\|_{L^3_x(\mathbb{R}^3)} < \delta$, then $\exists!$ solution u to (NSE) in $C([0,\infty), L^3_x) \cap C((0,\infty), W^{1,3}_x)$

(also, continuous dependence)

(ii) Same in $\dot{H}^{1/2}(\mathbb{R}^3)$ $\boxed{f \equiv 0}$ for now

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To prove LWP, we want to use mild formulation to define the solution map

$$\Gamma_{u_0}(u)(t) = e^{tL} u_0 - \int_0^t e^{(t-t')L} \Pi \left((u \cdot \nabla) u \right)(t') \, dt'$$

and show that it is a contraction.

| Fujita-Kato Theory 60's (?)

$$\nabla \Gamma_{u_0}(u)(t) = \nabla e^{tL} u_0 - \int_0^t \nabla e^{(t-t')L} \Pi \left((u \cdot \nabla) u \right)(t') \, dt'$$

$$\| \Gamma u(t) \|_{L^3_x} \overset{\text{Prop.}}{\leq} c \|u_0\|_{L^3_x} + c \int_0^t (t-t')^{-1/2} \underbrace{\| (u \cdot \nabla) u (t') \|_{L^{3/2}_x}}_{\lesssim \|u(t')\|_{L^3_x} \|\nabla u\|_{L^3_x}} dt'$$

$$\leq c \|u_0\|_{L^3_x} + c \int_0^t (t-t')^{-1/2} (t')^{-1/2} dt' \; \|u\|_{L^\infty_t(0,t; L^3_x)} \sup_{t' \in (0,t)} (t')^{1/2} \|\nabla u(t')\|_{L^3_x} .$$

Beta function : $B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt$, $Re(p), Re(q) > 0$

$$\int_0^t (t-t')^{-1/2} (t')^{-1/2} dt' = \int_0^1 (1-\tau)^{-1/2} (\tau)^{-1/2} d\tau = B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) < \infty .$$

$$\|\nabla \Gamma u(t)\|_{L^3} \overset{\text{Prop. } q=3, p=2}{\leq} c\, t^{-1/2} \|u_0\|_{L^3} + \int_0^t (t-t')^{-3/4} \underbrace{\| (u \cdot \nabla) u(t') \|_{L^2_x}}_{\lesssim \underbrace{\|u(t')\|_{L^6_x}}_{\lesssim \| |D|^{1/2} u \|_{L^3_x} \lesssim \|u\|^{1/2}_{L^3_x} \|\nabla u\|^{1/2}_{L^3_x}} \|\nabla u(t')\|_{L^3_x}} dt'$$

$$t^{1/2} \|\nabla \Gamma u(t)\|_{L^3} \leq c \|u_0\|_{L^3} + c\, t^{1/2} \int_0^t (t-t')^{-3/4} (t')^{-3/4} dt' \left(\sup_{t'} \|u(t')\|_{L^3_x} \right)^{1/2} \cdot \left(\sup_{t'} (t')^{1/2} \|\nabla u(t')\|_{L^3} \right)^{3/2}$$

Define X by $\|u\|_X = \|u\|_{L^\infty_t([0,\infty); L^3_x)} + \sup_{t \in (0,\infty)} t^{1/2} \|\nabla u(t)\|_{L^3_x}$

$$\leq c \, \| u_0 \|_{L^3_x} + c \int_0^t (t-t')^{-1/2} (t')^{-1/2} \, dt' \cdot \| u \|_{L^\infty_t (0,t; L^3_x)} \cdot \sup_{t' \in (0,t)} (t')^{1/2} \| D u(t') \|_{L^3_x} .$$