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Lec 1 15/01/18 (Mon)

• Nonlinear dispersive PDEs

examples:

① Nonlinear Schrödinger equations (NLS)

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

$$\partial_t = \frac{\partial}{\partial t}, \quad \Delta = \sum_{j=1}^d \partial_j^2 = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$$

$$u: (t, x) \in \mathbb{R} \times \mathbb{R}^d \rightarrow u(t, x) \in \mathbb{C}, \quad p > 1$$

② Nonlinear wave equations (NLW)

$$-\partial_t^2 u + \Delta u = \pm |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d$$

u, \mathbb{R} -valued

③ generalized Korteweg-de Vries equations (gKdV)

$$\partial_t u + \partial_x^3 u = \pm \partial_x(u^p), \quad (t, x) \in \mathbb{R} \times \mathbb{R}$$

u, \mathbb{R} -valued, $p = 2$, KdV

$p = 3$, modified KdV (mKdV)

Questions :

(2)

- Well-posedness?

We say (NLS) is locally well-posed in $B^s(\mathbb{R}^d)$

if given any $u_0 \in B^s(\mathbb{R}^d)$, $\exists!$ soln u to (NLS)
on $[-T, T]$, $T = T(u_0) > 0$, with $u|_{t=0} = u_0$.

- For NLW, initial data $(u, \partial_t u)|_{t=0} = (u_0, u_1) \in B^s \times B^{s-1}$.
- If solutions exist for short times, then
 - Does the solution exist globally in time?
 \Rightarrow if so, we say (NLS) is globally well-posed
(if we can take $T = \infty \Leftarrow T \gg 1$ arbitrarily large)
 - Otherwise, the solution may cease to exist at some time.
 \Leftarrow finite time blowup (formation of singularity)

- If GWP holds, then behavior of global-in-time solns? ③
 - scattering: $u(t) = \text{soln to (NLS)}$
 behaves asymptotically (as $t \rightarrow \pm\infty$)
 like a linear solution w : $i\partial_t w + \Delta w = 0$
 (but with different data)
 $\Rightarrow \|u(t)\| \rightarrow 0$ in some appropriate norm.
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- In some situations (i.e. if "s is too small"), we have ill-posedness results for these equations

Back to LWP: existence of unique solutions for short times
 with stability (continuity of the soln map):

$$\Phi : u_0 \in B^s(\mathbb{R}^d) \longmapsto u \in C([-T, T]; B^s(\mathbb{R}^d))$$

is continuous for $T = T(u_0) > 0$ small

If ill-posed, then what is the nature of the ill-posedness? ④

- discontinuity of the soln map (often $u_0 = 0$)
 - non-uniqueness
 - non-existence
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- Sec 0: Background materials:

- Lebesgue spaces: $L^p(X)$, $X = \mathbb{R}^d$

$$L^p(X) = \left\{ f \text{ on } X : \|f\|_{L^p(X)} \stackrel{\text{def}}{=} \left(\int_X |f(x)|^p dx \right)^{1/p} < \infty \right\}$$

- $\|\cdot\|_{L^p}$ is a norm
- $L^p(X)$ is a normed vector space ($p \geq 1$)
- Also complete \Rightarrow Banach space
- $p=2$: Hilbert space (Banach & inner product)

$$\langle f, g \rangle_{L^2} = \int_X f \bar{g} dx. \quad (\text{or } \underline{\underline{\int_X f \bar{g} dx}})$$

• $(X, dx), (Y, dy)$, measure spaces

⑤

$$\|f\|_{L_y^q L_x^r} = \left\| \underbrace{\|\cdot\|}_{\text{on } X} \right\|_{L_x^r} \| \cdot \|_{L_y^q}$$

$$= \left(\int_Y \left(\int_X |f(x, y)|^r dx \right)^{q/r} dy \right)^{1/q}.$$

$L^q(Y; L^r(X))$

function of x

$$C(\mathbb{R}_+; L^p(X)) = \left\{ f(t, x) : t \mapsto \underbrace{f(t)}_{\text{This map is continuous}} \in L^p(X) \right\}$$

$$(\mathbb{R}, |\cdot|_{\mathbb{R}}) \mapsto (X, \|\cdot\|_{L^p(X)})$$

Key inequalities:

q is the Hölder conjugate of p

(6)

① Hölder inequality: $\frac{1}{p} + \frac{1}{q} = 1$.

$p = q = 2$
Cauchy-Schwarz

$$\|fg\|_{L^r(X)} \leq \|f\|_{L^p(X)} \|g\|_{L^q(X)}$$

In general, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$: $\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$.

② Interpolation of L^p -spaces (log-convexity of L^p -norms)

$$0 < p < q \leq \infty, \quad \theta \in [0, 1].$$

$$\|f\|_{L^r} \leq \|f\|_{L^p}^\theta \|f\|_{L^q}^{1-\theta}, \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

$\cdot \underline{p = \infty}:$

$$\|f\|_{L^\infty(X)} = \underset{x \in X}{\text{ess sup}} |f(x)|$$

$$p \leq r \leq q$$

③ Minkowski's integral inequality : $1 \leq p \leq q \leq \infty$. (7)

$$\left\| \|f\|_{L^p(X)} \right\|_{L^q(Y)} \leq \left\| \|f\|_{L^q(Y)} \right\|_{L^p(X)}$$

usual Minkowski's ineq : $\left\| \sum_{j \in \mathbb{Z}} f_j \right\|_{L^q(Y)} \leq \sum_{j \in \mathbb{Z}} \|f_j\|_{L^q(Y)}$
 < Take $L^p(X) = \ell'(\mathbb{Z})$

④ Young's inequality:

convolution of f and g on \mathbb{R}^d .

$$\begin{aligned} f * g(x) &= \int_{\mathbb{R}^d} f(x-y) g(y) dy \\ &= \int_{\mathbb{R}^d} f(y) g(x-y) dy. \end{aligned}$$

Young's : $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$. $1 \leq p, q, r \leq \infty$.

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

(8)

⑤ Riesz - Thorin interpolation theorem

$$1 \leq p_j, q_j \leq \infty, \quad j = 0, 1$$

T , linear operator defined on $L^{p_j}(R^d) \rightarrow L^{q_j}(R^n)$

s.t.

$$\|Tf\|_{L^{q_j}} \leq A_j \|f\|_{L^{p_j}}, \quad j = 0, 1.$$

Then,

$$\|Tf\|_{L^q} \leq A_0^\theta A_1^{1-\theta} \|f\|_{L^p}, \quad \theta \in [0, 1].$$

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$$

• Fourier transform on \mathbb{R}^d

(9)

f on \mathbb{R}^d . The Fourier transform of f is given by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^d$$

Here, $x \cdot \xi = \sum_{j=1}^d x_j \xi_j$.

• well defined for $f \in L^1(\mathbb{R}^d)$

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^d} |f(x)| dx = \|f\|_{L^1(\mathbb{R}^d)} < \infty.$$

• Schwartz class $\underset{\uparrow}{\mathcal{S}(\mathbb{R}^d)} = C^\infty(\mathbb{R}^d) + \text{fast decay}$
 "rapidly decreasing".

$$\|f\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty$$

• multi-index notation: $\alpha \in (\mathbb{N} \cup \{0\})^d$ $\alpha = (\alpha_1, \dots, \alpha_d)$

$$x \in \mathbb{R}^d, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$$

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} \quad \partial_j = \frac{\partial}{\partial x_j}, \quad |\alpha| = \sum_{j=1}^d \alpha_j$$

(10)

Inverse Fourier transform:

$$\mathcal{F}^{-1}(f)(x) = \check{f}(x)$$

$$\stackrel{\text{def}}{=} \hat{f}(-x) = \int_{\mathbb{R}^d} f(\xi) e^{\frac{2\pi i}{\lambda} x \cdot \xi} d\xi$$

(different definitions: $\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int f(x) e^{-ix \cdot \xi} dx$, $\check{f} = \frac{1}{(2\pi)^{d/2}} \hat{f}(\xi)$)

$$\hat{f}(\xi) = L \cdot \int \dots , \quad \check{f} = \frac{1}{(2\pi)^{d/2}} \hat{f}(\xi)$$

Basic properties:

- $\|f\|_{L^2} = \|\hat{f}\|_{L^2} = \|\check{f}\|_{L^2}$ Plancherel's identity

- $\mathcal{F}: L^2 \rightarrow L^2$, bijection
 $f \rightarrow \hat{f}$

- $(\hat{f})^\vee = f = (\check{f})^\wedge$

- Parseval $\int f(x) \overline{g(x)} dx = \int \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi$

$$\langle f, g \rangle_{L^2_x} = \langle \hat{f}, \hat{g} \rangle_{L^2_\xi}$$

Hausdorff - Young's inequality: $\frac{1}{p} + \frac{1}{p'} = 1$, $p \geq 2$ (11)

$$\|\hat{f}\|_{L^p} \leq \|f\|_{L^{p'}}$$

$$\Leftarrow \begin{aligned} \|\hat{f}\|_{L^\infty} &\leq \|f\|_{L^1} \\ \|\hat{f}\|_{L^2} &= \|f\|_{L^2} \end{aligned} \quad \& \text{ Riesz - Thorin interpolation}$$

Lec 2 17 / 01 / 18 (Wed)

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- Folland, Real Analysis (Chap 6, 8, 9)
- Grafakos, Classical Fourier Analysis (Chap 1, 2, 5)
Modern = (Chap 6)
- Tao, Nonlinear Dispersive PDEs (Appendix)
- Cazenave, Semilinear Schrödinger equations
- Linares - Ponce, Intro to dispersive PDEs.

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- $(\partial^\alpha f)^\wedge(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$
 - $(\partial^\alpha \hat{f})^\vee(x) = (-2\pi i x)^\alpha f(x)$
 - $\widehat{f * g}(\xi) = \hat{f}(\xi) \hat{g}(\xi), \quad \hat{f} * \hat{g} = \widehat{fg}$
 - Let $f_\varepsilon(x) = \frac{1}{\varepsilon^d} f(\frac{x}{\varepsilon}) \Rightarrow \hat{f}_\varepsilon(\xi) = \hat{f}(\varepsilon \xi)$
 $\hat{g}^\varepsilon(x) = g(\frac{x}{\varepsilon}) \Rightarrow \widehat{\hat{g}^\varepsilon}(\xi) = \varepsilon^d \hat{g}(\varepsilon \xi)$

(2)

Riemann - Lebesgue Lemma

$$f \in L^1(\mathbb{R}^d) \Rightarrow \hat{f}(\xi) \rightarrow 0 \text{ as } |\xi| \rightarrow \infty$$

- $\mathcal{F}L^1(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$

\uparrow decaying to 0 at ∞ .

C_c = continuous functions with cpt support.

- $(\mathcal{S}(\mathbb{R}^d))^* = \mathcal{S}'(\mathbb{R}^d) = \text{tempered distributions}$

$T \in \mathcal{S}'(\mathbb{R}^d) : \mathcal{S} \rightarrow \mathbb{C}, \text{ lin \& conti.}$

- Topology on \mathcal{S}' : $T_k \rightarrow T$ in \mathcal{S}'

if $T_k(f) \rightarrow T(f), \forall f \in \mathcal{S}$

weak-* topology

- can define F.T. on \mathcal{S}' .

$$f \in \mathcal{S}', g \in \mathcal{S} \rightarrow \hat{f}(g) \stackrel{\mathcal{S}'}{\leftarrow} " = \langle \hat{f}, g \rangle" \stackrel{\text{def}}{=} f(\hat{g})$$

- Sobolev space $H^s(\mathbb{R}^d)$ = completion of \mathcal{S} w.r.t. ③

$$\|f\|_{H^s} = \left(\int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}, \quad s \in \mathbb{R}.$$

$$= \|\langle \xi \rangle^s \hat{f}(\xi)\|_{L_\xi^2(\mathbb{R}^d)}$$

$$\langle \xi \rangle = (1+|\xi|^2)^{1/2}$$

$$\cdot H^0 = L^2$$

↑ Plancherel

upto const

$$\cdot H^1 = L^2 \cap \{\partial f \in L^2\} \Leftarrow \text{recall } |\xi|^2 |\hat{f}(\xi)|^2 \stackrel{\downarrow}{=} |\nabla \hat{f}(\xi)|^2$$

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2.$$

$$\cdot \text{general } s > 0: H^s = f \in L^2 \text{ and derivatives up to order } s$$

$\langle \nabla \rangle^s f$ lies in L^2 are also in L^2 .

↑ fractional integration of order $-s$

Sobolev embedding theorem: If $s > \frac{d}{2}$ ($2s > d$), (4)

$$H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d) \quad (\text{and conti})$$

$$\Leftrightarrow \|f\|_{L^\infty} \leq C \|f\|_{H^s}, \quad \forall f \in H^s.$$

Pf: $\|f\|_{L^\infty_x} \leq \|\hat{f}\|_{L^1_{\xi}} = \int_{\mathbb{R}^d} |\hat{f}(\xi)| d\xi$

$$\int \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \stackrel{\text{Cauchy-Schwarz}}{\leq} \underbrace{\left(\int \frac{1}{|\xi|^{2s}} d\xi \right)^{1/2}}_{\leq C < \infty \text{ if } s > \frac{d}{2}} \|f\|_{H^s}$$

□

- Notations:
- $A \lesssim B$ if $A \leq CB$ for some $C > 0$
 - $A \sim B$ if $A \lesssim B$ and $B \lesssim A$
 - $A \ll B$ if $A \leq \varepsilon B$ for some small $\varepsilon > 0$

(5)

• Other Sobolev spaces

① L_s^p = Bessel potential space

$$\|f\|_{L_s^p} = \|\mathcal{F}^{-1}(\langle \xi \rangle^s \hat{f}(\xi))\|_{L_x^p}$$

Sobolev embedding holds
for $p > s > d$

$$p=2: L_s^2 = H^s$$

② $W^{s,p}$, $0 < s < 1$

$$\|f\|_{W^{s,p}} = \|f\|_{L^p} + \left(\iint \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} dx dy \right)^{\frac{1}{p}}$$

Stein's book: singular integrals...

$$N_s^p$$

③ H^s = inhomogeneous Sobolev space $\sim (1+|\xi|^2)^{s/2}$

\dot{H}^s = homogeneous Sobolev space $\sim |\xi|^s$

ex: $\dot{H}^1 = \{\nabla f \in L^2\}$ (no need to have $f \in L^2$)

$s \geq 0: H^s \subset \dot{H}^s$

\dot{L}_s^p = Riesz potential space

Rmk: \dot{H}^s up to polynomial ($s \geq 0$) $\|\langle \xi \rangle^s \hat{f}(\xi)\|_{L_x^2} = 0$

if $\text{supp } \hat{f} = \{0\} \rightarrow \hat{f} = \sum j^\alpha f_\alpha$

(6)

• Sobolev inequality $0 \leq \frac{s}{d} = \frac{1}{p} - \frac{1}{q}, \quad 1 < p \leq q < \infty$

$$\|f\|_{L^q} \leq C \|f\|_{L_s^{s,p}}$$

$\overset{\circ}{W}{}^{s,p}$

"controlling higher order integrability by the L^p -norm
of the s -derivative of f ".

• Convention: We use $W^{s,p}$ for L_s^p

i.e. for us, $\|f\|_{W^{s,p}} = \|\mathcal{F}^{-1}(\langle \xi \rangle^s \hat{f}(\xi))\|_{L_x^p}$

• Interpolation: $s = \theta s_1 + (1-\theta)s_2 \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$
 $\theta \in (0, 1)$

$$\|f\|_{H^s} \leq \|f\|_{H^{s_1}}^\theta \|f\|_{H^{s_2}}^{1-\theta}$$

Algebra property of H^s , $s > d/2$

We proved

(7)

$$f, g \in H^s \Rightarrow fg \in H^s$$

$$H^s \subset \mathcal{F}L^1 \subset C_0$$

\leftarrow Riemann-Lebesgue

$$(\Leftarrow \quad \underline{\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}}$$

Pf: $\langle \xi \rangle^s \lesssim \langle \xi + \xi_1 \rangle^s + \langle \xi_1 \rangle^s \Leftarrow \text{triangle ineq \&}$

$$\|fg\|_{H^s} = \|\langle \xi \rangle^s \widehat{fg}(\xi)\|_{L^2_\xi}$$

$$(a+b)^\theta \leq \begin{cases} a^\theta + b^\theta, & \theta \in (0, 1] \\ ca^\theta + cb^\theta, & \theta > 1 \end{cases}$$

$$= \widehat{f} * \widehat{g}(\xi) = \int \widehat{f}(\xi - \xi_1) \widehat{g}(\xi_1) d\xi_1$$

$$\lesssim \left\| \int (\langle \xi - \xi_1 \rangle^s |\widehat{f}(\xi - \xi_1)| |\widehat{g}(\xi_1)| d\xi_1 \right\|_{L^2_\xi}$$

$$+ \left\| \int |\widehat{f}(\xi - \xi_1)| (\langle \xi_1 \rangle^s |\widehat{g}(\xi_1)|) d\xi_1 \right\|_{L^2_\xi}$$

$$= \left\| (\langle \cdot \rangle^s |\widehat{f}|) * |\widehat{g}| \right\|_{L^2_\xi} + \left\| |\widehat{f}| * (\langle \cdot \rangle^s |\widehat{g}|) \right\|_{L^2_\xi}$$

Young's

$$\leq \left\| \langle \cdot \rangle^s \widehat{f} \right\|_{L^2_\xi} \|\widehat{g}\|_{L^1_\xi} + \|\widehat{f}\|_{L^1_\xi} \left\| \langle \cdot \rangle^s \widehat{g} \right\|_{L^2_\xi}$$

from the proof
of Sobolev embedding

$$\lesssim \|f\|_{H^s} \|g\|_{H^s} \quad \square$$

Lec 3 22/01/18 (Mon)

①

sec 1: LWP of NLS in $H^s(\mathbb{R}^d)$, $s > d/2$

$$(NLS) \quad i\partial_t u + \Delta u = \pm |u|^{p-1}u$$

First, consider the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

⇒ Take F.T. in x

$$\begin{cases} i\partial_t \hat{u}(\xi) - |\xi|^2 \hat{u}(\xi) = 0 \\ \hat{u}(t, \xi)|_{t=0} = \hat{u}_0(\xi) \end{cases}$$

$$\begin{aligned} \Delta &= \partial_1^2 + \cdots + \partial_d^2 \xrightarrow{\text{F.T.}} \hat{\Delta}(\xi) = -4\pi^2 (\xi_1^2 + \cdots + \xi_d^2) \\ &\downarrow \\ (2\pi i \xi_1)^2 &= -4\pi^2 |\xi|^2. \end{aligned}$$

omitted for simplicity

$$\text{For fixed } \xi, \quad \hat{u}(t, \xi) = e^{-it|\xi|^2} \hat{u}_0(\xi) \quad \textcircled{2}$$

$$\Rightarrow u(t, x) = \mathcal{F}^{-1}(e^{-it|\xi|^2} \hat{u}_0(\xi))(x)$$

$$=: (e^{it\Delta} u_0)(x) \quad \partial_t u = i\Delta u$$

• $S(t) = e^{it\Delta}$ = linear Schrödinger operator

Next, we consider the nonhomogeneous lin. Schrödinger eqn

$$\begin{cases} i\partial_t u + \Delta u = F(t, x), & F \text{ given (and "nice")} \\ u|_{t=0} = u_0 \end{cases}$$

F.T. in x

$$\Rightarrow \partial_t \hat{u}(\xi) + i|\xi|^2 \hat{u}(\xi) = -i \hat{F}(t, \xi), \quad \forall \xi \in \mathbb{R}^d.$$

$$\partial_t (e^{it|\xi|^2} \hat{u}(\xi)) = -i e^{it|\xi|^2} \hat{F}(t, \xi)$$

integrating factor

Integrate from 0 to t.

③

$$e^{it|\vec{z}|^2} \hat{u}(t, \vec{z}) - \hat{u}_0(\vec{z}) = -i \int_0^t e^{it'|\vec{z}|^2} \hat{F}(t', \vec{z}) dt'.$$

$$\Rightarrow \hat{u}(t, \vec{z}) = \underbrace{e^{-it|\vec{z}|^2} \hat{u}_0(\vec{z})}_{\text{given}} - i \int_0^t \underbrace{e^{-i(t-t')|\vec{z}|^2} \hat{F}(t', \vec{z})}_{\text{unknown}} dt'.$$

Take \mathcal{F}^{-1}

$$\Rightarrow u(t) = \underbrace{s(t) u_0}_{\text{given}} - i \int_0^t s(t-t') F(t') dt'$$

- Back to NLS with nonlinearity $N(u, \bar{u}) = u^{p_1} \bar{u}^{p_2}$

$$p_1 + p_2 = p$$

We say u is a soln to (NLS) if $p_j \in \mathbb{N} \cup \{0\}$.

u satisfies the following Duhamel formulation:

$$\begin{aligned} u(t) &= s(t) \underbrace{u_0}_{\text{given}} - i \int_0^t s(t-t') \underbrace{N(u, \bar{u})(t')}_{\text{unknown}} dt' \\ &=: \Gamma_{u_0}(u)(t) \end{aligned}$$

(4)

Goal: Given $u_0 \in H^s(\mathbb{R}^d)$, show that $\exists! u$

$$\text{s.t. } \underline{u} = \Gamma_{u_0}(\underline{u}) \text{ on } [-T, T], \quad T = T(u_0) > 0.$$

i.e. u is a fixed of Γ_{u_0} .

- Basic properties of $S(t)$

① $S(t)$ is unitary on $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$

$$\begin{aligned} \|S(t)f\|_{H^s} &= \left(\int \langle \xi \rangle^{2s} |e^{\cancel{i}t|\xi|^2} \hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &= \|f\|_{H^s} \end{aligned}$$

② $S(t)f \in C(\mathbb{R}_+; H_x^s(\mathbb{R}^d))$, $f \in H^s$

$$S(\cdot)f : t \mapsto S(t)f \in H^s$$

↑ H^s -valued function (in t)

(5)

Fix $t \in \mathbb{R}$.

Semigroup property

$$\| S(t+h)f - S(t)f \|_{H^s} \quad S(t_1 + t_2) = S(t_1)S(t_2)$$

$$= \| S(\cancel{t}) (S(h) - 1)f \|_{H^s}$$

← write down on the Fourier side

separate $\cdot |\tilde{\beta}| > N$ s.t. $\| \underbrace{P_{|\tilde{\beta}| > N} f}_{\mathcal{F}^{-1}(1_{|\tilde{\beta}| > N} \hat{f}(\tilde{\beta}))} \|_{H^s} < \frac{\varepsilon}{4}$

$$\cdot |\tilde{\beta}| \leq N, \text{ use mean value theorem}$$

$$|(S(h) - 1)^n(\tilde{\beta})| = |e^{-ih|\tilde{\beta}|^2} - 1|$$

$$\leq N^2 |h|.$$

- Fix $u_0 \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$ (6)

$$\| P_{u_0}(u) \|_{C_T H^s} \leq \| S(t) u_0 \|_{C_T H^s} + \left\| \int_0^t S(t-t') N(u, \bar{u})(t') dt' \right\|_{C_T H^s}$$

$C_T H^s = C([-T, T]; H^s(\mathbb{R}^d))$

$$\| u \|_{C_T H^s} = \| \| u(t) \|_{H^s} \|_{L_T^\infty} \rightarrow L^\infty([-T, T])$$

$$\leq \| u_0 \|_{H^s} + \int_0^T \| N(u, \bar{u}) \|_{C_T H^s} dt.$$

↑ unitarity of $S(t)$ & Minkowski's integral ineq. indep of t

$$\leq \| u_0 \|_{H^s} + T \| u^P \bar{u}^{P_2} \|_{C_T H^s}$$

$$\leq \| u_0 \|_{H^s} + CT \| u \|_{C_T H^s}^{P_1} \| \bar{u} \|_{C_T H^s}^{P_2} \stackrel{\text{WANT}}{\leq} 2 \| u_0 \|_{H^s} = : R$$

$$= \| u \|_{C_T H^s}^P$$

For $u \in \overline{B}_R =$ closed ball of radius R in $C_T H^s$

(7)

$$\begin{aligned} \Rightarrow \|\Gamma_{u_0}(u)\|_{C_T H^s} &\leq \|u_0\|_{H^s} + C T R^p \\ &\leq \frac{R}{2} + \underbrace{(2CTR^{p-1})}_{\leq 1 \text{ by choosing } T \leq (2CR^{p-1})^{-1}} \cdot \frac{R}{2} = R \end{aligned}$$

$$\Rightarrow \Gamma_{u_0} : \overline{B}_R \hookrightarrow$$

$$u, v \in \overline{B}_R \quad \|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{C_T H^s} \leq \int_0^T \|N(u) - N(v)\|_{C_T H^s} dt.$$

$$\begin{aligned} u^{p_1} \bar{u}^{p_2} - v^{p_1} v^{p_2} &= \cancel{(u-v)} u^{p_1-1} \bar{u}^{p_2} \\ &\quad + v \cancel{(u-v)} u^{p_1-2} \bar{u}^{p_2} \\ &\quad + \dots + v^{p_1} \bar{v}^{p_2-1} \cancel{(u-\bar{v})} \end{aligned}$$

Telescoping sum

(8)

$$\leq \tilde{C}T \left(\sum_{j=0}^{p-1} \|u\|_{CTH^s}^{p-1-j} \|v\|_{CTH^s}^j \right) \|u-v\|_{CTH^s}.$$

Young's ineq: $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$
 $j=1$: $\|u\|_{CTH^s}^{p-2} \|v\|_{CTH^s} \leq C \left(\|u\|_{CTH^s}^{\frac{(p-2)p}{p-1}} + \|v\|_{CTH^s}^{\frac{p-1}{p-1}} \right)^{\frac{p-2}{p-1} + \frac{1}{p-1}} = 1$
 $\leq \tilde{C}T \left(\|u\|_{CTH^s}^{p-1} + \|v\|_{CTH^s}^{p-1} \right) \|u-v\|_{CTH^s}$
 $\leq \underbrace{\tilde{C}T R^{p-1}}_{< 1} \|u-v\|_{CTH^s}$
 by choosing $T = \frac{1}{2\tilde{C}} R^{p-1}$

(9)

Summary: $\forall u, v \in \overline{B_R}$

$$\| \Gamma_{u_0}(u) \|_{C^s} \leq R$$

$$\| \Gamma_{u_0}(u) - \Gamma_{u_0}(v) \|_{C^s} \leq \frac{1}{2} \| u - v \|_{C^s},$$

by choosing $T = \min\left(\frac{1}{2CR^{p-1}}, \frac{1}{2\tilde{C}R^{p-1}}\right)$

$$\text{Recall that } R = 2 \| u_0 \|_{H^s}$$

$$\Rightarrow T = T(\| u_0 \|_{H^s}) > 0$$

• Banach fixed pt thm (Contraction mapping principle)
 A contraction on a closed ball in a complete metric space has a unique fixed pt.

$$T: \overline{B_R} \hookleftarrow \text{ and } \| T(u) - T(v) \| \leq \theta \| u - v \| \quad \theta < 1$$

(10)

$$\Rightarrow \exists! u \in \overline{B_R} \text{ s.t. } u = \Gamma_{u_0}(u)$$

Namely, u is a soln to (NLS).

Remarks: ① $u \in C_T H^s$ (\Leftarrow requirement for LWP)

$$u(t) = \underbrace{S(t)u_0}_{\text{in } C_T H^s} - i \int_0^t S(t-t') N(u, \bar{u})(t') dt'$$

$$\begin{aligned} G(t+h) - G(t) &= \int_0^{t+h} S(t+h-t') \cdots dt' - \int_0^t S(t-t') \cdots dt' \\ &= \underbrace{\int_t^{t+h} S(t+h-t') \cdots dt'}_{\text{use shortness of } [t, t+h]} - \underbrace{\int_0^t (S(t+h-t') - S(t-t')) \cdots dt'}_{= S(t-t')(S(h)-1)} \end{aligned}$$

② At this point, u is unique only in $\overline{B_R} \subset C_T H^s$.

Show u is unique in $C_T H^s$ (possibly by shrinking $T = T(R)$ a bit.)

OR (i) Gronwall's ineq.

(ii) Bootstrap argument.

- No lecture on 29/01/18 (Mon)

Sec 2: Conservation laws, global well-posedness
and persistence of regularity

$$(NLS) \quad i\partial_t u + \Delta u = \pm |u|^{p-1}u. \quad \Rightarrow \partial_t u = i\Delta u \mp i|u|^{p-1}u$$

- Conservation laws

Mass : $M(u) = \int_{\mathbb{R}^d} |u(x)|^2 dx$

If u is a soln to (NLS), then $M(u)(t) = M(u_0)$

$$\begin{aligned} \partial_t \int |u|^2 &= 2 \operatorname{Re} \int (\partial_t u) \bar{u} = 2 \operatorname{Re} i \int (\Delta u) \bar{u} \mp 2 \operatorname{Re} i \underbrace{\int |u|^{p+1}}_{=0} \\ &= -2 \operatorname{Re} i \int |\nabla u|^2 = 0. \end{aligned}$$

- For "rough" solns, we use well-posedness theory (continuous dependence) to prove the conservation of mass.

Given $u_0 \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$, take $\{u_{0,n}\}_{n=1}^\infty \subset H^\infty(\mathbb{R}^d)$

(2)

s.t. $u_{0,n} \rightarrow u_0$ in $H^s(\mathbb{R}^d)$.

Denote by u_n , the soln to (NLS) with $u_n|_{t=0} = u_{0,n}$, i.e.

$$u_n(t) = S(t)u_{0,n} + i \int_0^t S(t-t') |u_n|^{p-1} u_n(t') dt'.$$

$$\bullet \quad u(t) = S(t)u_0 + i \int_0^t S(t-t') |u|^{p-1} u(t') dt'.$$

- For $n \gg 1$, we have $\|u_{0,n}\|_{H^s} \leq \|u_0\|_{H^s} + 1 =: R/2$
 $\xrightarrow{\text{LWP}}$ u_n and u exist on $[-T, T]$, $T = T(R) > 0$.

Also,

$$\begin{aligned} \|u_n - u\|_{CTH^s} &\leq \|u_{0,n} - u_0\|_{H^s} + \underbrace{CTR^{p-1} \|u_n - u\|}_{CTH^s} \\ &\leq \frac{1}{2} \text{ by choosing } T = T(R) \text{ small} \end{aligned}$$

$$\Rightarrow \|u_n - u\|_{CTH^s} \lesssim \|u_{0,n} - u_0\|_{H^s}$$

(3)

\Rightarrow continuous dependence, i.e.

$\Phi = \text{soln map} : u_0 \in H^s \xrightarrow{\Phi(u_0)} u \in C_T H^s$
is continuous.

- Conservation of $M(u_n)$ and $u_n \rightarrow u$ in $C_T L^2$ implies the conservation of $M(u)$.

- Hamiltonian (= energy) : $H(u) = \frac{1}{2} \int |\nabla u|^2 dx \pm \frac{1}{p+1} \int |u|^{p+1} dx$
 - + = defocusing case (repulsive)
 - = focusing case (attractive)
- $H(u(t)) = H(u_0)$.
- Momentum : $P(u) = \text{Im} \int u \nabla \bar{u} dx \stackrel{\text{IBP}}{=} -i \int u \nabla \bar{u} \in \mathbb{C}^d$.
- $P(u)(t) = P(u_0)$

• Consider the 1-d defocusing NLS, $s > \frac{1}{2}$. (4)

Let $u_0 \in H^s(\mathbb{R})$.

$$\Rightarrow \|u(t)\|_{H^1}^2 = \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 < \infty \text{ by Sobolev}$$

$$\lesssim M(u)|t) + H(u(t))$$

$$\stackrel{\text{cons}}{=} M(u_0) + H(u_0) =: C^2(u_0) < \infty$$

$$\|f\|_{L^q} \lesssim \|u\|_{H^s}^{-\frac{1}{2}}$$

$$\begin{cases} \frac{s}{d} = \frac{1}{2} - \frac{1}{q}, q < \infty \\ \frac{s}{d} > \frac{1}{2}, q = \infty \end{cases}$$

\Rightarrow Set $R = 2C(u_0)$ ($\geq 2\|u(t)\|_{H^1}$) and repeat the LWP argument.

$$\frac{\|u(T)\|_{H^1} \leq R/2}{\overbrace{0 \rightarrow T \sim R^{1-p}}} \rightarrow \|u(jT)\|_{H^1} \leq R/2, \forall j \in \mathbb{Z}.$$

$\Rightarrow u$ exists globally in time.

Thm 1: $p \in 2N+1$. Then, 1-d def. NLS is globally well-posed
in $H^s(\mathbb{R})$.

\checkmark
 $H^s(\mathbb{R}), s \geq 1$.

(5)

• Persistence of regularity: If $u_0 \in H^s(\mathbb{R})$, $s > 1$,
then $u_0 \in H^1(\mathbb{R}) \Rightarrow u \in C(\mathbb{R}; H^1(\mathbb{R}))$

Q: $u \in C(\mathbb{R}; H^s(\mathbb{R}))$?

Yes: product estimate: $s \geq 0$

$$(*) \quad \|fg\|_{H^s} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}$$

(\Leftarrow follows from the para product formula.

$$\begin{aligned} d=1: \quad \| |u|^{p-1} u \|_{H^s} &\lesssim \|u\|_{L^\infty}^{p-1} \|u\|_{H^s} \lesssim \|u\|_{H^1}^{p-1} \|u\|_{H^s} \\ &\lesssim R^{p-1} \|u\|_{H^s} \end{aligned}$$

$$\Rightarrow \|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + CR^{p-1} \int_0^t \|u(t')\|_{H^s} dt'$$

Gronwall

$$\Rightarrow \|u(t)\|_{H^s} \leq e^{CRt} \|u_0\|_{H^s} < \infty \quad (**)$$

(6)

$$\Rightarrow u(t) \in H^s(\mathbb{R})$$

& By LWP, $u \in C(\mathbb{R}; H^s(\mathbb{R}))$.

Rmk: ~~**~~ provides an exponential upper bound on the growth of a soln.

- It is of importance to improve this bound.

Bourgain, Staffilani, mid 90's: $\|u(t)\|_{H^s} \leq C(u_0) t^{\alpha(s-1)}$

Q: subpolynomial upper bound?

Q: Can we construct a soln s.t. the H^s -norm actually grows?

- $d=2$: defocusing cubic NLS ($p=3$)

(7)

- LWP in $H^s(\mathbb{R}^2)$, $s > 1$.

Thm 2: The 2-d def cubic NLS is globally well-posed in $H^2(\mathbb{R}^2)$.

- Brezis-Gallouët inequality '80 : $s > d/2$

$$\|f\|_{L^\infty} \leq c_s \|f\|_{H^{d/2}} \left[\log \left(2 + \frac{\|f\|_{H^s}}{\|f\|_{H^{d/2}}} \right) \right]^{1/2}$$

(Sobolev's embedding thm : $\|f\|_{L^\infty} \lesssim \|f\|_{H^s}$, $s > d/2$)

Pf: $g = f / \|f\|_{H^{d/2}}$. $\Rightarrow \|g\|_{H^{d/2}} = 1$.

WTS : $\|g\|_{L^\infty} \leq c_s \left(\log \left(2 + \|g\|_{H^s} \right) \right)^{1/2}$.

$$\|g\|_{L^\infty} \leq \|\hat{g}\|_{L^1} = \int_{|\tilde{z}| \leq R} |\hat{g}(\tilde{z})| d\tilde{z} + \int_{|\tilde{z}| > R} |\hat{g}(\tilde{z})| d\tilde{z} \quad (8)$$

$$= \int_{|\tilde{z}| \leq R} \langle \tilde{z} \rangle^{d/2} |\hat{g}(\tilde{z})| \frac{1}{\langle \tilde{z} \rangle^{d/2}} d\tilde{z} + \int_{|\tilde{z}| > R} \langle \tilde{z} \rangle^s |\hat{g}(\tilde{z})| \frac{1}{\langle \tilde{z} \rangle^s} d\tilde{z}.$$

$$\stackrel{C-S}{\leq} \underbrace{\|g\|_{H^{d/2}}}_{\| \cdot \|_1} \left(\log(2+R) \right)^{1/2} + \|g\|_{H^s} \underbrace{\left(\int_{|\tilde{z}| > R} \frac{1}{\langle \tilde{z} \rangle^{2s}} d\tilde{z} \right)^{1/2}}_{\sim R^{d/2-s}}$$

$$\text{Set } R = \|g\|_{H^s}^\theta ; \quad \theta = \frac{1}{s-d/2} > 0$$

$$\lesssim \left(\log(2 + \|g\|_{H^s}) \right)^{1/2}$$

□

$$\|u(t)\|_{H^2} \stackrel{*}{\leq} \|u_0\|_{H^2} + C \int_0^t \|u(t')\|_{H^2} \|u(t')\|_{L^\infty}^2 dt'. \quad (9)$$

(***)

$$\begin{aligned} &\stackrel{B-G}{\leq} \|u_0\|_{H^2} + C \int_0^t \|u(t')\|_{H^2} \left(1 + \log(1 + \|u(t')\|_{H^2}) \right) dt' \\ &=: F(t). \end{aligned}$$

depends on $\|u(t)\|_{H^1}$ appearing in B-G meg.

$$\Rightarrow F'(t) = \|u(t)\|_{H^2} (1 + \log(1 + \|u(t)\|_{H^2}))$$

$$\leq F(t) (1 + \log(1 + F(t))).$$

$$\Rightarrow \frac{d}{dt} \log(1 + \log(1 + F(t))) \leq C$$

$$\Rightarrow \|u(t)\|_{H^2} \leq F(t) \leq e^{e^{ct}} C(u_0)$$

\Rightarrow Thm 2.

Sec 3: Scaling heuristics

- Scaling symmetry (dilation symmetry)

If u is a soln to (NLS) with $u|_{t=0} = u_0$,

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x^d,$$

then set $u^\lambda(t, x) = \frac{1}{\lambda^a} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$

$$u_0^\lambda(x) = \frac{1}{\lambda^a} u_0\left(\frac{x}{\lambda}\right)$$

$$\Rightarrow a+2 = \alpha p \Rightarrow a = \frac{2}{p-1}$$

$$\Rightarrow u^\lambda(t, x) = \frac{1}{\lambda^{2/p-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \text{ is also a soln to (NLS)}$$

with the scaled initial data $u^\lambda|_{t=0} = u_0^\lambda$.

(2)

This scaling symmetry induces the so-called scaling-critical Sobolev index $s_c = s_{\text{crit.}}$

$$\|f^\lambda\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|f\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$$

- $\|f^\lambda\|_{\dot{H}^s} = \left(\int |\xi|^{2s} |\hat{f}^\lambda(\xi)|^2 d\xi \right)^{1/2}$

$$\left(f^\lambda(x) = \frac{1}{\lambda^{2/p-1}} f\left(\frac{x}{\lambda}\right) \Rightarrow \widehat{f^\lambda}(\xi) = \lambda^{d-\frac{2}{p-1}} \widehat{f}(\lambda \xi) \right)$$

$$= \left(\int |\lambda \xi|^{2s} |\widehat{f}(\lambda \xi)|^2 d(\lambda^d \xi) \right)^{1/2} \lambda^{d - \frac{2}{p-1} - s - \frac{d}{2}}$$

$$= \lambda^{\frac{d}{2} - \frac{2}{p-1} - s} \|f\|_{\dot{H}^s}, \quad \forall \lambda > 0.$$

$$\Rightarrow \boxed{s_c = \frac{d}{2} - \frac{2}{p-1}} \quad (< \frac{d}{2})$$

Given $u_0 \in H^s(\mathbb{R}^d)$, the Cauchy problem (NLS) is ③

- subcritical (w.r.t. scaling) if $s > s_c = s_c(d, p)$
⇒ expect good behavior, LWP, etc.

- critical if $s = s_c$

delicate balance between

lin. dispersion and nonlinear concentration

- supercritical if $s < s_c$

⇒ expect ill-posedness

• subcritical case: $\|u_0^\lambda\|_{\dot{H}^s} = \lambda^{\frac{s_c - s}{2}} \|u_0\|_{\dot{H}^s}$

$$u \text{ on } [0, T] \longleftrightarrow u^\lambda \text{ on } [0, \lambda^2 T] \quad \lambda \gg 1$$

$$u_0 \gg u_0^\lambda \text{ (in } \dot{H}^s \text{)}$$

small data ⇒ soln lives longer.

(4)

• supercritical case

$$u \text{ on } [0, T] \longleftrightarrow u^2 \text{ on } [0, \lambda^2 T]$$

$$u_0 \ll u_0^2 \text{ (in } H^s\text{)}$$

\Rightarrow larger initial data , longer time of existence

\Leftarrow Too good to be true.

• critical case: $s = s_c$

need more info than the H^s -norm of initial data.

Other symmetries: time translation: $t \rightarrow t + t_0$

spatial translation: $x \rightarrow x + x_0$

$u \mapsto e^{i\theta} u$, $\theta \in \mathbb{R}$ $\xrightarrow{\text{Noether}}$ mass conservation

Galilean symmetry: $u(t, x) \mapsto e^{i\frac{V}{2} \cdot x} e^{-i\frac{|V|^2}{4}t} u(t, x + vt)$

time reversal: $u(t, x) \mapsto \overline{u(-t, x)}$

induces another critical regularity $s_c^\infty = 0$.

- If we prove LWP by a fixed pt argument
(contraction)

(5)

for $\|u\|^{p+1} u$, $p \in 2N + 1$ (algebraic), then

$$\Phi : u_0 \in H^s \mapsto u \in C_T H^s$$

is analytic.

So, if we know that the soln map is not smooth, then this means that we can not prove LWP by a fixed pt argument. (but does not say that it's ill-posed.)

- We will discuss more about ill-posedness later in the course.

(6)

Sec 4 : Strichartz estimate

① dispersive estimate

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases} \Rightarrow u(t) = \mathcal{F}^{-1}(e^{-4\pi^2 i t |\xi|^2} \hat{u}_0(\xi)) = S(t) u_0$$

$$\begin{aligned} \Rightarrow S(t) u_0(x) &= \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} e^{-4\pi^2 i t |\xi|^2} \hat{u}_0(\xi) d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot (x - 2\pi \xi t)} \hat{u}_0(\xi) d\xi \end{aligned}$$

If \hat{u}_0 is "localized" around $\xi = \xi_0 \in \mathbb{R}^d$,
 $\Rightarrow 2\pi \xi (\sim 2\pi \xi_0) = \text{phase velocity.}$

Consider $u_0 = e^{2\pi i x \cdot \xi_0}$ ← plane wave at freq ξ_0 . (7)

$$\Rightarrow s(t) u_0(x) = e^{2\pi i x \cdot \xi_0 - 4\pi^2 i t |\xi_0|^2} \quad \left(\begin{array}{l} \text{As a distribution,} \\ \mathcal{F}(e^{2\pi i x \cdot \xi_0})(\xi) = \delta_{\xi_0}(\xi) \end{array} \right)$$

$$= e^{2\pi i \xi_0 \cdot \frac{(x - 2\pi \xi_0 t)}{\cancel{t}}} \quad \begin{matrix} \text{phase velocity} = \text{velocity for the propagation} \\ \text{of oscillation} (= \text{how phase changes}) \end{matrix}$$

Next, let's consider the following spatially localized wave:

$$v_0 = \underbrace{e^{-\frac{|x|^2}{4\sigma^2}}}_{\text{spatial localization}} e^{2\pi i x \cdot \xi_0}$$

$$\Rightarrow \hat{v}_0(\xi) = (4\pi\sigma^2)^{d/2} e^{-4\pi^2 \sigma^2 |\xi - \xi_0|^2}$$

$\left/ \begin{array}{l} \Leftarrow \text{follows from FACT:} \\ \text{① } g(x) = e^{-\pi|x|^2} \Rightarrow \hat{g}(\xi) = e^{-\pi|\xi|^2} \\ \text{② } g_\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \Rightarrow \hat{g}_\varepsilon(\xi) = \varepsilon^d \hat{g}(\varepsilon \xi) \end{array} \right.$

Pf of ① : Let $F(s) = \int_{\mathbb{R}} e^{-\pi(x+is)^2} dx, s \in \mathbb{R}$ ⑧

$$\Rightarrow \frac{d}{ds} F(s) = \int_{\mathbb{R}} -2\pi i (x+is) e^{-\pi(x+is)^2} dx \\ = \int_{\mathbb{R}} i \frac{d}{dx} (e^{-\pi(x+is)^2}) dx = 0 \Rightarrow F(s) \equiv \text{const.}$$

$$\bullet \mathcal{F}(e^{-\pi|x|^2})(\vec{\gamma}) = \int_{\mathbb{R}^d} e^{-\pi|x|^2} e^{-2\pi i x \cdot \vec{\gamma}} dx = \prod_{j=1}^d \int_{\mathbb{R}} e^{-\pi(y_j + i\vec{\gamma}_j)^2} e^{\pi(i\vec{\gamma}_j)^2} dy_j \\ = \underbrace{\left(\int_{\mathbb{R}} e^{-\pi y^2} dy \right)^d}_{=1} e^{-\pi|\vec{\gamma}|^2}$$

$$S(t) V_0 = (4\pi \sigma^2)^{d/2} \int e^{2\pi i x \cdot \vec{\gamma} - 4\pi i t |\vec{\gamma}|^2 - 4\pi^2 \sigma^2 |\vec{\gamma} - \vec{\gamma}_0|^2} d\vec{\gamma}$$

$$\text{phase} = \frac{-4\pi^2(\sigma^2 + it)|\vec{\gamma}|^2 + 2\pi \vec{\gamma} \cdot (ix + 4\pi \sigma^2 \vec{\gamma}_0) - 4\pi^2 \sigma^2 |\vec{\gamma}_0|^2}{\text{complete square}}$$

$$= -4\pi^2(\sigma^2 + it) \left| \vec{\gamma} - \frac{ix + 4\pi \sigma^2 \vec{\gamma}_0}{4\pi(\sigma^2 + it)} \right|^2 \\ + \frac{|ix + 4\pi \sigma^2 \vec{\gamma}_0|^2}{4(\sigma^2 + it)} - 4\pi^2 \sigma^2 |\vec{\gamma}_0|^2$$

(9)

Note that $\frac{|ix + 4\pi\sigma^2 \bar{z}_0|^2}{4(\sigma^2 + it)} - 4\pi^2 \sigma^{-2} |\bar{z}_0|^2$

$$= \frac{1}{\sigma^2 + it} \left(-\frac{|x|^2}{4} + 2\pi i \sigma^2 x \cdot \bar{z}_0 + 4\pi^2 \sigma^4 |\bar{z}_0|^2 \right) - 4\pi^2 \sigma^{-2} |\bar{z}_0|^2$$

$$= 2\pi i x \cdot \bar{z}_0 + \frac{1}{\sigma^2 + it} \left(-\frac{|x|^2}{4} + 2\pi i x \cdot \bar{z}_0 - 4\pi^2 \sigma^2 it |\bar{z}_0|^2 \right)$$

Complete square

$$= 2\pi i x \cdot \bar{z}_0 + \frac{1}{\sigma^2 + it} \left\{ \left(-\frac{1}{4} \right) |x - 4\pi t \bar{z}_0|^2 - 4\pi i t (\sigma^2 + it) |\bar{z}_0|^2 \right\}$$

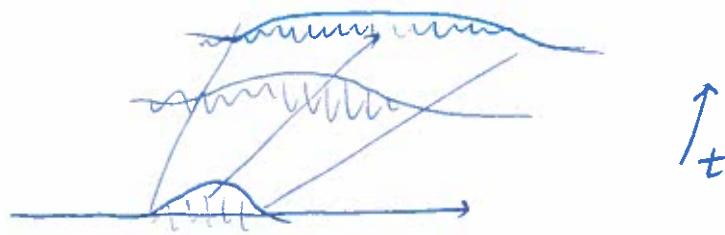
$$= 2\pi i x \cdot \bar{z}_0 - 4\pi i t |\bar{z}_0|^2 - \frac{|x - 4\pi t \bar{z}_0|^2}{4(\sigma^2 + it)} \quad \leftarrow \text{indep of } \bar{z}$$

$$\Rightarrow S(t) V_0 = e^{2\pi i x \cdot \bar{z}_0 - 4\pi i t |\bar{z}_0|^2 - \frac{|x - 4\pi t \bar{z}_0|^2}{4(\sigma^2 + it)}} \\ \times (4\pi \sigma^2)^{d/2} \int e^{-4\pi^2 (\sigma^2 + it) |\bar{z} - \frac{ix + 4\pi \sigma^2 \bar{z}_0}{4\pi(\sigma^2 + it)}|^2} d\bar{z}$$

$$= \left(\frac{\sigma^2}{\sigma^2 + it} \right)^{d/2} e^{2\pi i x \cdot \bar{z}_0 - 4\pi i t |\bar{z}_0|^2 - \frac{|x - 4\pi t \bar{z}_0|^2}{4(\sigma^2 + it)}}$$

localized wave travel at speed $4\pi \bar{z}_0$ = group velocity

- width of the wave packet increase since $\operatorname{Re} \frac{1}{4(\sigma^2 + it)} < \frac{1}{4\sigma^2}$ (10)
- amplitude of the wave packet decreases $\left(\frac{\sigma^2}{\sigma^2 + it}\right)^{d/2} \rightarrow 0$ as $t \rightarrow \infty$



Dispersive relation: lin. Schrödinger equation

$$i \partial_t u + \Delta u = 0$$

space-time F.T.

$$\Rightarrow -2\pi \tau \hat{u}(\tau, \vec{x}) - 4\pi^2 |\vec{x}|^2 \hat{u}(\tau, \vec{x}) = 0$$

$$\Rightarrow -2\pi (\tau + 2\pi |\vec{x}|^2) \hat{u}(\tau, \vec{x}) = 0.$$

i.e. The space-time Fourier transform $\hat{u}(\tau, \vec{x})$ of a soln to the linear Schrödinger eqn is a distribution supported on the paraboloid: $\tau + 2\pi |\vec{x}|^2 = 0$.

\Rightarrow Dispersive relation for Schrödinger egn: $\omega(\vec{z}) = 2\pi|\vec{z}|^2$

(11)

• phase velocity: $\omega(\vec{z}) \frac{\vec{z}}{|\vec{z}|^2} = 2\pi\vec{z}$

• group velocity: $\nabla\omega(\vec{z}) = 4\pi\vec{z}$

Aside: * on page ⑨: Let $G(z) = \int_R e^{-z|x|^2} dx$, $z = a+ib$
with $a > 0$.

For $a > 0$, we have

$$G(a) = \int e^{-a|x|^2} dx \quad a^{1/2}x = \pi^{1/2}y$$

$$= \sqrt{\frac{\pi}{a}} \underbrace{\int e^{-\pi|y|^2} dy}_{=} = \sqrt{\frac{\pi}{a}}$$

Hence, two analytic functions $G(z)$ and $H(z) = \sqrt{\frac{\pi}{z}}$ on $\{Re z > 0\}$
agree on the positive real axis.

$$\Rightarrow G(z) = \int_R e^{-z|x|^2} dx = \sqrt{\frac{\pi}{z}} \text{ on } \{Re z > 0\}$$

\Rightarrow * on p. ⑨

The limiting case is given by the following

(12)

**

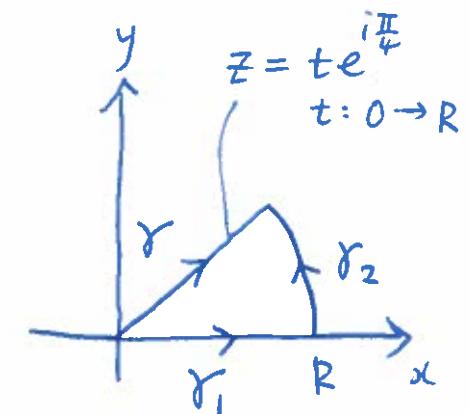
Fresnel integral:

$$\int_{\mathbb{R}} e^{-ix^2} dx = \sqrt{\frac{\pi}{i}}$$

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\gamma=\gamma(R)} e^{-z^2} dz = e^{i\frac{\pi}{4}} \int_0^\infty e^{-it^2} dt \\ & \quad || \\ & \quad = \frac{\sqrt{i}}{2} \int_{\mathbb{R}} e^{-ix^2} dx \end{aligned}$$

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\gamma_1} e^{-z^2} dz + \underbrace{\lim_{R \rightarrow \infty} \int_{\gamma_2} e^{-z^2} dz}_{=0} \\ & \quad = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \end{aligned}$$

\Rightarrow **



$$\begin{aligned}
 \Rightarrow K_t(x) &= \mathcal{F}^{-1}(e^{-4\pi^2 it |\xi|^2})(x) \\
 &= \int_{\mathbb{R}^d} e^{-4\pi^2 it |\xi|^2} e^{2\pi i x \cdot \xi} d\xi \\
 &= \prod_{j=1}^d \left(e^{-x_j^2/4it} \int_{\mathbb{R}} e^{-4\pi^2 it (\xi_j - \frac{x_j}{4\pi t})^2} d\xi_j \right) \\
 &= \frac{1}{(4\pi i t)^{d/2}} e^{-|x|^2/4it} \underbrace{\prod_{j=1}^d \int_{\mathbb{R}} e^{-4\pi^2 it (\xi_j - \frac{x_j}{4\pi t})^2} d\xi_j}_{= \frac{1}{2\pi} \sqrt{\frac{\pi}{it}} = \sqrt{\frac{1}{4\pi i t}}} \\
 &\quad \xrightarrow{\text{so as } t \rightarrow 0}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow S(t)u_0(x) &= K_t * u_0(x) \\
 &= \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} u_0(y) dy
 \end{aligned}$$

Lemma (Dispersive estimate):

$$\|S(t)f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{d/2} \|f\|_{L_x^1(\mathbb{R}^d)}$$

Lec 6 05/02/18 (Mon)

①

- Dispersive estimate: $\|S(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L_x^1} \leftarrow \theta$

- Unitarity of $S(t)$: $\|S(t)f\|_{L_x^2} = \|f\|_{L_x^2} \leftarrow 1-\theta$

interpolation
⇒

* $\|S(t)f\|_{L_x^p} \lesssim \frac{1}{|t|^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}} \|f\|_{L_x^{p'}} \quad \text{for } 2 \leq p \leq \infty$
 $\frac{1}{p} + \frac{1}{p'} = 1$

Pf: $\frac{1}{p} = \frac{\theta}{\infty} + \frac{1-\theta}{2} = \frac{1}{2} - \frac{1}{2}\theta$

$\left(\frac{1}{p'} = \frac{\theta}{1} + \frac{1-\theta}{2} \right)$

(2)

Prop (Strichartz estimates)

We say (q, r) is Schrödinger admissible

if $2 \leq q, r \leq \infty$, $(q, r, d) \neq (2, \infty, 2)$,

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad (\text{scaling condition})$$

(i) (homogeneous estimate) (q, r) admissible

$$\|S(t)f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|f\|_{L_x^2}, \quad \# f \in L_x^2$$

(ii) (dual homog est) \downarrow func of (t, x)

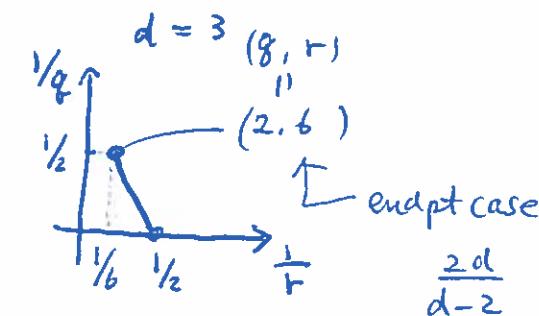
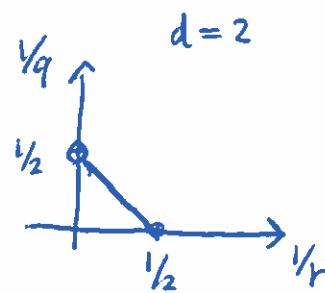
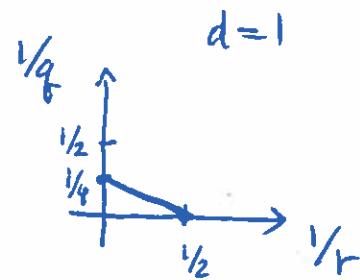
$$\left\| \int_{\mathbb{R}} S(-t') F(t') dt' \right\|_{L^2(\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \quad \frac{1}{q} + \frac{1}{q'} = 1$$

$$\frac{1}{r} + \frac{1}{r'} = 1$$

(iii) non-homog est (retarded est)

$$\left\| \int_0^t S(t-t') F(t') dt' \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \quad (\tilde{q}, \tilde{r}) \text{ admissible}$$

(3)



• In the time-averaged sense (L_t^q),

$S(t)f$ gains integrability (i.e. smoothing effect.)

b/c $S(t)f \in L_x^r$, a.e. t

$$f^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$$

↑
NOT in terms of differentiability
but smoothes out high peaks

Pf (non-endpoint case)

TT* argument

$$T: H \rightarrow B$$

$$T^*: B' \rightarrow H$$

$$TT^*: B' \rightarrow B$$

H, Hilbert space

B, Banach space

Then, $\|T\| < \infty \Leftrightarrow \|T^*\| < \infty \Leftrightarrow \|TT^*\| < \infty$

(4)

$$T = S(t)$$

$$\begin{aligned} \langle S(t)f, G(t,x) \rangle_{t,x} &= \iint_{\mathbb{R}^2} S(t)f \overline{G(t,x)} dt dx \\ &= \int_x f(x) \underbrace{\int_t \overline{S(t)G(t,x)} dt}_{\text{Parseval in } x} dx && \text{Parseval in } x \\ &= \left\langle f, \underbrace{\int_{\mathbb{R}} S(-t)G(t) dt}_{= T^*G} \right\rangle_x \end{aligned}$$

$$\underline{TT^*F} = S(t) \int_{\mathbb{R}} S(t') F(t') dt' = \int_{\mathbb{R}} S(t-t') F(t') dt'$$

Lemma : Hardy-Littlewood-Sobolev inequality

$$1 < p, q, r < \infty, \quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

$$\| \frac{1}{|x|^{\frac{1}{p}}} * f \|_{L_x^r} \lesssim \| f \|_{L_x^q}$$

$\left(\text{Young's ineq} : (\text{LHS}) \lesssim \| f \|_{L_x^q} \| \frac{1}{|x|^{\frac{1}{p}}} \|_{L_x^p} = \left(\int \frac{1}{|x|^{\frac{1}{p}}} dx \right)^{\frac{1}{p}} = \infty \right)$

(5)

H-L-S ineq \Leftrightarrow Sobolev ineq

$$\|TT^*F\|_{L_t^q L_x^r} \stackrel{\text{Mink}}{\leq} \left\| \int \|S(t-t')F(t')\|_{L_x^r} dt' \right\|_{L_t^q}, \quad r \geq 2$$

$$\stackrel{\oplus}{\sim} \left\| \int \frac{1}{|t-t'|^{d(\frac{1}{2}-\frac{1}{r})}} \|F(t')\|_{L_x^r} dt' \right\|_{L_t^q} \frac{1}{|t-t'|^{1/p}}$$

$$\stackrel{\text{H-L-S}}{\sim} \|F\|_{L_t^{q'} L_x^{r'}} \xrightarrow[\text{TT* argument}]{} \text{(i) \& (ii)}$$

Check the condition for H-L-S ineq. (in t with $d=1$)

$$\begin{aligned} \frac{1}{q} + 1 &= d\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{1}{q'} \\ &= \frac{d}{2} - \frac{d}{r} + 1 - \frac{1}{q} \end{aligned}$$

$$\Leftrightarrow \frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad \text{Schrödinger admis. cond.}$$

Need $0 < \frac{1}{q}$, $d\left(\frac{1}{2} - \frac{1}{r}\right)$, $\frac{1}{q'} < 1$ for H-L-S.

(6)

$$\Leftrightarrow 1 < g < \infty$$

and $2 < r < \frac{2d}{d-2}$

missing the endpt cases

- When $r=2$, $g=\infty \Rightarrow$ (i) follows from the unitarity of $S(t)$)
- For $(g, r) = (2, \frac{2d}{d-2})$, $d \geq 3$, see Keel-Tao Amer J. Math '98

It remains to prove (iii) :

$$\left\| \int_0^t S(t-t') F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}^{q'}$$

$(q, r), (\tilde{q}, \tilde{r})$, Sch. admissible.

$$(q, r) = (\tilde{q}, \tilde{r})$$

$$\left\| \int_0^t S(t-t') F(t') dt' \right\|_{L_t^q L_x^r} = \left\| \int_R S(t-t') \underbrace{\mathbf{1}_{[0,t]}(t')}_{\text{char. func / indicator func}} F(t') dt' \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}}$$

as before

$$\lesssim \left\| \int_R \frac{1}{|t-t'|^{d(\frac{1}{2}-\frac{1}{r})}} \underbrace{\mathbf{1}_{[0,t]}(t')}_{\text{char. func / indicator func}} F(t') dt' \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}}$$

H-L-S

$$\lesssim \|F\|_{L_t^{q'} L_x^{r'}}$$

$$\textcircled{a} \quad \left\| \int_{t' < t} S(t-t') F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \quad \textcircled{2}$$

$$\textcircled{b} \quad \left\| \int_{t' < t} S(t-t') F(t') dt' \right\|_{L_t^\infty L_x^2} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$$

$$\textcircled{a} \quad \mathbb{1}_{(-\infty, t]} = \mathbb{1}_{(-\infty, 0)} + \mathbb{1}_{[0, t]}$$

Pf of \textcircled{b}

$$\begin{aligned} (\text{LHS})^2 &= \left\langle \underbrace{\int_{t_1 < t} S(t-t_1) F(t_1) dt_1}, \underbrace{\int_{t_2 < t} S(t-t_2) F(t_2) dt_2} \right\rangle_{L_x^2} \\ &\quad S(t) S(-t_1) \quad S(t) S(-t_2) \end{aligned}$$

$$\left(\int (S(t)f) \bar{g} dx = \int f \overline{S(t)g} dx \right)$$

$$= \int_{t_1 < t} \left\langle F(t_1), \int_{t_2 < t} S(t_1 - t_2) F(t_2) dt_2 \right\rangle_{L_x^2} dt_1,$$

$$\stackrel{\text{H\"older}}{\leq} \int_R \|F(t_1)\|_{L_x^{r'}} \left\| \int_{t_2 < t} S(t_1 - t_2) F(t_2) dt_2 \right\|_{L_x^r} dt_1,$$

$$\stackrel{\text{H\"older}}{\leq} \|F\|_{L_t^{q'} L_x^{r'}} \left\| \int_{t_2 < t} S(t-t_2) F(t_2) dt_2 \right\|_{L_t^q L_x^r} \stackrel{\textcircled{a}}{\lesssim} (\text{RHS})^2$$

□

- WTS (iii)

(3)

$$\Leftrightarrow \left| \iint_{t' < t} \langle S(t-t') F(t'), G(t) \rangle_{L_x^2} dt' dt \right|$$

④

$$\lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \|G\|_{L_t^{q'} L_x^{r'}}$$

where the implicit const is indep of G . (and F)

- Fix (\tilde{q}, \tilde{r}) .

a) \Rightarrow ④ holds for $q = \tilde{q}$, $r = \tilde{r}$.

b) \Rightarrow ④ holds for $q = \infty$, $r = 2$.

interpolation

\Rightarrow ④ holds for any $q \geq \tilde{q}$ (and $r = r(q)$.)

- For ④ with $q < \tilde{q}$, we use symmetry in ④.

④ with $q \geq \tilde{q}$ implies

$$\left| \int_{t'} \left\langle F(t'), \int_{t > t'} S(t-t') G(t) dt \right\rangle_{L_x^2} dt' \right| \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \|G\|_{L_t^{q'} L_x^{r'}}$$

view F as a duality var.

duality

$$\Leftrightarrow \left\| \int_{t>t'} s(t'-t) G(t) dt \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \lesssim \|G\|_{L_t^{q'} L_x^{r'}} \quad (4)$$

By relabelling,

$$\left\| \int_{t'>t} s(t-t') F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} , \text{ for } q \leq \tilde{q}$$

Now, write $\mathbb{1}_{[0,t]} = \underbrace{\mathbb{1}_R(t)}_{\equiv 1} - \mathbb{1}_{(t,\infty)}$

$\Rightarrow \textcircled{*}$ holds for $q \leq \tilde{q}$

□

Rmk: Christ - Kiselev lemma (for inserting $\mathbb{1}_{[0,t]}$). See Tao's book.

$$\begin{aligned} & \left\| \int_R s(t-t') F(t') dt' \right\|_{L_t^q L_x^r} = \left\| s(t) \int_R s(-t') F(t') dt' \right\|_{L_t^q L_x^r} \\ & \stackrel{(i)}{\lesssim} \left\| \int_R s(-t') F(t') dt' \right\|_{L_x^2} \stackrel{(ii)}{\lesssim} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \end{aligned}$$

(5)

Sec 5: LWP of NLS, part II

Ex 1: $d=1, p=3$ (cubic NLS on \mathbb{R})

$$S_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1} = -\frac{1}{2}$$

Thm 1: Let $s \geq 0$. cubic NLS on \mathbb{R} is locally well-posed in $H^s(\mathbb{R})$

Pf: $s=0$. Let $u_0 \in L^2(\mathbb{R})$

$$(NLS) \Leftrightarrow u(t) = \Gamma_{u_0}(u)(t) := S(t)u_0 + i \int_0^t S(t-t')|u|^2 u(t') dt'$$

\uparrow
on $[-T, T]$

\uparrow $L^{\frac{8}{3}}_{[-T, T]}$

- Note: $(g, r) = (8, 4)$ is admiss.

$$\frac{2}{8} + \frac{1}{4} = \frac{1}{2}$$

- Set $X_T = \underbrace{C([-T, T]; L_x^2(\mathbb{R}))}_{= C_T L_x^2} \cap L_T^{\frac{8}{3}} L_x^4$

$\leftarrow \|u\|_{X \cap Y}$
 $= \|u\|_X + \|u\|_Y$

$\downarrow L_T^{\frac{8}{3}}([-T, T]; L_x^4)$

$$\|\Gamma_{u_0}(u)\|_{X_T} \stackrel{\text{Str.}}{\leq} C_1 \|u_0\|_{L_x^2} + C_2 \underbrace{\||u|^2 u\|_{L_T^{8/7} L_x^{4/3}}}_{\substack{\text{H\"older int} \\ \leq C_3 T^{1/2} \||u|^2 u\|_{L_T^{8/3} L_x^{4/3}}}} \quad (8.4), \text{admis}$$

$$\frac{7}{8} = \frac{1}{2} + \frac{3}{8}$$

$$\underbrace{\leq_{\text{H\"older int}}}_{\leq C_4 \|u\|_{L_T^8 L_x^4}^3} \leq \|u\|_{X_T}^3$$

$$\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{4}$$

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

For $u \in \overline{B}_R \subset X_T$,

$$\|\Gamma_{u_0}(u)\|_{X_T} \leq C_1 \|u_0\|_{L_x^2} + \underbrace{C_3 T^{1/2} R^2 \cdot R}_{\leq 1/2}$$

$$\leq 2 C_1 \|u_0\|_{L_x^2} =: R \quad \nwarrow \text{by choosing } T = T(R) \text{ suff. small}$$

provided

$$\underline{C_3 T^{1/2} R^2 \leq 1/2.}$$

(7)

- Similarly, for $u, v \in X_T$

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} \leq C_3 T^{1/2} \underbrace{\|(u^2 u - v^2 v)\|_{L_T^{8/3} L_x^{4/3}}}_{\text{telescoping sum}}$$

$$u\bar{u}u - v\bar{v}v = (u-v)\bar{u}u + v(\bar{u}-\bar{v})u + v\bar{v}(u-v)$$

$$\stackrel{\text{Young's}}{\leq} C_4 T^{1/2} \left(\|u\|_{X_T}^2 + \|v\|_{X_T}^2 \right) \|u-v\|_{X_T}$$

$$\text{Young's: } ab \leq \frac{ap}{p} + \frac{b^p}{p}$$

$$\leq \boxed{2 C_4 T^{1/2} R^2} \leq \frac{1}{2}$$

By choosing $T^{1/2} \sim R^{-2}$ i.e. $T \sim R^{-4}$, we conclude that

Γ_{u_0} is a contraction on $\overline{B_R} \subset X_T$, $R = 2C_1 \|u_0\|_{L_x^2}$

\Rightarrow Banach fixed pt thm $\Rightarrow \exists! u \in \overline{B_R} \subset X_T$ s.t. $u = \Gamma_{u_0}(u)$



Rmk : ① local existence time $T = T(\|u_0\|_{L^2}) \sim \frac{1}{\|u_0\|_{L^2}^4}$ ⑧

\Rightarrow Using the L^2 -conservation, we conclude that
Cubic NLS on \mathbb{R} is global well-posed in $L^2(\mathbb{R})$.



② Uniqueness holds in $\overline{B_R} \subset C_T L_x^2 \cap L_T^\infty L_x^4$

\Rightarrow uniqueness in $C_T L_x^2 \cap L_T^\infty L_x^4$

conditional uniqueness.

(unconditional uniqueness \Leftarrow uniqueness in the entire $C_T H^s(\mathbb{R}^d)$)

③ should show continuity of u in t . (with values in L^2)
- omitted

- No class next week

- 1-d cubic NLS: $i\partial_t u + \Delta u = \pm |u|^2 u$, $x \in \mathbb{R}$

We prove LWP in $L^2(\mathbb{R})$. $\xrightarrow{\text{mass conservation}}$ GWP in $L^2(\mathbb{R})$.

What if $u_0 \in H^s(\mathbb{R})$ for some $s > 0$?

If $s \in \mathbb{N}$, we can use Leibniz rule: $\partial_x(fg) = \partial_x f \cdot g + f \cdot \partial_x g$.

For general $s > 0$, we need the fractional Leibniz rule:

$$s \in (0, 1], \quad 1 < r, p_1, p_2, q_1, q_2 < \infty \text{ s.t.}$$

Then,

$$\| |\nabla|^s (fg) \|_{L_x^r} \lesssim \| |\nabla|^s f \|_{L^{p_1}} \| g \|_{L^{q_1}} + \| f \|_{L^{p_2}} \| |\nabla|^s g \|_{L^{q_2}} \quad \frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}, \quad j = 1, 2.$$

$\left\{ \begin{array}{l} |\nabla|^s \\ \end{array} \right.$

$$\underline{\text{Moral}}: |\nabla|^s(fg) \approx (|\nabla|^s f)g + f(|\nabla|^s g) \quad (2)$$

- $s=0$: Hölder
- $s=1$: "Leibniz rule" & Hölder

↑ really need $s \in 2\mathbb{N}$: $|\nabla|^{2s} = (-\Delta)^{s/2}$

$$\textcircled{*} \Rightarrow \|\langle \nabla \rangle^s (fg)\|_{L_x^r} \lesssim \|\langle \nabla \rangle^s f\|_{L_x^{p_1}} \|g\|_{L_x^{q_1}} + \|f\|_{L_x^{p_2}} \|\langle \nabla \rangle^s g\|_{L_x^{q_2}}$$

$$\text{b/c } \langle \xi \rangle^s \sim 1 + |\xi|^s$$

- LWP of cubic NLS in $H^s(\mathbb{R})$, $s \geq 0$

Given $T > 0$, set $X_T^s = C_T H^s \cap L_T^\infty W_x^{s,4}$

$$\begin{aligned} \|\Gamma_{u_0}(u)\|_{X_T^s} &= \|\langle \nabla \rangle^s \Gamma_{u_0}(u)\|_{X_T^0} \stackrel{\text{str.}}{\lesssim} \|\langle \nabla \rangle^s u_0\|_{L_x^2} + \|\langle \nabla \rangle^s (|u|^2 u)\|_{L_T^{8/3} L_x^{4/3}} \\ &\lesssim \|u_0\|_{H^s} + T^{1/2} \|\langle \nabla \rangle^s (|u|^2 u)\|_{L_T^{8/3} L_x^{4/3}} \end{aligned}$$

(*) & Hölder int

$$\lesssim \|u_0\|_{H^s} + T^{1/2} \underbrace{\|u\|_{L_T^\infty W_x^{s,4}}^3}_{\lesssim \|u\|_{X_T^s}^3} \lesssim \|u\|_{X_T^s}^3$$

By a similar computation,

(3)

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T^s} \lesssim T^{1/2} \left(\|u\|_{X_T^s}^2 + \|v\|_{X_T^s}^2 \right) \|u - v\|_{X_T^s}$$

\Rightarrow For $T = T(\|u_0\|_{H^s}) > 0$ sufficiently small,

Γ_{u_0} is a contraction $\overline{B_R} \subset X_T^s$ where $R \sim \|u_0\|_{H^s}$

\Rightarrow 1-d cubic NLS is LWP in $H^s(\mathbb{R})$, $s \geq 0$

\Rightarrow GWP in $H^s(\mathbb{R})$, ($s \geq 0$) (mass conservation & persistence of reg.)
 improvement \leftarrow need to prove

$$\begin{aligned} & \|\langle v \rangle^s (|u|^2 u)\|_{L_T^{8/3} L_x^{4/3}} \\ & \lesssim \|u\|_{L_T^8 W_x^{s, 4}} \|u\|_{L_T^2 L_x^4}^2 \end{aligned}$$

$$\cdot \frac{d=2}{p=3} : \quad S_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1} = 1 - 1 = 0. \quad (4)$$

L^2 -critical / mass-critical.

Thm 2: Cubic NLS on \mathbb{R}^2 is locally well-posed in $L^2(\mathbb{R}^2)$ and is also globally well-posed in $L^2(\mathbb{R}^2)$ with small initial data. ($\|u_0\|_{L^2}$ is suff. small)

$$(q, r) = (4, 4) \text{ is Schrödinger admissible: } \frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

$$\cdot X_T = X_T^\circ = G L^2 \cap L_T^4 L_x^4.$$

$$\| \Gamma_{u_0}(u) \|_{X_T} \leq \| S(t) u_0 \|_{X_T} + C_2 \| |u|^2 u \|_{L_T^{4/3} L_x^{4/3}}$$

$$\leq \underline{\underline{C_1 \|u_0\|_{L^2}}} + C_2 \|u\|_{L_T^4 L_x^4}^3$$

No power of T .

(**)

(5)

$$\begin{aligned} \|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} &\leq C_2 \| |u|^2 u - |v|^2 v \|_{L_T^{4/3} L_x^{4/3}} \\ &\leq C_3 \left(\|u\|_{L_T^4 L_x^4}^2 + \|v\|_{L_T^4 L_x^4}^2 \right) \|u - v\|_{L_T^4 L_x^4} \end{aligned}$$

We need this to be less than 1.

Set $R = 2C_1 \|u_0\|_{L^2}$ and let $u \in \overline{B}_R \subset X_T$.

$$\Rightarrow \cdot \|\Gamma_{u_0}(u)\|_{X_T} \leq \frac{1}{2} R + C_2 R^3 \leq R \text{ if } R \ll 1. \quad (C_2 R^2 \leq \frac{1}{2})$$

$$\begin{aligned} \cdot \|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} &\leq 2C_3 R^2 \|u - v\|_{X_T} \\ &\leq \frac{1}{2} \|u - v\|_{X_T} \text{ if } R \ll 1 \end{aligned}$$

\Leftarrow These estimates hold true for $T = \infty$.

\Rightarrow GWP for small initial data in $L^2(\mathbb{R}^2)$
 (by running a fixed pt argument in $\overline{B}_R \subset X_\infty$.)

(6)

- What about large L^2 -data?

$$\|S(t)u_0\|_{L^4(\mathbb{R}_t; L_x^4)} \stackrel{\text{Str.}}{\lesssim} \|u_0\|_{L^2}$$

DCT

$$\Rightarrow \lim_{T \rightarrow 0} \|S(t)u_0\|_{L_T^4 L_x^4} = 0.$$

but $\|S(t)u_0\|_{L_T^\infty L_x^2} (= \|u_0\|_{L_x^2}) \rightarrow 0$. as $T \rightarrow 0$.

$$\|\Gamma u_0(u)\|_{L_T^4 L_x^4} \leq \|S(t)u_0\|_{L_T^4 L_x^4} + C_2 \| |u|^2 u \|_{L_T^{4/3} L_x^{4/3}}$$

We do NOT apply the Strichartz esti.

Fix $\gamma > 0$ small. Then, given $u_0 \in L^2(\mathbb{R}^2)$, $\exists T = T(u_0) > 0$ small s.t.

$$\|S(t)u_0\|_{L_T^4 L_x^4} \leq \frac{1}{2} \gamma.$$

(7)

Let $u \in \overline{B_y} \subset L_T^4 L_x^4$.

$$\Rightarrow \|\Gamma_{u_0}(u)\|_{L_T^4 L_x^4} \leq \frac{1}{2} y + C_2 y^3 \leq y$$

Also,

$$\begin{aligned} \|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{L_T^4 L_x^4} &\leq C_3 \left(\|u\|_{L_T^4 L_x^4}^2 + \|v\|_{L_T^4 L_x^4}^2 \right) \|u - v\|_{L_T^4 L_x^4} \\ &\leq \underbrace{2C_3 y^2}_{\leq 1/2} \|u - v\|_{L_T^4 L_x^4}, \quad \forall u, v \in \overline{B_y} \end{aligned}$$

\Rightarrow Contraction mapping principle

$\Rightarrow \exists! u \in \overline{B_y} \subset L_T^4 L_x^4$, soln to NLS. ($u = \Gamma_{u_0}(u)$)

By (**), $\|u\|_{C_T L^2} = \|\Gamma_{u_0}(u)\|_{C_T L^2} \leq C_1 \|u_0\|_{L^2} + C_2 y^3 < \infty$

$\Rightarrow u \in C_T L^2 \cap L_T^4 L_x^4$

- $T = T(u_0)$ depends on "the profile of u_0 " ⑧

\Leftarrow critical nature of the problem.

- We could simply run a contraction argument in

$$\overline{A_{R,\gamma}} = \{ u \in \overline{B_R} \subset X_T, \text{ and } u \in \overline{B_\gamma} \subset L_T^4 L_x^4 \}$$

$$R \sim \|u_0\|_{L^2}, \quad \gamma \ll 1,$$

endowed with the X_T -norm.

$$\textcircled{**} \Rightarrow \|\Gamma_{u_0}(u)\|_{X_T} \leq C_1 \|u_0\|_{L^2} + C_2 \gamma^3$$

$$\leq 2C_1 \|u_0\|_{L^2} = R$$

$$\textcircled{***} \Rightarrow \|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} \leq \underbrace{2C_3 \gamma^2}_{\leq \frac{1}{2}} \|u - v\|_{X_T} \Rightarrow \Gamma_{u_0} : \overline{A_{R,\gamma}} \hookrightarrow$$

Also, $\|\Gamma_{u_0}(u)\|_{L_T^4 L_x^4} \stackrel{\leq \frac{1}{2}}{\leq} \frac{1}{2} \gamma + \gamma^3 \leq \gamma$.

Sec 6: More on estimates

6.1 Dispersive estimate for the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = f \end{cases}$$

$$\Rightarrow S(t)f(x) = e^{it\Delta}f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} f(y) dy$$

$$\Rightarrow \|S(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L_x^1}, \quad t \neq 0.$$

Goal: Prove the dispersive estimate WITHOUT the explicit formula.

• 1-d case: $S(t)f = K_t * f$

$$K_t = \mathcal{F}^{-1}(e^{-it|\xi|^2}) = \int_{\mathbb{R}} e^{-it|\xi|^2 + ix\xi} d\xi$$

$$\underline{\text{Claim}}: \|K_t(x)\|_{L_x^\infty} \lesssim \frac{1}{|t|^{1/2}}, \quad (t \neq 0)$$

(2)

Then, by Young's ineq.

$$\begin{aligned}\|S(t)f\|_{L_x^\infty} &= \|K_t * f\|_{L_x^\infty} & \frac{1}{\infty} + 1 &= \frac{1}{\infty} + 1 \\ &\leq \|K_t\|_{L_x^\infty} \|f\|_{L_x'} \\ &\lesssim \frac{1}{|t|^{1/2}} \|f\|_{L_x'}.\end{aligned}$$

Pf of Claim: Assume $t > 0$

$$\begin{aligned}K_t(x) &= \frac{1}{\sqrt{t}} \int_R e^{-i\zeta^2 + i\frac{x}{\sqrt{t}}\zeta} d\zeta & \zeta = t^{1/2} \xi \\ &= \frac{1}{\sqrt{t}} K_1\left(\frac{x}{\sqrt{t}}\right)^*\end{aligned}$$

$$\Rightarrow \|K_t\|_{L_x^\infty} = \frac{1}{\sqrt{t}} \|K_1\|_{L_y^\infty} \quad (t \text{ is fixed}).$$

(3)

Tool : Method of stationary phase

$$K_1(x) = \int_{\mathbb{R}} e^{-i\zeta^2 + ix\zeta} d\zeta \\ = \int_{\mathbb{R}} e^{-i\phi(\zeta)} d\zeta \quad \leftarrow \text{oscillatory integral}$$

where $\phi(\zeta) = \zeta^2 - x\zeta \quad (x \text{ is fixed.})$

Idea : Integration by parts.

$$e^{-i\phi(\zeta)} = \frac{\partial_{\zeta} e^{-i\phi(\zeta)}}{-i\phi'(\zeta)} \quad \phi'(\zeta) = 2\zeta - x$$

← If $\phi'(\zeta)$ is not small, good.

$$\int_{\mathbb{R}} e^{-i\phi(\zeta)} d\zeta = \int_{\mathbb{R}} \partial_{\zeta} e^{-i\phi(\zeta)} \cdot \frac{1}{-i\phi'(\zeta)} d\zeta$$

IBP

$$= \int_{\mathbb{R}} e^{-i\phi(\zeta)} \left(\frac{1}{i\phi'(\zeta)} \right)' d\zeta$$

Let $\Psi \in C^\infty(\mathbb{R}; [0, 1])$ s.t. $\Psi(\xi) = \begin{cases} 1, & |\xi| \leq 1 \\ 0, & |\xi| \geq 2 \end{cases}$ ④



$$\Rightarrow K_1(x) = \int_{\mathbb{R}} e^{-i\phi(\xi)} \underbrace{\Psi(2\xi-x)}_{\phi(\xi)} d\xi$$

$$+ \int_{\mathbb{R}} e^{-i\phi(\xi)} (1-\Psi)(2\xi-x) d\xi =: I(x) + II(x)$$

$$\circ |I(x)| \leq \int |\Psi(2\xi-x)| d\xi \lesssim 1.$$

$$\cdot II(x) \stackrel{IBP}{=} \int e^{-i\phi(\xi)} \partial_{\xi} \left(\frac{(1-\Psi)(2\xi-x)}{i\phi'(\xi)} \right) d\xi$$

$$= \int e^{-i\phi(\xi)} \left\{ -2 \frac{(1-\Psi)(2\xi-x)}{i(2\xi-x)^2} \right\} d\xi \quad (\phi'(\xi)) = |2\xi-x| \geq 1$$

$$+ \int e^{-i\phi(\xi)} \frac{(1-\Psi)'(2\xi-x)}{i\phi'(\xi)} d\xi \quad (1-\Psi)' = -\Psi'$$

≥ 1 in abs value

supported on
 $[-2, -1] \cup [1, 2]$

$$\Rightarrow |\mathbb{I}(x)| \leq c \int_{|2\bar{z}-x| \geq 1} \frac{1}{(2\bar{z}-x)^2} d\bar{z}$$

$$+ \int_{2\bar{z}-x \in \text{supp } \Psi'} |(1-\psi)'(2\bar{z}-x)| d\bar{z} \lesssim 1.$$

□

6.2 Glimpse on oscillatory integrals

$$I(\lambda) = \int_a^b e^{i\lambda \Phi(x)} \psi(x) dx.$$

phase $\Phi(x)$, real-valued

$\psi(x)$, complex-valued, smooth, with cpt support.

Lemma: $\text{supp } \psi \subset \underbrace{(a, b)}$

$\Phi'(x) \neq 0$ for all $x \in [a, b]$.
cpt subset

Then, $I(\lambda) = \Theta(\lambda^{-N})$ as $\lambda \rightarrow \infty$. $\forall N \in \mathbb{N}$.

(6)

Big O notation, $f = \Theta(g)$ if $\lim \frac{|f|}{g} \leq c$
 Little o notation, $f = o(g)$ if $\lim \frac{|f|}{g} = 0$.

Pf: Let $Df(x) = \frac{1}{i\lambda\phi'(x)} \frac{df}{dx}$

$$\Rightarrow D(e^{i\lambda\phi}) = e^{i\lambda\phi}$$

$$\int (Df) g dx = \int f D^T g dx$$

Transpose: $D^T f = -\frac{d}{dx} \left(\frac{f}{i\lambda\phi'(x)} \right)$

$$\Rightarrow I(\lambda) = \int_a^b \underbrace{e^{i\lambda\phi}}_{D^N(e^{i\lambda\phi})} \cdot \psi dx \stackrel{\substack{\text{IBP} \\ N \text{ times}}}{=} \int_a^b e^{i\lambda\phi} (D^T)^N \psi dx$$

↑ No bdry term ψ has cpt supp in (a, b)

$$|\phi'(x)| \geq c \text{ on } [a, b]$$

$$\Rightarrow |I(\lambda)| \lesssim_{N, \psi, \phi} \lambda^{-N}$$

□

Prop (van der Corput) ϕ , real-valued, smooth on (a, b) ⑦

Suppose $\exists k \in \mathbb{N}$ st. $|\phi^{(k)}(x)| \geq 1$ for all $[a, b]$

Then, $\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq C_k \underline{\lambda^{-1/k}}$, indep of a, b .

provided (i) $k \geq 2$,

much worse decay due to non-vanishing
at the endpoints.

or (ii) $\phi'(x)$ is monotonic when $k=1$.

Pf: (ii) $k=1$

$$\begin{aligned} \int_a^b e^{i\lambda\phi} dx &= \int_a^b D(e^{i\lambda\phi}) \cdot 1 dx \\ &= \underbrace{\int_a^b e^{i\lambda\phi} D^T(1) dx}_{+} + \underbrace{\frac{e^{i\lambda\phi}}{i\lambda\phi'} \Big|_a^b}_{1 \cdot | \leq 2/\lambda} \end{aligned}$$

$$\Rightarrow \left| \int_a^b e^{i\lambda\phi} D^T(1) dx \right| \leq 2/\lambda.$$

$$\text{φ' monotonic} \quad = \frac{1}{\lambda} \left| \int_a^b e^{i\lambda\phi} \frac{d}{dx} \left(\frac{1}{\phi'} \right) dx \right| \leq \frac{1}{\lambda} \int_a^b \left| \frac{d}{dx} \left(\frac{1}{\phi'} \right) \right| dx.$$

$$\Rightarrow \frac{1}{\phi'} \text{ monotonic} \quad = \frac{1}{\lambda} \left| \int_a^b \frac{d}{dx} \left(\frac{1}{\phi'} \right) dx \right| \stackrel{\text{ETC}}{\leq} \frac{1}{\lambda} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \lesssim 1/\lambda.$$

(i) $k \geq 2$: We proceed by induction on k .

(f)

Suppose that the result holds for k .

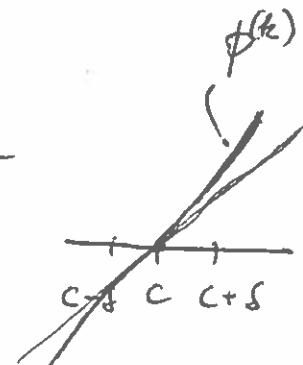
WLOG, assume $\phi^{(k+1)}(x) \geq 1$, $\forall x \in [a, b]$

- Let $x = c$ be the (unique) point in $[a, b]$

s.t. $|\phi^{(k)}(x)|$ attains its min.

- If $\phi^{(k)}(c) = 0$, then $|\phi^{(k)}(x)| \geq \delta$ on $(c-\delta, c+\delta)^c$

write $\int_a^b = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b$



$$\left| \frac{\phi^{(k)}}{\delta} \right| \geq 1$$

$$i \lambda \phi = i \lambda \delta \cdot \frac{\phi}{\delta}$$

By inductive hypothesis,

$$\left| \int_a^{c-\delta} + \int_{c+\delta}^b e^{i \lambda \phi} dx \right| \leq C_k (\lambda \delta)^{-1/k}$$

On the other hand,

$$\left| \int_{c-\delta}^{c+\delta} \dots \right| \leq 2\delta$$

equate them

$$\Rightarrow \delta \sim (\lambda \delta)^{-1/k} \Leftrightarrow \delta \sim \lambda^{-1/k+1}$$

(When $k+1=2$,
 $\phi^{(k+1)} \geq 1 \Rightarrow \phi'$ is monotonic.)

If $\Phi^{(k)}(c) \neq 0$, then $c = a$ (or $c = b$) ⑨

write $\int_a^b = \int_a^{a+\delta} + \int_{a+\delta}^b$ and proceed as before



Cor: Same assumption as in the previous prop.

$$\left| \int_a^b e^{i\lambda \Phi(x)} \underline{\Psi(x)} dx \right| \leq C_k \lambda^{-1/k} \left[|\Psi(b)| + \int_a^b |\Psi'(x)| dx \right]$$

Pf: Let $F(x) = \int_a^x e^{i\lambda \Phi(y)} dy$

\Rightarrow By Prop, $|F(x)| \leq C_k \lambda^{-1/k}$

$$\Rightarrow \int_a^b e^{i\lambda \Phi(x)} \Psi(x) dx \stackrel{\text{IBP}}{=} \underline{F(b)\Psi(b)} - \cancel{F(a)\Psi(a)} - \int_a^b \underline{F(x)\Psi'(x)} dx$$

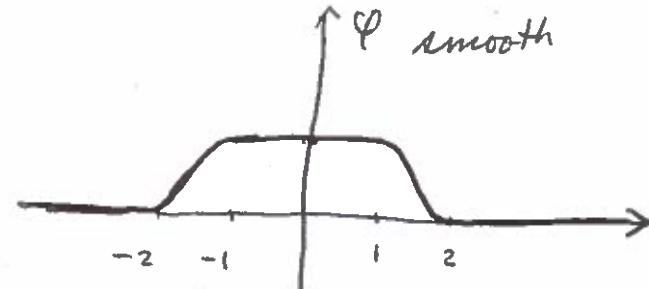


(1)

Lec 10 26/02/18 (Mon)

6.3 Glimpse on the Littlewood-Paley decomposition

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \leq 1 \\ 0, & |\xi| > 2 \end{cases}$$



Given dyadic N ,

$$\left(\begin{array}{l} N = 2^j \text{ for some } j \in \mathbb{Z} \\ N, \text{ dyadic} \iff N \in 2^{\mathbb{Z}} \end{array} \right) \quad \left| \begin{array}{l} \text{dyadic } N \geq 1 \\ \iff N \in 2^{N \vee 0} \end{array} \right.$$

define

$$P_{\leq N} f = \underbrace{\mathcal{F}^{-1}(\varphi(\frac{\xi}{N}) \hat{f}(\xi))}_{= \begin{cases} 1, & |\xi| \leq N \\ 0, & |\xi| > 2N \end{cases}}$$

Littlewood-Paley projection

$$P_N f = P_{\leq N} f - P_{\leq \frac{N}{2}} f = \text{localization onto } \{|\xi| \sim N\}$$

$$\begin{cases} 1, & |\xi| \leq N \\ 0, & |\xi| > 2N \end{cases} \quad \begin{cases} 1, & |\xi| \leq \frac{N}{2} \\ 0, & |\xi| > N \end{cases} \quad \frac{1}{2}N \leq |\xi| \leq 2N.$$

(2)

- $P_{\leq N} f = \sum_{M \leq N} P_M f$
 - $f = \lim_{N \rightarrow \infty} P_{\leq N} f$
 - $= \sum_{\substack{M \\ \text{dyadic}}} P_M f$
- i.e. $P_{\leq N} \rightarrow \text{Id}$

Thm : Littlewood - Paley theory : $1 < p < \infty$.

Then, $\|f\|_{L^p} \sim \left\| \left(\sum_{\substack{N \\ \text{dyadic}}} |P_N f(x)|^2 \right)^{1/2} \right\|_{L^p}$

$\quad \quad \quad = \text{square function of } f.$

$p=2$: $(RHS) = \left(\sum_N \|P_N f\|_{L_x^2}^2 \right)^{1/2}$

$$= \left(\sum_N \int_{|\xi| \sim N} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \sim \left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \stackrel{\text{Plancheral}}{=} \|f\|_{L_x^2}$$

Bernstein's inequalities : $1 \leq p \leq q \leq \infty$

(3)

$$(i) \| P_{\leq N} |\nabla|^s f \|_{L^p} \lesssim N^s \| P_{\leq N} f \|_{L^p}, \quad s \geq 0$$

$$= \mathcal{F}^{-1}(|\vec{\zeta}|^s \hat{f}(\vec{\zeta}))$$

$$|\nabla| = \sqrt{-\Delta} \sim |\vec{\zeta}| = \sqrt{\zeta_1^2 + \zeta_2^2 + \dots + \zeta_d^2}$$

$$(ii) \| P_N |\nabla|^s f \|_{L^p} \sim N^s \| P_N f \|_{L^p}, \quad s \in \mathbb{R}$$

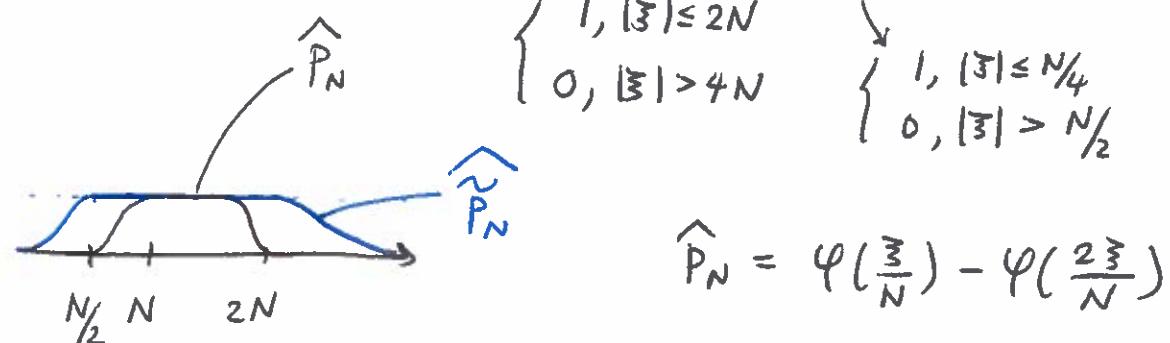
$$(iii) \| P_{\leq N} f \|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_{\leq N} f \|_{L^p} \quad \left. \begin{array}{l} \text{freq localized analogue} \\ \text{of Sobolev inequality: } s = \frac{d}{p} - \frac{d}{q} \end{array} \right\}$$

$$(iv) \| P_N f \|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_N f \|_{L^p} \quad \left. \begin{array}{l} \text{freq localized analogue} \\ \text{of Sobolev inequality: } s = \frac{d}{p} - \frac{d}{q} \end{array} \right\}$$

Pf of (ii)

$$\tilde{P}_N = P_{N/2} + P_N + P_{2N} = P_{\leq 2N} - P_{\leq \frac{N}{4}}$$

$$\Rightarrow P_N = \tilde{P}_N P_N$$



$$\underline{\text{Rmk}}: \quad \sum_N \tilde{P}_N = 3 \text{Id} \quad (4)$$

$$\|f\|_{L^p} \sim \left\| \left(\sum_N |\tilde{P}_N f|^2 \right)^{1/2} \right\|_{L^p}, \quad 1 < p < \infty \quad \text{just like } P_N$$

$$P_N |\nabla|^s f = |\nabla|^s P_N f = \underbrace{|\nabla|^s \tilde{P}_N}_{\psi(\frac{\xi}{N})} P_N f$$

$$\begin{aligned} & |\xi|^s \left(\underbrace{\psi\left(\frac{\xi}{2N}\right) - \psi\left(\frac{4\xi}{N}\right)}_{=: \psi\left(\frac{\xi}{N}\right)} \right) \quad \psi(\xi) = \psi\left(\frac{\xi}{2}\right) - \psi(4\xi) \\ &= N^s \underbrace{\left| \frac{\xi}{N} \right|^s \psi\left(\frac{\xi}{N}\right)}_{\text{smooth}} \end{aligned}$$

- Let $K_s = \mathcal{F}^{-1} \left(|\xi|^s \psi(\xi) \right)$

supported away from $\xi = 0$

$\Rightarrow |\xi|^s \psi(\xi)$ is smooth (not smooth at $\xi = 0$
but $\psi(0) = 0$)

$\Rightarrow K_s$ decays rapidly

$\Rightarrow \|K_s\|_{L_x^1} < \infty$

$$\begin{aligned} \mathcal{F}^{-1}\left(\left|\frac{x}{N}\right|^s \psi\left(\frac{x}{N}\right)\right) &= \mathcal{F}^{-1}\left(\widehat{K}_s\left(\frac{x}{N}\right)\right) \\ &= N^d K_s(Nx) =: K_{N,s}(x) \end{aligned} \quad (5)$$

• Note: $\|K_{N,s}\|_{L_x^1} = N^d \int |K_s(Nx)| dx = \|K_s\|_{L_x^1} < \infty$

$$\begin{aligned} \Rightarrow \|P_N |\nabla|^s f\|_{L^p} &= N^s \|K_{N,s} * (P_N f)\|_{L^p} \\ &\stackrel{\text{Young}}{\leq} N^s \|K_{N,s}\|_{L_x^1} \|P_N f\|_{L^p} \\ &\lesssim N^s \|P_N f\|_{L^p} \\ &\underset{\text{indep of } N.}{\square} \end{aligned}$$

This proves (LHS) \lesssim (RHS). The other inequality follows in a similar manner.

or Let $g = |\nabla|^s f$, then $f = |\nabla|^{-s} g$

and by what we proved

$$\begin{array}{c} \|P_N |\nabla|^{-s} g\|_{L^p} \lesssim N^{-s} \|P_N g\|_{L^p} \\ \parallel \\ \|P_N f\|_{L^p} \qquad \qquad \qquad N^{-s} \|P_N |\nabla|^s f\|_{L^p} \end{array}$$

(6)

$$(i) \quad \underset{\substack{s > 0 \\ 1 < p < \infty}}{P_{\leq N} |\nabla|^s f} = \sum_{\substack{M \leq N \\ \text{dyadic}}} P_M |\nabla|^s f.$$

$$\| P_{\leq N} |\nabla|^s f \|_{L^p} \leq \sum_{M \leq N} \| P_M |\nabla|^s f \|_{L^p} \stackrel{(ii)}{\leq} c_s \sum_{M \leq N} M^s \| P_M f \|_{L^p}$$

$$\left(\sum_{M \leq N} M^s = \sum_{j=0}^{\infty} (N 2^{-j})^s = N^s \sum_{j=0}^{\infty} 2^{-js} \right) \underset{s > 0}{\approx} N^s$$

$$\lesssim N^s \sup_{\substack{M \text{ dyadic} \\ M \leq N}} \| P_M f \|_{L^p} \leq N^s \| \left(\sum_{M \leq N} |P_M f|^2 \right)^{1/2} \|_{L^p}$$

$\xrightarrow{\text{Mink}}$

$$\underset{1 < p < \infty}{\sim} N^s \| P_{\leq N} f \|_{L^p}.$$

(One can argue directly as in (ii) and prove (i))



(6.4) Back to dispersive estimate

(7)

Goal: Prove

$$\text{(*)} \quad \|S(t)P_N f\|_{L_x^\infty(\mathbb{R}^d)} \leq \frac{C}{|t|^{d/2}} \|P_N f\|_{L_x^1}, \quad \forall N, \text{dyadic}$$

(*) is enough to go through the proof of the Strichartz estimates.

• By unitarity, $\|S(t)P_N f\|_{L_x^2} = \|P_N f\|_{L_x^2}$

interpolation

$$\text{(**)} \quad \|S(t)P_N f\|_{L_x^p} \lesssim \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \|P_N f\|_{L_x^{p'}}, \quad 2 \leq p \leq \infty$$

For $p < \infty$, $\|S(t)f\|_{L_x^p} \stackrel{\text{theory}}{\sim} \left\| \left(\sum_N |S(t)P_N f|^2 \right)^{1/2} \right\|_{L_x^p}$

$$= \|S(t)P_N f\|_{L_x^p L_N^2}$$

Mink

$$\stackrel{p \geq 2}{\leq} \|S(t)P_N f\|_{L_N^2 L_x^p} \stackrel{**}{\sim} \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \|P_N f\|_{L_N^2 L_x^{p'}}$$

(8)

$$\underset{p' \leq 2}{\stackrel{\text{Mink}}{\leq}} \frac{1}{|t|^{d(\frac{1}{2} - \frac{1}{p})}} \| P_N f \|_{L_x^{p'} l_N^2} \stackrel{\text{theory}}{\sim} \frac{1}{|t|^{d(\frac{1}{2} - \frac{1}{p})}} \| f \|_{L_x^{p'}}.$$

Strichartz estimates.

It remains to prove \star .

$$\begin{aligned} S(t) P_N f &= S(t) \widehat{P}_N P_N f = \widehat{P}_N S(t)(P_N f) \\ &= (\widehat{P}_N K_t) * (P_N f) \quad S(t)f = K_t * f. \end{aligned}$$

$$K_t(x) = \int_{\mathbb{R}^d} e^{i(-4\pi|\xi|^2 t + 2\pi i \omega \cdot \xi)} d\xi$$

$$\Rightarrow \|S(t) P_N f\|_{L_x^\infty} \leq \|\widehat{P}_N K_t\|_{L_x^\infty} \|P_N f\|_{L_x'^1} \quad \frac{1}{\infty} + 1 = \frac{1}{\infty} + \frac{1}{1}$$

Goal: $\|\widehat{P}_N K_t\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}}$

$$\underline{\text{Goal}}: \|\tilde{P}_N K_t\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}}, \quad \tilde{P}_N = P_{N/2} + P_N + P_{2N}$$

$$K_t(x) = \int_{\mathbb{R}^d} e^{i(-4\pi^2 |\xi|^2 t + 2\pi i x \cdot \xi)} d\xi$$

$$\text{s.t. } \tilde{P}_N P_N = P_N$$

For simplicity, drop \sim in \tilde{P}_N

$$\Rightarrow P_N K_t(x) = \int_{\mathbb{R}^d} e^{i(-4\pi^2 |\xi|^2 t + 2\pi i x \cdot \xi)} \psi\left(\frac{\xi}{N}\right) d\xi$$

Case 1: $t \lesssim N^{-2}$

By change of var: $t^{1/2} \xi \mapsto \xi$.

$$\psi(\cdot) = \varphi(\cdot) - \varphi(2\cdot)$$

\uparrow smooth bump func supported

$$|P_N K_t(x)| = \frac{1}{|t|^{d/2}} \left| \int_{\mathbb{R}^d} e^{i(-4\pi^2 |\xi|^2 + 2\pi \left(\frac{x}{t^{1/2}}\right) \cdot \xi)} \psi\left(\frac{\xi}{t^{1/2} N}\right) d\xi \right|$$

$$\lesssim \frac{1}{|t|^{d/2}}$$

$$\uparrow |\xi| \lesssim 1$$

$$t^{1/2} N \lesssim 1 \text{ and } |\xi| \sim t^{1/2} N \lesssim 1$$

case 2 : $t \gg N^{-2}$

$$\bar{z} = \bar{z}/N$$

(2)

$$P_N K_t(x) = N^d \int_{\mathbb{R}^d} e^{2\pi i N x \cdot \bar{z}} e^{-4\pi^2 i t N^2 |\bar{z}|^2} \underbrace{\psi(\bar{z})}_{\text{radial}} d\bar{z}$$

$= \psi(|\bar{z}|)$

Subcase 2.a : $|x| \ll tN$

By polar coord,

$$P_N K_t(x) = N^d \int_{S^{d-1}} \int_0^\infty e^{2\pi i N r x \cdot \omega} e^{-4\pi^2 i t N^2 r^2} \underbrace{\psi(r)}_{i t N^2 \phi(r)} r^{d-1} dr d\sigma(\omega)$$

$\hookrightarrow r \sim 1 \quad (\frac{1}{2} < r < 2)$

$\bar{z} = r\omega, \quad r \geq 0$
 $\omega \in S^{d-1}$

Note

$$\phi(r) = 2\pi r \frac{x}{tN} \cdot \omega - 4\pi^2 r^2$$

$$\Rightarrow |\phi'(r)| = |2\pi \left(\frac{x}{tN}\right) \cdot \omega - 8\pi^2 r| \sim 1.$$

$|x| \ll 1$

\Rightarrow By IBP k times (no bdry terms ψ has cpt supp)

$$|P_N K_t(x)| \lesssim \frac{N^d}{(tN^2)^k} \lesssim \frac{1}{|t|^{d/2}}$$

$k = \frac{d}{2}$ if d is even
 $k = \frac{d+1}{2}$ if d is odd.

(3)

Subcase 2-b: $|x| \gtrsim +N$

radial function on \mathbb{R}^d : $f(x) = f(|x|)$

FACT: f radial. Then,

$$\hat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{d-2}{2}}} \int_0^\infty f_0(r) \underbrace{J_{\frac{d-2}{2}}(2\pi|\xi|r)}_{J_\nu} r^{d/2} dr$$

J_ν = Bessel function of order ν

See Appendix B of Grafakos

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + \underbrace{\mathcal{O}(r^{-3/2})}_{\text{error}}, \quad r \rightarrow \infty$$

$$\Rightarrow P_N K_t(x) = \frac{N^{\frac{d+2}{2}}}{|x|^{\frac{d-2}{2}}} \int_{r \sim 1} e^{-4\pi^2 i t N^2 r^2} \psi(r) J_{\frac{d-2}{2}}(2\pi N|x|r) r^{d/2} dr$$

(4)

$$\begin{aligned}
 \text{error term :} &\lesssim \frac{N^{\frac{d+2}{2}}}{|x|^{\frac{d-2}{2}}} \cdot \frac{1}{(N|x|)^{3/2}} = \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d+1}{2}}} \cdot \frac{1}{N} \stackrel{|x| \gtrsim N}{\lesssim} \frac{1}{t^{\frac{d+1}{2}}} \frac{1}{N} \\
 &\stackrel{t \gg N^{-2}}{\sim} \frac{1}{t^{d/2}}.
 \end{aligned}$$

Main contribution :

$$I_t^\pm(x) = \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} \int_{r \sim 1} e^{-4\pi^2 i \frac{t+N^2 r^2}{t+N^2} r} e^{\pm 2\pi i N|x|r} \underbrace{\psi(r) r^{\frac{d-1}{2}} dr}_{\text{smooth func.}}$$

$$\begin{cases} \text{Let } \Phi(r) = -4\pi^2 r^2 \pm 2\pi \frac{N^{\frac{d+1}{2}}}{t+N^2} r \\ \Rightarrow |\Phi''(r)| \gtrsim 1 \end{cases}$$

Cor to Vander Corput

$$\lesssim \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} \cdot \frac{1}{(t+N^2)^{1/2}} \stackrel{|x| \gtrsim N}{\lesssim} \frac{1}{|t|^{d-1/2}} \cdot N^{\frac{1}{2}} \frac{1}{t^{1/2} N} = \frac{1}{t^{d/2}}$$

□

Rmk: dispersive estimate (with P_N) for the wave egn ⑤
 can be obtained in a similar but simpler manner.

only need $\widehat{d\sigma}(\vec{z}) = \int_{S^{d-1}} e^{-2\pi i \vec{z} \cdot \omega} d\sigma(\omega) = \frac{2\pi}{|\vec{z}|^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(2\pi |\vec{z}|)$

See Monica Visan's lec note
 (Oberwolfach).

6.5 Maximal function estimate

Prop: $\|S(t)f\|_{\underline{L_x^4 L_t^\infty}} \lesssim \|D^{\frac{1}{4}} f\|_{\underline{L_x^2}}, \quad d=1, D=|\partial_x|$

$$\left(\|D^{-\frac{1}{2}} \int_0^t S(t-t') F(t') dt'\right)_{\underline{L_x^4 L_t^\infty}} \lesssim \|F\|_{\underline{L_x^{4/3} L_t^1}}$$

$$\|S(t)f(x)\|_{L_t^\infty} = \sup_{t \in \mathbb{R}} |S(t)f(x)| \leftarrow \text{maximal function}$$

Q: $U(t, x) = S(t)f(x) \rightarrow f(x)$ a.e.? ⑥

= Carleson '80, $f \in H^{1/4}(\mathbb{R})$

Dahlberg - Kenig '82, false if $s < \frac{1}{4}$

Sjölin - Vega '85, $s > \frac{1}{2}$, $\forall d$

- Bourgain, Vargas - Vega, Tao - V-V, T - Vargas '90's

- Sanghyuk Lee '06: $d = 2, s > \frac{3}{8}$

- Bourgain '12: $d \geq 3$ suff. cond $s > \frac{1}{2} - \frac{1}{4d}$
 $d \geq 4$ nec. cond $s \geq \frac{1}{2} - \frac{1}{2d}$

- Luca - Rogers '15, Bourgain

Also, can study Hausdorff dimensions of bad sets.

Pf of a.e. conv ($d=1, s \geq \frac{1}{4}$)

$$\limsup_{t \downarrow 0} |S(t)f(x) - f(x)| \leq \sup_t |S(t)(f(x) - g(x))|$$

$$+ |f(x) - g(x)| \quad g, \text{ smooth.}$$

$$+ \limsup_{t \downarrow 0} \cancel{|S(t)g(x) - g(x)|}$$

$$= 0 \iff S(t)g(x) \rightarrow g(x), \forall x$$

(7)

Given $\alpha > 0$, let

$$A = \{x \in \mathbb{R} : \limsup_{t \downarrow 0} |s(t)f(x) - f(x)| > \alpha\}$$

$$\subset \left\{ x \in \mathbb{R} : \sup_t |s(t)(f(x) - g(x))| > \frac{\alpha}{2} \right\}$$

$$\cup \{ |f(x) - g(x)| > \frac{\alpha}{2} \} =: B \cup C$$

$$\boxed{|B| = \int_x \mathbb{1}_B(x) dx \leq \int_x \left(\frac{2}{\alpha} \sup_t |s(t)(f(x) - g(x))| \right)^p dx}$$

↑ Chebychev's ineq

$$|A| \leq |B| + |C| \leq \left(\frac{2}{\alpha}\right)^4 \left\| \sup_t |s(t)(f(x) - g(x))| \right\|_{L_x^4}^4$$

$$+ \left(\frac{2}{\alpha}\right)^4 \|f(x) - g(x)\|_{L_x^4}^4$$

$$\stackrel{\text{Max. func est.}}{\leq} C \left(\frac{2}{\alpha}\right)^4 \|f - g\|_{\dot{H}^{\frac{1}{4}}(\mathbb{R})}^4$$

$$\lesssim \frac{\varepsilon}{\alpha^4} \quad < \varepsilon$$

$$\dot{H}^{\frac{1}{4}}(\mathbb{R}) \subset L^4(\mathbb{R})$$

True for any $\varepsilon > 0 \Rightarrow |A_\alpha| = 0 \quad \square$

Prop (maximal func estimate) $S(t) = e^{it\partial_x^2}$

$$\begin{aligned} \textcircled{\ast} \quad \|S(t)f\|_{L_x^4 L_t^\infty} &\lesssim \|D^{\frac{1}{4}} f\|_{L_x^2}, \quad d=1. \\ &= \|f\|_{\dot{H}_x^{\frac{1}{4}}(\mathbb{R})}, \quad D = |\partial_x| \sim |\Im| \end{aligned}$$

Pf: $\textcircled{\ast} \iff$

$$\left\| \int_{\mathbb{R}} D^{-\frac{1}{4}} S(-t) G(t) dt \right\|_{L_x^2} \lesssim \|G\|_{L_x^{4/3} L_t^1}$$

$$T = \underline{D^{-\frac{1}{4}} S(t)}, \quad T^* G$$

Want: $T: L_x^2 \rightarrow L_x^4 L_t^\infty$

$$\textcircled{**} \iff \left\| \int_{\mathbb{R}} D^{-\frac{1}{2}} S(t-t') G(t') dt' \right\|_{L_x^4 L_t^\infty} \lesssim \|G\|_{L_x^{4/3} L_t^1}$$

$$\underline{\text{Lemma}} : \left| \int_{\mathbb{R}} e^{i(x\zeta - t\zeta^2)} \frac{1}{|\zeta|^{\frac{1}{2}}} d\zeta \right| \lesssim \frac{1}{|x|^{\frac{1}{2}}} \quad (2)$$

Assuming Lemma, we prove $\star\star$.

implicit const indep of t .

$$\left\| \int_{\mathbb{R}} D^{-\frac{1}{2}} S(t-t') G(t) dt' \right\|_{L_x^4 L_t^\infty} \quad S(t)f = K_t *_x f$$

Young

$$\lesssim \left\| \int_y \left\| D^{-\frac{1}{2}} K(\cdot, x-y) \right\|_{L_t^\infty} \times \left\| G(\cdot, y) \right\|_{L_t^1} dy \right\|_{L_x^4}$$

$$\text{Young: } \frac{1}{\infty} + 1 = \frac{1}{\infty} + 1$$

Lemma

$$\lesssim \left\| \frac{1}{|x|^{\frac{1}{2}}} *_x \left\| G(t, \cdot) \right\|_{L_t^1} \right\|_{L_x^4} \quad \text{H-L-S: } \frac{1}{4} + 1 = \frac{3}{4} + \frac{1}{2}$$

$$\lesssim \|G\|_{L_x^{4/3} L_t^1}$$

$$\left\| \frac{1}{|x|^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R})} = \infty \text{ but H-L-S works}$$



(3)

Pf of Lemma :

$$\textcircled{1} \quad |\xi| \lesssim |x|^{-1}.$$

$$(\text{LHS}) \lesssim |\xi|^{\frac{1}{2}} \int_0^{|x|^{-1}} = \frac{1}{|x|^{\frac{1}{2}}}.$$

$$\textcircled{2} \quad |\xi| \gg |x|^{-1}. \quad \text{We only consider } |\xi| \gtrsim |x|^{-1}.$$

$$\text{Change of var: } \xi^2 = y$$

$$\textcircled{+} \quad (\text{LHS}) \sim \int e^{ixy^{\frac{1}{2}} - ity} \frac{1}{y^{\frac{3}{4}}} dy. \quad y \gtrsim |x|^{-2}.$$

change of viewpoint: view t as a spatial var
 x as a temporal var. $\rightarrow i\partial_t u = D^{\frac{1}{2}} u$

$$\text{Let } \phi(y) = y^{\frac{1}{2}}$$

$$\phi'(y) \sim y^{-\frac{1}{2}}$$

$$\phi''(y) \sim y^{-\frac{3}{2}}$$

$$x\phi(y) - ty = x \left(\underbrace{\phi(y)}_{= \Phi} - \frac{t}{x} y \right)$$

$$\textcircled{2.a} \quad \Omega_1 = \left\{ y \geq |x|^{-2} : \left| \phi'(y) - \frac{t}{x} \right| \leq \left| \frac{t}{2x} \right| \right\} \quad \textcircled{4}$$

$$\Rightarrow \phi'(y) \sim \frac{t}{x} \quad \rightarrow \quad y \sim \left(\frac{x}{t} \right)^2$$

\downarrow
 $y^{-1/2}$

$$\textcircled{+} \sim \int e^{i \underbrace{\frac{x \min |\Phi''|}{2}}_{\downarrow} \cdot \boxed{\frac{\Phi}{\min |\Phi''|}} \cdot |\phi''|^{1/2} d\gamma}$$

$$\Phi(\gamma) = \phi(\gamma) - \frac{t}{x}\gamma$$

$$\begin{aligned} \Phi''(\gamma) &= \phi''(\gamma) \\ &\sim \gamma^{-3/2} \end{aligned}$$

Cor to van der Corput

$$()'' \geq 1$$

$$\phi''' \sim \gamma^{-5/2}$$

$$\begin{aligned} &\lesssim (x \cdot \min |\Phi''|)^{-1/2} \left\{ \underbrace{\max |\phi''|^{1/2}}_{\sim \left(\frac{t}{x} \right)^{3/2}} + \int_{y \sim \left(\frac{x}{t} \right)^2} |\phi''|^{-1/2} |\phi'''| d\gamma \right\} \\ &\sim \left(x \cdot \frac{t^3}{x^3} \right)^{-1/2} \quad \sim \left(\frac{t}{x} \right)^{3/2} \quad \sim \left(\frac{x}{t} \right)^{3/2} \left(\frac{t}{x} \right)^5 \left(\frac{x}{t} \right)^2 = \left(\frac{t}{x} \right)^{3/2} \\ &\sim x/t^{3/2} \end{aligned}$$

$$\sim \frac{1}{|x|^{1/2}}$$

(5)

$$②.b \quad \Omega_2 = \left\{ y \gtrsim |x|^{-2} : |\phi'(y) - \frac{t}{x}| > \left| \frac{t}{2x} \right| \right\}.$$

$$\Rightarrow |\phi' - \frac{t}{x}| \stackrel{\text{triangle}}{\geq} |\phi'| - \left| \frac{t}{x} \right| \\ \stackrel{\text{on } \Omega_2}{>} |\phi'| - 2 \left| \phi' - \frac{t}{x} \right|$$

$$\Rightarrow \underbrace{\left| \phi' - \frac{t}{x} \right|}_{= |\Phi'|} \gtrsim |\phi'| \sim y^{-1/2}$$

$$\int e^{i \underbrace{x \Phi(y)}_{\Psi(y)}} \frac{1}{y^{3/4}} dy = \int \partial_y e^{i \Psi(y)} \frac{|\phi''|^{1/2}}{i \partial_y \Psi(y)} dy$$

$$\stackrel{\text{IBP}}{\leq} \int \left| \partial_y \left(\frac{|\phi''|^{1/2}}{\partial_y \Psi(y)} \right) \right| dy$$

$$\lesssim \frac{1}{|x|} \int \underbrace{\frac{1}{|\phi''|^{1/2}} \frac{|\phi'''|}{\left| \phi' - \frac{t}{x} \right|}}_{\sim y^{-5/4}} + \underbrace{\frac{|\phi''|^{3/2}}{\left| \phi' - \frac{t}{x} \right|^2}}_{\sim y^{-5/4}} dy$$

$$\sim \frac{-1}{|x|} y^{-\frac{1}{4}} \Big|_{|x|^{-2}}^{\infty} = \frac{1}{|x|^{1/2}} \quad \square$$

$$\phi'' \sim y^{-3/2} \\ \phi''' \sim y^{-5/2}$$

(6)

6.6 Local smoothing estimate

Prop (local smoothing) $d=1$

$$\|D^{\frac{1}{2}}S(t)f\|_{L_x^\infty L_t^2} \sim \|f\|_{L_x^2} \quad S(t) = e^{it\partial_x^2}$$

Note: $S(t)$ is an isometry on H^s & time reversible
 \Rightarrow No global gain of differentiability.

Pf:

$$\begin{aligned} D^{\frac{1}{2}}S(t)f(x) &= \int |\xi|^{1/2} e^{ix\xi - it\xi^2} \hat{f}(\xi) d\xi & y = \xi^2 \\ &\sim \int |y|^{-1/4} e^{ixy^{1/2} - ity} \hat{f}(y^{1/2}) dy & \frac{1}{2} \frac{1}{y^{1/2}} dy = d\xi \end{aligned}$$

$$\Rightarrow \|D^{\frac{1}{2}}S(t)f(x)\|_{L_t^2} \underset{\text{Plancherel int}}{\sim} \|\int |y|^{-1/4} e^{ixy^{1/2}} \hat{f}(y^{1/2}) dy\|_{L_y^2}$$

$$= \left(\int \frac{1}{|y|^{1/2}} |\hat{f}(y^{1/2})|^2 dy \right)^{1/2} \underset{\substack{\text{undo} \\ \text{ch of var}}}{\sim} \left(\int |\hat{f}(\xi)|^2 d\xi \right)^{1/2} = \underset{\substack{\text{Plancherel} \\ \text{in } x}}{\|f\|_{L_x^2}}$$



Rmk: • Local smoothing estimate & maximal func

(7)

estimate are used in pair to study NLS
with a derivative nonlinearity.

- \exists proof without using F.T. (short note by Tao)
"A physical space proof of --";

- For $d \geq 1$, local smoothing est:

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} |\nabla|^{1/2} S(t) f(x)|^2 e^{-|x|^2} dx dt \right)^{1/2} \lesssim \|f\|_{L_x^2}$$

spatial localization

\Leftarrow See Vişan's lec note. (Also, course note from '16.)

- See the course note from '16 for LPW of gKdV.

$$\partial_t u + \partial_x^3 u = \partial_x u^5,$$

Sec 7 : Global-in-time behavior of solns to NLS

Strichartz est

\Rightarrow LWP of NLS on \mathbb{R}^d if $s \geq \max(\frac{1}{2}, 0)$

(if $p \in 2\mathbb{N} + 1$. Otherwise, need extra cond.)

Q1: Does the soln exists globally in time (Global well-posedness)

or does it cease to exist at some finite time?
(finite time blowup soln)

Q2: If u exists globally in time, then

what is the behavior of the soln u as $t \rightarrow \pm\infty$?

• Scattering: "asymptotic linear behavior" $\lim_{t \rightarrow \pm\infty} \|u(t)\|_{L_x^\infty} \rightarrow 0$

$$\exists u_\pm \in H^s(\mathbb{R}^d) \text{ s.t. } \lim_{t \rightarrow \pm\infty} \underbrace{\|u(t) - S(t)u_\pm\|_{H^s}}_{\text{lin soln}} = 0$$

$$\lim_{t \rightarrow \pm\infty} u(t) = S(t)u_0$$

- non-scattering soln such as solitons:

(2)

$$U(t) = e^{it} \underbrace{Q(x)}_{\text{indep of time}}$$



$$\text{In particular, } \|U(t)\|_{L_x^\infty} = \|Q\|_{L_x^\infty} \not\rightarrow 0.$$

Conjecture: soliton resolution conjecture.

For "generic" initial data, $U(t)$ decouples into a sum of solitons + radiation (= scattering part) as $t \rightarrow \pm\infty$.

still open: except for "integrable equations" such as KdV and NLW (radial, Kenig-Merle et al. '12~).

- We discussed the conservation of

(3)

$$\text{Mass : } M(u) = \int |u|^2 dx$$

$$\text{Momentum : } P(u) = \text{Im} \int \bar{u} \nabla u \quad \leftarrow \text{not sign definite}$$

Hamiltonian / Energy :

$$H(u) = \frac{1}{2} \int |\nabla u|^2 dx \pm \frac{1}{p+1} \int |u|^{p+1} dx$$

$$(\Leftarrow i \partial_t u + \Delta u = \pm |u|^{p-1} u)$$

- \pm ... defocusing case / repulsive case

- \pm ... focusing case / attractive case

ex: 1-d cubic NLS, $\text{scrit} = -\frac{1}{2}$ ④

LWP in $L^2(\mathbb{R})$ in the subcritical sense ($T \sim \|u_0\|_{L^2}^{-\theta}$)
 mass cons
 \Rightarrow GWP in $L^2(\mathbb{R})$

but not on \mathbb{R}^2 : $\text{scrit} = 0$
 T was given s.t. $\|\delta(t)u_0\|_{L_T^4 L_x^4} \ll 1$.
 GWP in $L^2(\mathbb{R}^2)$ holds true but the proof is much more complicated. Dodson '12?

• 3-d cubic NLS (defocusing): $\text{scrit} = \frac{1}{2}$

HW: LWP in $H^1(\mathbb{R}^3)$ in the subcritical sense ($s=1 > \frac{1}{2}$)

$$\|u(t)\|_{H^1}^2 \leq \int |u|^2 dx + \int |\nabla u|^2 dx + \frac{2}{4} \int |u|^4 dx$$

$$= M(u(t)) + 2H(u(t)) \underset{\text{'cons.'}}{\circlearrowleft} M(u_0) + 2H(u_0) < \infty$$

(5)

\Rightarrow GWP in $H^1(\mathbb{R}^3)$. \leftarrow We'll prove scattering.

- NOT true in the focusing case
 - GWP (and scattering) in $H^{1/2}(\mathbb{R}^3)$ is open.
-

• We say NLS is

$$S_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1}$$

$$\begin{cases} \text{mass-critical if } S_{\text{crit}} = 0. & p = 1 + \frac{4}{d} \quad \begin{matrix} 1-d, \text{quintic} \\ 2-d, \text{cubic} \end{matrix} \\ \text{mass-subcritical if } S_{\text{crit}} < 0 & (\text{ex: 1-d cubic NLS}) \\ \text{mass-supercritical if } S_{\text{crit}} > 0. & \end{cases}$$

$$\begin{cases} \text{energy-critical if } S_{\text{crit}} = 1 & \frac{d \geq 3}{p = 1 + \frac{4}{d-2}} : \begin{matrix} 3-d: \text{quintic} \\ 4-d: \text{cubic} \end{matrix} \\ \text{energy-subcritical if } S_{\text{crit}} < 1 & (\text{ex: 3-d cubic}) \\ \text{energy-supercritical if } S_{\text{crit}} > 1 & \end{cases}$$

(6)

- energy-critical defocusing NLS

GWPs & scattering: Bourgain '99, CKSTT '08, Vigan

- energy-supercritical defocusing NLS

LWP in $\dot{H}^{\text{crit}}(\mathbb{R}^d)$ but GWP is open even for smooth solns.

(analogous to Navier-Stokes eqn: $\dot{H}^{1/2}(\mathbb{R}^3)$)

but only "energy" $\int |u|^2 dx$ is conserved

too weak to control

$\dot{H}^{1/2}$ -norm.

(7.1)

Solitons & finite time blowup solutions

Consider the focusing NLS:

$$i\partial_t u + \Delta u = -|u|^{p-1}u$$

Solitons (solitary wave solution)

$$u(t, x) = e^{it} \underbrace{\phi(x)}_{\text{profile}}$$

- Such u solves (NLS) iff ϕ solves the following elliptic PDE: ⑦

$$\textcircled{*} \quad \Delta \phi - \phi + |\phi|^{p-1} \phi = 0, \quad \phi \in H^1(\mathbb{R}^d)$$

FACT: $d=1$: all solns to $\textcircled{*}$ are translates of

Raphaël's
clay lec note.

$$Q(x) = \left(\frac{p+1}{2 \cosh^2 \frac{(p-1)x}{2}} \right)^{p-1}$$



$d \geq 2$: \exists seq $\{Q_n\}_{n \geq 0}$ of real radial solns to $\textcircled{*}$

with increasing L^2 -norms s.t.

$Q_n(r)$ vanishes n times on \mathbb{R}_+

• Q_0 , radially sym, pos (ground state)

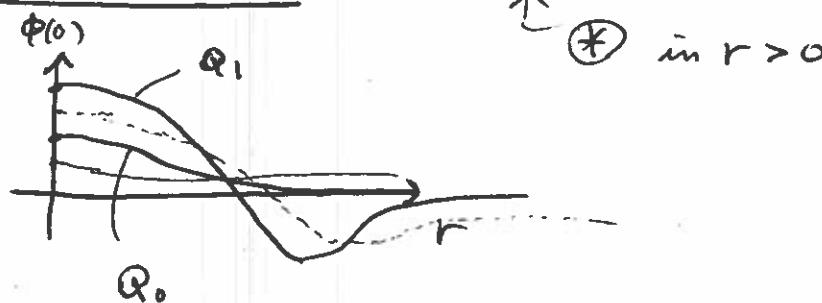
uniqueness: $\phi > 0, \phi \in H^1$

radial, C^2 , exp decaying (Gidas - Ni - Nirenberg '79
Kwong '89)

existence:

Berestycki - Lions - Peletier
'81

- shooting method (on ODE, $r > 0$)



- Ground states play an important role in elliptic, parabolic, dispersive PDEs
variational problem
functional inequality, etc.

mass - subcritical : $p < 1 + \frac{4}{d}$. $\text{Scrit} < 0$

$$\text{NLS scaling : } Q^{\lambda}(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$$

Prop: (variational characterization of Q). $d \geq 1$, $1 < p < 1 + \frac{4}{d}$
 $M > 0$ fixed. Then, the minimization problem

$$\min_{\|u\|_{L^2} = M} H(u)$$

(9)

has min attained at

$$Q^{\text{ADM}}(\cdot - x_0) e^{i\gamma_0} \quad \text{for all } x_0 \in \mathbb{R}^d, \gamma_0 \in \mathbb{R}$$

↑ rescale of Q s.t. $\|Q^{\text{ADM}}\|_{L^2} = M$

$\text{Scrit} < 0$.

Scaling preserves H^{Scrit} -norm but not L^2 -norm

- minimization problem

← Lagrange problem : $\frac{d}{d\varepsilon} H(u + \varepsilon v) \Big|_{\varepsilon=0} \leftarrow \text{Gâteaux deriv.}$

↳ Euler-Lagrange eqn : $\Delta\phi - \lambda\phi + |\phi|^{p-1}\phi = 0$

↑ Lagrange multiplier.

• mass-critical case: $S_{\text{crit}} = 0$, $p = 1 + \frac{4}{d}$

$$\text{Let } J(u) = \frac{\left(\int |\nabla u|^2 \right) \left(\int |u|^{2/d} \right)^{2/d}}{\int |u|^{2+4/d}}, \quad u \neq 0$$

Prop: (i) $\min_{\substack{u \in H^1 \\ u \neq 0}} J(u)$ is attained at

$$\lambda_0^{d/2} Q(\lambda_0 x + x_0) e^{i\varphi_0}, \quad (\lambda_0, x_0, \varphi_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$$

\uparrow unique ground state

In particular, we have the following sharp Gagliardo-Nirenberg ineq:

$$\int |u|^{2+4/d} \leq \underline{J(Q)}^{-1} \int |\nabla u|^2 \left(\int |u|^{2/d} \right)^{2/d}$$

optimal const

$$(\Leftarrow J(Q) \leq J(u))$$

(2)

(ii) "Rigidity". Let $u \in H^1$ s.t.

$$\int |u|^2 = \int Q^2, \quad H(u) = 0 \quad (= H(Q)) \quad \begin{cases} \text{For mass-subcrit,} \\ H(Q) < 0 \end{cases}$$

Then, $u(x) = A_0^{d/2} Q(\lambda_0 x + x_0) e^{i\gamma_0}$

for some $(\lambda_0, x_0, \gamma_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$.

Rmk: $\nabla u \in H^1$

$$H(u) \geq \frac{1}{2} \int |\nabla u|^2 \left(1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}} \right)^{4/d} \right) dx$$

Let $u_0 \in H^1$ s.t. $\|u_0\|_{L^2} < \|Q\|_{L^2}$

$$H(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{2+4/d} \int |u|^{2+4/d}$$

By mass conservation,

$$H(u) \gtrsim \int |\nabla u|^2 dx$$

↑
indep of t (but depends on $\|u_0\|_{L^2}$)

$$\Rightarrow H(u) + M(u) \gtrsim \|u\|_{H^1}^2$$

\Rightarrow GWP in $H^1(\mathbb{R}^d)$, provided that $\|u_0\|_{L^2} \neq \|Q\|_{L^2}$ (3)
 (also scattering)

Note: mass-subcrit NLS is GWP in $H^1(\mathbb{R}^d)$
 (regardless of $\|u_0\|_{L^2}$)

• Q: $\|u_0\|_{L^2} = \|Q\|_{L^2}$?

- soliton: $u(t) = e^{it} Q$ is a global non-scattering soln.
- pseudo-conformal symmetry ($\text{Scrit} = 0$)

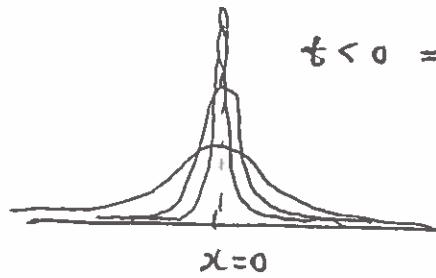
$$u(t, x) \xrightarrow{\quad} v(t, x) = \frac{1}{|t|^{d/2}} u\left(-\frac{1}{t}, \frac{x}{t}\right) e^{i \frac{|x|^2}{4t}}, (t \neq 0)$$

Apply PC symmetry to $e^{it} Q$

$$\Rightarrow Q^*(t, x) = \frac{1}{|t|^{d/2}} Q\left(\frac{x}{t}\right) e^{-i \frac{|x|^2}{4t} + \frac{i}{t}}$$

\uparrow

sln to (NLS) for $t < 0$.



$$t < 0 \Rightarrow t = 0^-$$

(4)

Q^* blows up at time $t=0$ (starting at $t=-1$)

- $\|Q^*(t)\|^2 \rightarrow \|Q\|_{L^2}^2 \delta_{x=0}$ as $t \nearrow 0$

- $\|\nabla Q^*(t)\|_{L^2} \sim \frac{1}{|t|} \Leftarrow \text{blowup speed}$

Q^* , not stable

$\Leftarrow Q^*$ is the minimal mass blowup soln

- unique (Merle '93)

(if $M(u) = M(Q)$ and u blows up in a finite time,
then $u = Q^*$ (up symmetry))

- Other finite time blowup solns?

$$M(Q) < M(u_0) < M(Q) + \varepsilon \quad \text{Merle-Raphaël '00's} \sim$$

"log log" blowup soln $\sim \sqrt{\frac{\log \log (T^*-t)}{T^*-t}}$

\Leftarrow stable

Rmk: slowest possible blowup speed $\gtrsim \frac{1}{\sqrt{T-t}}$ (by scaling) (5)

7.2 Virial identity & Morawetz estimate

viriel = "force"

$$i\partial_t u + \Delta u = \lambda |u|^{p-1}u, \quad \lambda = \pm 1$$

$$(\partial_t u = i\Delta u - i\lambda |u|^{p-1}u)$$

$\lambda = 1$, defocusing

$\lambda = -1$, focusing

Virial potential

$$V_a(t) = \int a(x) |u|^2 dx$$

↑
soln

$a(x)$ "nice" func

$$a(x) = |x|^2$$

$$a(x) = |x|$$

Compute $\partial_t^2 V_a(t)$.

See Lec note from '16

$$\Rightarrow \partial_t^2 V_a(t) = 4 \int \operatorname{Re} (\partial_k u \partial_j \bar{u}) \partial_k \partial_j a$$

$$+ 2\lambda \frac{p-1}{p+1} \int |u|^{p+1} \Delta a$$

use eqn.

*

$$- \int |u|^2 \Delta^2 a$$

Einstein's summation notation:
sum over repeated indices

ex: Virial identity: $a(x) = |x|^2 = \sum_{j=1}^d x_j^2$. $\Delta a = 2d$ (6)

$$\Delta^2 a = 0$$

$$\partial_k \partial_j a = 2 \delta_{jk} \quad \text{Kronecker delta}$$

• $V(t) = \int |x|^2 |u(t, x)|^2 dx$

(= variance if we view $|u(t)|^2 dx$ as a prob meas.)

$$\begin{aligned} \Rightarrow \partial_t^2 V(t) &= 8 \int |\nabla u|^2 + 4d \lambda \frac{p-1}{p+1} \int |u|^{p+1} \\ &= 16 H(u) + \frac{4Ad}{p+1} \left(p - \underbrace{\left(1 + \frac{4}{d} \right)}_{\text{mass-critical power}} \right) \int |u|^{p+1} \end{aligned}$$

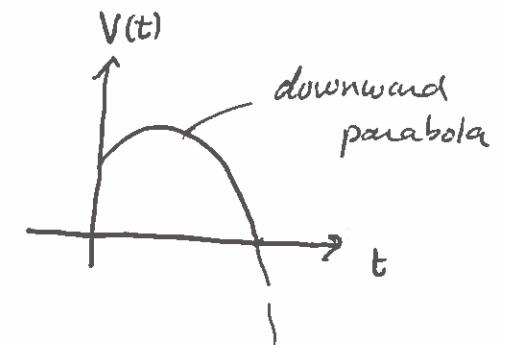
• mass-crit case: $\partial_t^2 V(t) = 16 H(u)$. \leftarrow conserved

If $H(u_0) < 0$, then $\partial_t^2 V(t) = 16 H(u_0) < 0$

• $V(t^*) < 0$ for some $t^* > 0$.

but $V(t) = \int |x|^2 |u(t, x)|^2 dx \geq 0$

$\Rightarrow u$ must blow up before time t^* (Glassey's argument,
Zakharov's



A similar argument works for mass-supercritical case.

(7)

($\lambda = -1$: focusing)

ex: Morawetz estimate : $a(x) = |x|$, $\partial_j a = \frac{x_j}{|x|}$

$$\Rightarrow \partial_j^2 a = \frac{1}{|x|} - \frac{x_j}{|x|^2} \cdot \frac{x_j}{|x|} \Rightarrow \Delta a(x) = \frac{d-1}{|x|}$$

$$\Rightarrow \dots \Rightarrow \Delta^2 a = -\frac{(d-1)(d-3)}{|x|^3} \leq 0 \text{ if } d \geq 3.$$

($\underline{d=3}$: $\frac{1}{|x|}$ = fundamental soln to $-\Delta \Rightarrow \Delta^2 a = -8\pi f$.

$$\begin{aligned} \Rightarrow \partial_t^2 V_a(t) &= \boxed{4 \int \frac{|t u|^2}{|x|} dx} \geq 0 \\ &\quad + 2 \lambda \frac{p-1}{p+1} \int \frac{|u|^{p+1}}{|x|} dx \\ &\quad - \underbrace{\int |u|^2 \Delta^2 a}_{\geq 0} \end{aligned}$$

$$\begin{aligned} |tu|^2 &= |\nabla u|^2 - \underbrace{\left| \frac{x}{|x|} \cdot \nabla u \right|^2}_{\text{angular component of grad.}} \\ &\quad \text{radial component} \end{aligned}$$

$$\Delta^2 a = -8\pi f \text{ when } d=3$$

(8)

On the other hand,

$$\partial_t^2 V_\alpha(t) = \partial_t \cdot 2 \operatorname{Im} \int \nabla u \cdot \frac{x}{|x|} \bar{u}$$

$\lambda = 1$ defocusing

$$\int_{t_1}^{t_2} \int \frac{|u|^{p+1}}{|x|} dx dt \lesssim \sup_{t=t_1, t_2} \left| \operatorname{Im} \int \nabla u \cdot \frac{x}{|x|} \bar{u} dx \right|$$

$$\lesssim \sup_{t_0, t_1} \|u(t)\|_{\dot{H}^{1/2}}^2$$

or

$$\lesssim_{C-S} M(u_0)^{1/2} H(u_0)^{1/2}$$

\Rightarrow Send $t_1 \rightarrow -\infty$

$t_2 \rightarrow +\infty$

Morawetz: $\int_{R_t} \int_{R_x^q} \frac{|u(t, x)|^{p+1}}{|x|} dx dt \lesssim \sup_{t+} \|u(t)\|_{\dot{H}^{1/2}}^2$

or $M(u_0)^{1/2} H(u_0)^{1/2}$ defocusing
case $\lambda = 1$

useful for radial case

- By repeating the derivation centered at y . ($d=3$) ⑨

$$\text{(*)} \quad \partial_t \operatorname{Im} \int \nabla u(x) \cdot \frac{x-y}{|x-y|} \bar{u}(x) dx = 2 \int \frac{|t \nabla_y u(x)|^2}{|x-y|} dx + 2 \frac{\alpha p-1}{p+1} \int \frac{|u(x)|^{p+1}}{|x-y|} dx \\ + 4\pi |u(y)|^2$$

- Multiply (*) by $|u(y)|^2$ and $\int \cdot dy$.

$$\Rightarrow \int_{\mathbb{R}^+} \int_{B_y^3} |u(y)|^4 dy dt \lesssim \sup_t \|u(t)\|_{L^2}^2 \|u(t)\|_{H^{\frac{1}{2}}}^2 \\ \text{or } M(u_0)^{3/2} H(u_0)$$

Interaction Morawetz estimate (Colliander - Keel - Staffilani - Takaoka - Tao med '00)

Lec 15 14/03/18 (Wed)

①

7.3

Scattering for energy-subcritical defocusing

cubic NLS on \mathbb{R}^3

(GWP in $H^1(\mathbb{R}^3)$)

We only consider $t \rightarrow +\infty$.

WTS: $\exists u_+ \in H^1(\mathbb{R}^3)$ s.t. $\|u(t) - S(t)u_+\|_{H^1} \rightarrow 0$ as $t \rightarrow +\infty$

$$\|S(-t)u(t) - u_+\|_{H^1}$$

$$S(-t)u(t) = u_0 - i \int_0^t S(-t')|u|^2 u(t') dt'$$

↓

u_+

⇒ Suffices to make sense (in H^1) of

$$\int_0^\infty S(-t)|u|^2 u(t) dt.$$

• Existence of wave operator: $\Omega_+ : u_+ \in H^1 \rightarrow u_0 \in H^1$. ②

Given $u_+ \in H^1$, can we find $u_0 \in H^1$ s.t.
the correxp soln u scatters to $S(t)u_+$?

Rmk: • If Ω_+ exists, it is injective

(\Leftarrow follows from the uniqueness part of WP theory)

• If Ω_+ is invertible, we say that we have asymptotic completeness
(= scattering as on page ①.)

$$u_+ = u_0 - i \int_0^\infty S(-t') |u|^2 u(t') dt'$$

$$S(-t) u(t) = u_0 - i \int_0^t S(-t') |u|^2 u(t') dt'.$$

Take a difference

✳

$$u(t) = \underbrace{S(t) u_+}_{\text{"value at ∞"}} + i \int_t^\infty S(t-t') |u|^2 u(t') dt'$$

Terminal value
problem

Pf of existence of wave op:

(3)

$$\underline{t = +\infty \rightarrow t = T \rightarrow t = 0}$$

① Well-posedness on $[T, \infty)$

$$\tilde{\mathcal{S}}' = L_{t,x}^5 \cap L_t^{\frac{10}{3}} W_x^{1, \frac{10}{3}}, \quad \left(\frac{10}{3}, \frac{10}{3}\right), \text{admissible}$$

$$\|u\|_{L_{t,x}^5} \stackrel{\text{Sob in } x}{\lesssim} \|u\|_{L_t^5 W_x^{1, \frac{30}{11}}}, \quad (5, \frac{30}{11}), \text{admissible}$$

By Strichartz,

$$\|S(t)u_+\|_{\tilde{\mathcal{S}}'(\mathbb{R}_t)} \lesssim \|u_+\|_{H^1} < \infty \quad \frac{11}{30} - \frac{1}{5} = \frac{1}{6} \leq \frac{1}{3}$$

\Rightarrow By MCT, $\exists T > 0$ s.t.

$$\|S(t)u_+\|_{\tilde{\mathcal{S}}'([T, \infty))} \leq \varepsilon.$$

Define $\tilde{\Gamma} u(t)$ by $\tilde{\Gamma} u(t) = (\text{RHS})$ of \circledast

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

$$\left(\frac{10}{3}, \frac{10}{3}\right), \text{admissible}$$

$$\frac{s}{d} \geq \frac{1}{q} - \frac{1}{p} : \|f\|_{L_x^p} \lesssim \|f\|_{W_x^{s,q}}$$

$$\Rightarrow \|\tilde{\Gamma}u\|_{\tilde{S}'([T, \infty))} \leq \varepsilon + C \underbrace{\|\langle \nabla \rangle(|u|^2 u)\|_{L_{t,x}^{10/7}([T, \infty))}}_{\lesssim \|u\|_{L_{t,x}^5}^2 \|\langle \nabla \rangle u\|_{L_{t,x}^{10/3}}}^{\frac{7}{10}} = \frac{1}{5} + \frac{1}{5} + \frac{3}{10}$$

$$\lesssim \|u\|_{L_{t,x}^5}^3 \|\langle \nabla \rangle u\|_{L_{t,x}^{10/3}}^3 \stackrel{(2\varepsilon)^3 \cdot C = \varepsilon}{\longrightarrow} \varepsilon$$

• Similarly,

$$\|\tilde{\Gamma}u - \tilde{\Gamma}v\|_{\tilde{S}'([T, \infty))} \leq C \left(\|u\|_{\tilde{S}'([T, \infty))}^2 + \|v\|_{\tilde{S}'([T, \infty))}^2 \right) \|u - v\|_{\tilde{S}'([T, \infty))}$$

$\Rightarrow \tilde{\Gamma}$ is a contraction on $\overline{B_{2\varepsilon}} \subset \tilde{S}'([T, \infty))$, $\varepsilon \ll 1$

• Now, apply LWP and the conservation of mass and Hamiltonian to extend u onto $[0, T]$



$$\|u(t)\|_{H^1} \lesssim (H(u(T)) + M(u(T)))^{1/2}$$

$$\Rightarrow \delta \sim (H(u(T)) + M(u(T)))^{-\theta}$$



- Morawetz: $[T, \infty)$: small data theory.
- $[0, T]$: iteration of LWP theory.

(5)

Scattering (asymptotic completeness)

$$\begin{aligned}
 & \left\| \int_0^\infty S(-t) |u| u(t) dt \right\|_{H^1} \stackrel{\text{dual str}}{\lesssim} \| \langle \nabla \rangle (|u|^2 u) \|_{L_{t,x}^{10/7}} \\
 & \lesssim \|u\|_{L_{t,x}^5}^2 \|u\|_{L_t^{10/3} W_x^{1, 10/3}} \\
 & \leq \underbrace{\left(\sup_{\substack{(q,r) \\ \text{admissible}}} \| \langle \nabla \rangle u \|_{L_t^q L_x^r} \right)^3}_{=: \|u\|_{S^1}} = \|u\|_{S^1}^3 \\
 \Rightarrow & \underbrace{\|u\|_{S^1(R_t)}}_{\text{strong space-time bound}} < \infty \text{ implies scattering.}
 \end{aligned}$$

Claim : "weak" space-time bound

(6)

$$\|u\|_{L^q_{t,x}} \lesssim 1 \quad \text{for some } q \in [\frac{10}{3}, 10]$$

implies "strong" space-time bound $\|u\|_{S^1} \lesssim 1$

(which in turn implies scattering.)

Rmk: Interaction Morawetz \Rightarrow "weak" space-time bd ($q=4$)

Morawetz estimate in the radial setting $\Rightarrow \|u\|_{L^5_{t,x}} \lesssim 1$

$$\leftarrow \text{Radial Sobolev ineq: } \||x|^s u\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \|u\|_{H^1}$$

$$\frac{d}{2}-1 \leq s \leq \frac{d-1}{2}, \quad u, \text{ radial}$$

$\left(\begin{array}{l} \leftarrow \text{localize around fixed} \\ \text{apply 1-d Gagliardo-Nirenberg (in } r) \end{array} \right)$

$$\iint_{\mathbb{R}^d} |u|^5 = \iint_{\mathbb{R}^d} |x| |u| \cdot \frac{|u|^4}{|x|} dx dt \stackrel{\substack{\text{rad Sob.} \\ \text{Morawetz}}} {\leq} \| |x| u \|_{L_{t,x}^\infty} \int_{\mathbb{R}^d} \frac{|u|^4}{|x|} dx \stackrel{\text{polar coord}}{\leq} C(\|u_0\|_{H^1})$$

Pf of Claim: Given $\varepsilon > 0$, divide $\mathbb{R}_+ = \bigcup_{j=1}^N I_j$ finite # (7)

s.t. $\|u\|_{L_t^q L_x^8} \leq \varepsilon$. ($\Leftarrow \|u\|_{L_t^q L_x^q(\mathbb{R}_+ \times \mathbb{R}_x^d)} < \infty$)

$$I_j = [t_j, t_{j+1})$$

$$u(t) = S(t-t_j)u(t_j) - i \int_{t_j}^t S(t-t')|u|^2 u(t') dt', \quad t \in I_j.$$

$$\Rightarrow \|u(t)\|_{S'(I_j)} \stackrel{\text{Str}}{\lesssim} \|u(t_j)\|_{H^1} + \|u\|_{L_{I_j}^5 L_x^5}^2 \|u\|_{L_{I_j}^{10/3} W_x^{1,10/3}}$$

$$\begin{aligned} \text{Note: } \|u\|_{L_{t,x}^{10/3}(I)} &\stackrel{\text{adm}}{\rightarrow} \sup_{(q,r)} \|u\|_{L_t^q L_x^r} \\ &\leq \|u\|_{S^0(I)}, \quad (\frac{10}{3}, \frac{10}{3}), \text{ adm} \\ &\leq \|u\|_{S'(I)} \end{aligned}$$

$$\|u\|_{L_{t,x}^{10}(I)} \stackrel{\text{Sob}}{\lesssim} \|u\|_{L_I^{10} W_x^{1,10/3}} \leq \|u\|_{S'(I)}. \quad \frac{13}{30} - \frac{1}{10} = \frac{1}{3} = \frac{s}{d}$$

(8)

By interpolation,

$$\|u\|_{L^5_{t,x}(I_j)} \leq \|u\|_{L^8_{t,x}(I_j)}^\theta \|u\|_{L^{10}_{t,x}(I_j)}^{1-\theta}$$

or

$$\|u\|_{L^q_{t,x}(I_j)}^\theta \|u\|_{L^{10/3}_{t,x}(I_j)}^{1-\theta}$$

$$\textcircled{B} \Rightarrow \|u\|_{L^5_{t,x}(I_j)} \leq \varepsilon^\theta \|u\|_{S'(I)}^{1-\theta}$$

$$\textcircled{A} \& \textcircled{B} \Rightarrow \|u\|_{S(I_j)} \lesssim \|u(t_j)\|_{H^1} + \varepsilon^{2\theta} \|u\|_{S'(I_j)}^{3-2\theta}$$

$\textcircled{C} \xrightarrow{\text{next page}}$
cont'd on next page

$$\Rightarrow \|u\|_{S'([0, \infty))} \lesssim 1$$

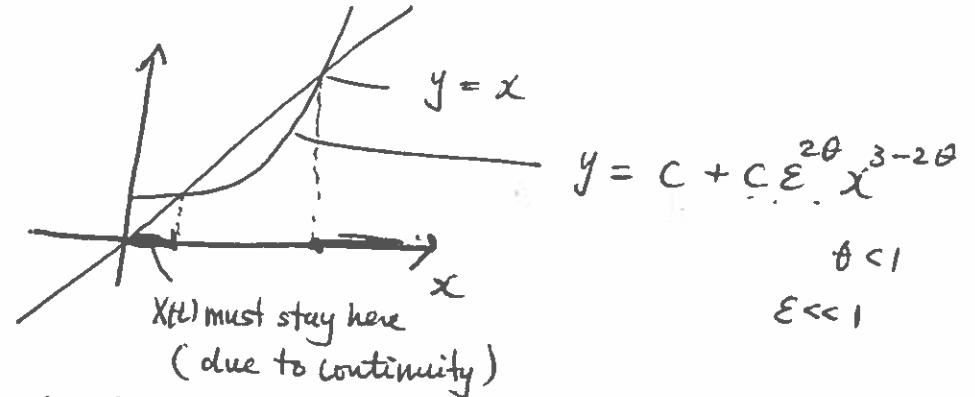
\Rightarrow scattering.

$$\textcircled{C} : X(t) = \|u\| S^1([t_j, t]) \quad t \in I_j \quad \textcircled{9}$$

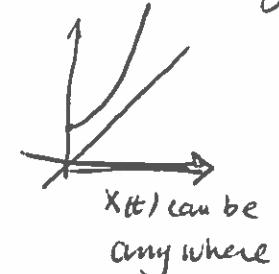
\textcircled{A} \& \textcircled{B} \Rightarrow

$$\textcircled{\ast\ast} \quad X(t) \approx 1 + \varepsilon^{2\theta} X(t)^{3-2\theta}$$

- $X(t)$, continuous in t . (but $X(t) \rightarrow 0$ as $t \rightarrow t_j$)
 - At $t = t_j$,
- $$X(t_j) \approx 1.$$
- $\Rightarrow \textcircled{\ast\ast}$ is satisfied
- $\Rightarrow X(t) \approx 1$ and $\textcircled{\ast\ast}$ is satisfied, $\forall t \in I_j$.



If ε is not small enough,



Sec 8: Ill-posedness of NLS in low regularities

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^{p-1} u = 0 \\ u|_{t=0} = u_0 \in H^s(M) \end{cases}, \quad M = \mathbb{R}^d \text{ or } \mathbb{T}^d$$

- Bad behaviors for $s \leq s_0$ (or $s < s_0$)
- ① failure of the nonlinear estimate.

lin est: $\left\| \int_0^t S(t-t') |u|^{p-1} u(t') dt' \right\|_{X_T^s} \lesssim \| |u|^{p-1} u \|_{N_T^s}$

X_T^s = solution space $\subset C_T H_x^s$

nonlin: $\| |u|^{p-1} u \|_{N_T^s} \lesssim \| u \|_{X_T^s}^p \leftarrow \text{This fails for } s < s_0$

ex: 2-d cubic NLS on \mathbb{R}^2 .

(2)

nonlin esti: $\| \langle \nabla \rangle^s (|u|^2 u) \|_{L_{T,x}^{4/3}} \lesssim \| \langle \nabla \rangle^s u \|_{L_{T,x}^4}^3, \quad s \geq 0$
 $\Leftarrow \underline{\text{fails for } s < 0}$

② failure of C^k -smoothness of the soln map:

$$\Phi: u_0 \in H^s \mapsto u \in C_T H^s$$

- If $p \in 2\mathbb{N} + 1$, the nonlin $|u|^{p-1} u$ is algebraic.
 $\Rightarrow \Phi$ is analytic if we can solve the fixed pt problem
 $\Gamma_{u_0}(u) = u$ by the standard contraction argument.
- failure of C^k -smoothness does not show ill-posedness
but says that we can not use a contraction argument.

(for example, one would need to use a more robust
energy method (say, in the context of the short-time
Fourier restriction norm method.))

③

③ failure of uniform continuity of the soln map.

(on bounded sets in H^s . i.e. local unif conti)

- same comment as in ②

- mild ill-posedness

④ failure of continuity of the soln map

- ill-posed.

⑤ failure of uniqueness or existence

(4)

Back to

③ failure of C^3 -smoothness of cubic NLS
on $H^s(\mathbb{T}^d)$, $s < 0$.

$$\textcircled{*} \quad \begin{cases} i\partial_t u + \Delta u \pm |u|^2 u = 0 \\ u|_{t=0} = \underbrace{\delta \phi}_{\text{(smooth)}} \end{cases}$$

for some $\underbrace{\phi \in H^s(\mathbb{T}^d)}_{\text{(smooth)}}$.

$\therefore u(t, x; \delta) = \text{soln to } \textcircled{*} \text{ with parameter } \delta \in \mathbb{R}$.

Note: $u(t, x; 0) \equiv 0$.

Let $\Phi(t) : u_0 \in H^s \mapsto u(t) \in H^s$

Suppose $\Phi(t)$ is C^k -smooth for some small $t > 0$, (5)

(Note: $\Phi(t)$ is well-defined on smooth functions.)

By the smoothness around the zero func,

$$\begin{aligned} \Phi(t)(\underbrace{u_0(\delta)}_{=\delta\phi}) &= \Phi(t)(\underbrace{u_0(\delta=0)}_{\equiv 0}) + \nabla \Phi(t)(0) \cdot \delta\phi \\ &\quad + \frac{1}{2} \nabla^2 \Phi(t)(0)(\delta\phi, \delta\phi) + \dots \end{aligned}$$

$$\Rightarrow \left\| \frac{d^k}{ds^k} \Phi(t)(\delta\phi) \right\|_{s=0, H^s} \lesssim \underbrace{\|\nabla^k \Phi(t)(0)\|_{(H^s)^{\otimes k} \rightarrow H^s}}_{\leq C < \infty} \|\phi\|_{H^s}^k$$

Namely,

$$** \quad \left\| \partial_s^k \Phi(t)(\delta\phi) \right\|_{s=0, H^s} \lesssim \|\phi\|_{H^s}^k. \quad \begin{matrix} \uparrow \\ \text{chain rule} \end{matrix}$$

$$\partial_s^k \Phi(t)(\delta\phi) = \nabla^k \Phi(t)(\delta\phi)(\underbrace{\phi, \dots, \phi}_{k\text{-times}})$$

(b)

$$\cdot \quad u(t) = S(t)\phi \pm i \int_0^t S(t-t') \underbrace{|u|^2 u(t')}_{\text{depends on } \delta. \text{ (higher order in } \delta)} dt'$$

$$\frac{\partial u}{\partial \delta} \Big|_{\delta=0} = S(t)\phi$$

$$\frac{\partial^2 u}{\partial \delta^2} \Big|_{\delta=0} = \pm i \int_0^t S(t-t') \underbrace{\partial_s^2 (|u|^2 u(t'))}_{\delta=0} \Big|_{\delta=0} dt'.$$

Apply the product rule: $u_s^2 \bar{u}$, $u_{ss} \bar{u} u$, ...
 - contains at least one u or \bar{u} but $u(\delta=0) \equiv 0$.

$$= 0.$$

$$\frac{\partial^3 u}{\partial \delta^3} \Big|_{\delta=0} \sim \int_0^t S(t-t') u_s^2 \bar{u}_s \Big|_{\delta=0} dt'$$

$$= \int_0^t S(t-t') |S(t)\phi|^2 s(t') \phi dt'$$

Given $N \in \mathbb{N}$, let $\phi = N^{-s} e^{iN \cdot x_1} \leftarrow \text{supp on}$ (7)

$$\Rightarrow \|\phi\|_{H^s} \sim 1 \quad Ne_1 = (N, 0, \dots, 0)$$

$$\Rightarrow S(t)\phi = N^{-s} e^{iN \cdot x_1 - iN^2 t}$$

$$\cdot (S(t')\phi)^2 S(t')\phi = N^{-3s} e^{iN \cdot x_1 - iN^2 t'}$$

$$\Rightarrow S(t-t') \left(|S(t')\phi|^2 S(t')\phi \right) = \underbrace{N^{-3s} e^{iN \cdot x_1 - iN^2 t}}_{\text{indep of } t'}.$$

$$\Rightarrow \frac{\partial^3 u(t)}{\partial \delta^3} \Big|_{\delta=0} \sim t N^{-3s} e^{iN \cdot x_1 - iN^2 t}$$

$$\Rightarrow \left\| \frac{\partial^3 u(t)}{\partial \delta^3} \Big|_{\delta=0} \right\|_{H^s} \sim t N^{-2s} \text{ but } \|\phi\|_{H^s} \sim 1$$

$\Rightarrow \textcircled{**}$ can not hold for $s < 0$.

(This argument first appeared in the KdV, mKdV context
by Bourgain '97.)

(8)

③ failure of local uniform continuity

Construct a family of pairs of smooth initial data
(and solns) s.t.

Given any small $t > 0$ and $\varepsilon > 0$,

$\exists u_{0,\varepsilon}$ and $v_{0,\varepsilon}$ s.t.

$$\|u_{0,\varepsilon} - v_{0,\varepsilon}\|_{H^s} < \varepsilon \text{ but } \|u_\varepsilon(t) - v_\varepsilon(t)\|_{H^s} \gtrsim 1.$$

On \mathbb{R} : Kenig - Ponce - Vega '01 : focusing
family of soliton solutions with parameters.

Christ - Colliander - Tao '03 : defocusing
family of approximate solutions

On \mathbb{T}^d : Burg - Gérard - Tzvetkov '02 (cubic NLS) ⑨

\Leftarrow enough to consider the $d=1$ case.

(For general $d \geq 1$, set $u(x) = u(x_1, 0, \dots, 0)$)

Consider

$$u_{N,a}(t, x) = a e^{i(Nx - N^2 t \pm |a|^2 t)}$$

for $a \in \mathbb{C}$ and $N \in \mathbb{N}$.

- $u_{N,a}$ is a soln to the cubic NLS.
- Choose $a = N^{-s}\alpha$ and $a' = N^{-s}\alpha'$.

$$\|u_{N,a}(0) - u_{N,a'}(0)\|_{H^s} \sim |\alpha - \alpha'| \xrightarrow{N \rightarrow \infty} 0$$

- For small $t_0 > 0$,

$$\begin{aligned} \|u_{N,a}(t_0) - u_{N,a'}(t_0)\|_{H^s} &\sim |\alpha e^{\pm i N^{-2s} |\alpha|^2 t_0} - \alpha' e^{\pm i N^{-2s} |\alpha'|^2 t_0}| \\ &= |\alpha - \alpha' e^{\underbrace{\pm i N^{-2s} (|\alpha'|^2 - |\alpha|^2) t_0}_{= \pi + o(1)}}| \sim |\alpha| + |\alpha'| \sim 1. \end{aligned}$$

$= \pi + o(1)$ by choose $N = N(t_0, \alpha, \alpha') \gg 1$

(10)

Choose $|\alpha|, |\alpha'| \sim 1$

$$|\alpha - \alpha'| \sim \varepsilon.$$

$$\begin{aligned} \text{Note: } |\alpha'|^2 - |\alpha|^2 &= (|\alpha'| - |\alpha|)(|\alpha'| + |\alpha|) \sim \varepsilon. \\ \Rightarrow N &\sim \varepsilon^{-\frac{1}{2s}} \rightarrow \infty \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

$$\Rightarrow \| u_{N,\alpha}(t) - u_{N,\alpha'}(t) \|_{H^s} \begin{cases} < \varepsilon, & t = 0 \\ \sim 1, & t = t_0 > 0. \end{cases}$$

We showed "mild ill-posedness" of NLS in $H^s(\mathbb{T}^d)$, $s < 0$
 (failure of C^3 -smoothness / local unif. continuity).

- Why is $s = 0$ important for NLS?

scaling critical regularity : $s_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1}$

cubic ($p=3$) : $s_{\text{crit}} = \frac{d}{2} - 1$ ($= 0 \Leftrightarrow d=2$)

- Galilean invariance : u is a soln to NLS on \mathbb{R}^d

$$\Rightarrow u^\beta(t, x) = e^{i\frac{\beta}{2}x} e^{-i\frac{\beta p^2}{4}t} u(t, \underline{x + \beta t}) \text{ is also a soln.}$$

↑
shift by $\frac{\beta}{2}$ on the Fourier side

- Galilean invariance makes sense on \mathbb{T}^d if $\beta \in 2\mathbb{Z}^d$.

$$\Rightarrow \hat{u}^\beta(t, \tilde{z}) = e^{-i\frac{3}{4}\beta^2 t} e^{i\tilde{z}\beta t} \hat{u}(t, \tilde{z} - \frac{\beta}{2}) \quad (2)$$

$$|\hat{u}^\beta(t, \tilde{z})| = |\hat{u}(t, \tilde{z} - \frac{\beta}{2})|$$

$\Rightarrow L_x^2$ -norm is invariant under Galilean symmetry.

i.e. $s_{\text{crit}}^\infty = 0$ is another critical regularity for NLS
(associated with Galilean sym.)

Also, the Fourier-Lebesgue space $\mathcal{FL}^{s,p}$

$$\|f\|_{\mathcal{FL}^{s,p}} = \|\langle \tilde{z} \rangle^s \hat{f}(\tilde{z})\|_{L_p^p} \quad (H^s = \mathcal{FL}^{s,2})$$

is invariant under Galilean symmetry when $s=0$.

- $d=1$: cubic NLS

$$S_{\text{crit}} = -\frac{1}{2} < 0 = S_{\text{crit}}^{\infty}$$

- $s < 0$: failure of uniform continuity.

LWP on \mathbb{R} is open. (only existence : Christ - Colliander - Tao '08
Koch - Tataru '07, '12.)

- On \mathbb{T} : cubic NLS is ill-posed in $H^s(\mathbb{T})$, $s < 0$.

• CCT '03 : discontinuity of soln map in $H^s(\mathbb{T})$, $s < 0$

• Molinet '09 : discontinuity in $L_x^2(\mathbb{T})$ endowed with weak topology into the space of distributions $((C^\infty(\mathbb{T}))')$

⇒ discontinuity in $H^s(\mathbb{T})$, $s < 0$.

• Guo - Oh '16 : non-existence of solns for $u_0 \notin L^2(\mathbb{T})$

On \mathbb{T} , the "correct" eqn to study outside $L^2(\mathbb{T})$ is ④

$$\begin{array}{l} (\text{WNLS}) \\ \uparrow \end{array} \quad i\partial_t u + \partial_x^2 u \pm \underbrace{(|u|^2 - 2f)u^2}_{= 0} = 0.$$

Wick ordered NLS: renormalized NLS

- If u solves (NLS), then $\underbrace{e^{-2it\int f|u|^2 dx}}_{=: G(u)} u$ solves (WNLS).
 $u \in C(\mathbb{R}; L^2(\mathbb{T}))$

- G is invertible on $C(\mathbb{R}; L^2(\mathbb{T}))$.

But G does not make sense outside $L^2(\mathbb{T})$ (for initial data)

- It turned out that (WNLS) is a better eqn to study outside $L^2(\mathbb{T})$ and a good a priori bound on solns to (WNLS) combined with G^{-1} gives the non-existence result for the original NLS in $H^s(\mathbb{T})$, $s < 0$.

$$(WNLS) : i\partial_t u + \partial_x^2 u + (|u|^2 - 2 \cdot \infty) u = 0. \quad (5)$$

Back to

④ failure of continuity of soln map for the cubic NLS
on $M = \mathbb{R}^d$ or \mathbb{T}^d .

- In order to show that the soln map is NOT conti at u_0 with the H^s -topology, it suffices to construct, for each $\varepsilon > 0$,
soln u_ε to (NLS) and $t_\varepsilon \in (0, \varepsilon)$ s.t.

$$\|u_\varepsilon(0) - u_0\|_{H^s(M)} < \varepsilon \text{ but } \|u_\varepsilon(t_\varepsilon) - u(t_\varepsilon)\|_{H^s} \gtrsim 1.$$

soln s.t. $u|_{t=0} = u_0$

In general, we take $u_0 = 0$
 $\Rightarrow u \equiv 0$.

(6)

Norm inflation (CCT '03): Given $\varepsilon > 0$,

\exists soln u_ε to (NLS) and $t_\varepsilon \in (0, \varepsilon)$ s.t.

$$\|u_\varepsilon(0)\|_{H^s} < \varepsilon \quad \text{but} \quad \|u_\varepsilon(t_\varepsilon)\|_{H^s} > \frac{1}{\varepsilon}$$

• Norm inflation \Rightarrow discontinuity at $u_0 = 0$.

• Norm inflation at general initial data: $d \geq 1$, $M = \mathbb{R}^d$ or \mathbb{T}^d

Suppose that $s \in \mathbb{R}$ satisfies

(i) $s \leq -\frac{1}{2} = s_{\text{crit}}$ when $d = 1$ (includes s_{crit})

(ii) $s < 0$ when $d \geq 2$. (When $d = 2$, $s_{\text{crit}} = 0$)

Fix $u_0 \in H^s(M)$. Then, given $\varepsilon > 0$, \exists soln u_ε to the cubic NLS
(or WNLS) and $t_\varepsilon \in (0, \varepsilon)$ s.t smooth

$$\|u_\varepsilon(0) - u_0\|_{H^s} < \varepsilon \quad \text{but} \quad \|u_\varepsilon(t_\varepsilon)\|_{H^s} > \frac{1}{\varepsilon}$$

• Kishimoto
(preprint)
• Oh'17.

- Power series expansion indexed by trees

cubic NLS : $u(t) = S(t)u_0 \pm i \underbrace{\int_0^t S(t-t')|u|^2 u(t') dt'}_{= I[u]} = I[u, u, u]$

$$I[u_1, u_2, u_3] = \pm i \int_0^t S(t-t') u_1 \bar{u}_2 u_3(t') dt'.$$

We proved LWP in $H^s(M)$, $s > \frac{d}{2}$.

A similar argument yields LWP in the Wiener algebra

$$\mathcal{FL}'(M) = \mathcal{FL}^{0,1}(M) \quad \|f\|_{\mathcal{FL}^p} = \|\hat{f}^{(3)}\|_{L_3^p}$$

\uparrow ($\overset{''}{A}(M)$)

$$\mathcal{FL}^p = \mathcal{FL}^{0,p} \text{ i.e. } s = 0.$$

algebra.

- Local existence time $T \sim \|u_0\|_{\mathcal{FL}'}^{-2} > 0$.

⑧

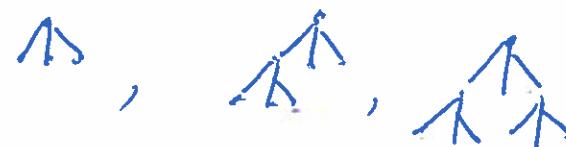
\Rightarrow Picard iteration converges.

$$P_0(\phi) = S(t)\phi$$

$$P_j(\phi) = S(t)\phi + I[P_{j-1}(\phi)], j \geq 1$$

\uparrow converges

Tree (= ternary tree)



• 0 or 3 children

\uparrow \uparrow
non-terminal node

terminal node

$T(j)$ = collection of trees T of j^{th} generation

$$|T| = 3^{j+1}$$

\downarrow
 j parental nodes

(9)

Fix $\phi \in \mathcal{FL}'$,

Given $T \in T(j)$, $j \geq \mathbb{Z}_{\geq 0}$, associate a multilin operator ($\bar{\in} \Phi$) by

- replace a non-terminal node by the Duhamel integral operator $\mathcal{I}[u_1, u_2, u_3]$, u_j = three children.
- replace a terminal node by the linear soln $s(t)\phi$.

Denote the map by $\Sigma : \bigcup_{j=0}^{\infty} T(j) \rightarrow \mathcal{D}'([-\tau, \tau] \times M)$

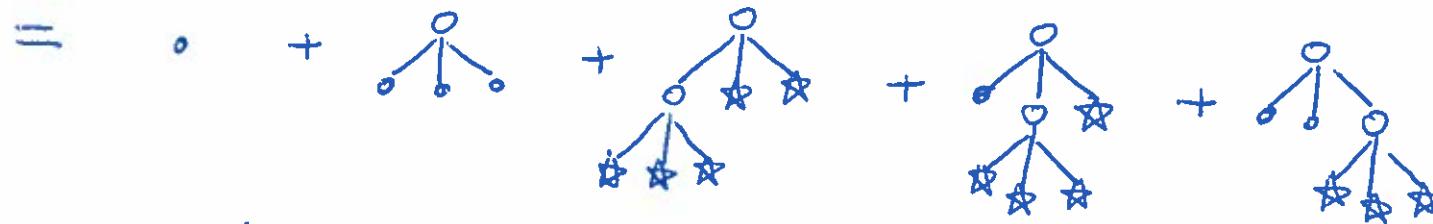
\circ = non-terminal node

\cdot = terminal node

\star = soln. u

$$(NLS) \iff \Phi = \circ + \underbrace{\begin{array}{c} \circ \\ | \\ \star \star \star \end{array}}_{=} = \circ + \begin{array}{c} \circ \\ | \\ \star \end{array} + \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \star \star \star \end{array}$$

(10)



\Leftarrow repeat this process indefinitely

$$U = \sum_{j=0}^{\infty} E_j(\phi) = \sum_{j=0}^{\infty} \sum_{T \in \Pi(j)} \Psi_{\phi}(T) \leftarrow \begin{array}{l} \text{Power series} \\ \text{expansion} \end{array}$$

Lec 18 26/03/18 (Mon)

①

- Lecture on Wednesday from 11:30 am

Trees = ternary trees.

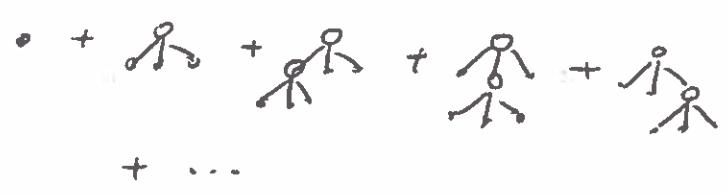
$$\Psi_\phi : \text{trees} \rightarrow \mathcal{L}'_{t,x}$$

- Rule:
- non-terminal node $\rightarrow I[u_1, u_2, u_3]$
 - terminal node \rightarrow lin soln $s(t)\phi$.
 $u_j = j^{\text{th}}$ child

\Rightarrow Power series expansion:

$$u = \sum_{j=0}^{\infty} \sum_{T \in T(j)} \Psi_\phi(T)$$

$$=: \sum_{j=0}^{\infty} \Sigma_j(\phi)$$



Lemma 1: $T(j)$ = collection of trees of j^{th} generation. (2)

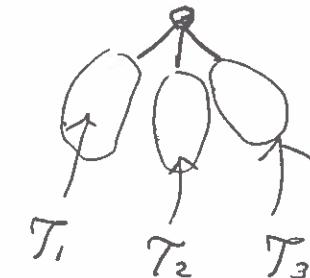
Then, $\# T(j) \leq C_0^j$.

Pf: $\# T(0) = \# T(1) = 1$.

Let $j \geq 2$.

$$\# T(j) = \sum_{\substack{j_1 + j_2 + j_3 = j-1 \\ j_1, j_2, j_3 \geq 0}} \# T(j_1) \cdot \# T(j_2) \cdot \# T(j_3)$$

\swarrow 2 sums



Assume a stronger bound:

$$T_i \in T(j_i)$$

$$\# T(k) \leq \frac{C_0^k}{(1+k)^2}, \quad \forall k \leq j-1.$$

$$\begin{aligned} \Rightarrow \# T(j) &\leq \sum_{i=1}^3 \frac{C_0^{j_i}}{(1+j_i)^2} && 3 \max(j_i+1) \geq j+1 \\ &\leq 3^2 \left(\underbrace{\sum_{k=1}^j \frac{1}{(1+k)^2}}_{\text{2 sum}} \right)^2 \frac{C_0^{j-1}}{(1+j)^2} && \leq \frac{C_0^j}{(1+j)^2}. \\ &= C_0^j \end{aligned}$$

□

• Basic multilinear estimates : $\|f\|_{\mathcal{FL}^p} = \|\hat{f}(3)\|_{L_{\frac{p}{3}}^p}$ ③

Lemma 2 :

$$\cdot \|\square_j(\phi)(t)\|_{\mathcal{FL}^1} \leq C^j t^j \|\phi\|_{\mathcal{FL}^1}^{2j+1}$$

$$\cdot \|\square_j(\phi)(t)\|_{\mathcal{FL}^\infty} \leq \underbrace{C^j t^j}_{\uparrow} \|\phi\|_{\mathcal{FL}^1}^{2j-1} \|\phi\|_{L^2}^2.$$

• j iterated (Duhamel) integral

$$\cdot \text{Young's ineq} : \frac{1}{\infty} + 1 = \frac{1}{2} + \frac{1}{2}$$

$$\frac{1}{2} + 1 = \frac{1}{2} + 1$$

$$1 + 1 = 1 + 1$$

Lemma 3 :

$$\|\square_j(u_0 + \phi)(t) - \square_j(\phi)(t)\|_{\mathcal{FL}^p} \quad 1 \leq p \leq \infty$$

$$\leq \underbrace{C^j t^j}_{\substack{\text{Lemma 1 and} \\ \text{Young's ineq}}} \|u_0\|_{\mathcal{FL}^p} \left(\|u_0\|_{\mathcal{FL}^1}^{2j} + \|\phi\|_{\mathcal{FL}^1}^{2j} \right).$$

• $\sum_{T \in T(j)} \sum_{\phi_1, \dots, \phi_{2j+1}} \Psi(T; \phi_1, \dots, \phi_{2j+1}) \leq C^j$

$\Psi(T; \phi_1, \dots, \phi_{2j+1}) \Rightarrow \Phi_i = u_0 \text{ or } \phi \text{ but at least one } u_0.$

(4)

- By a standard density argument, we may assume $u_0 \in \mathcal{F}(M)$.
- Want to construct, given $n \in \mathbb{N}$,
a soln u_n and $t_n \in (0, \frac{1}{n})$

$$\|u_n(0) - u_0\|_{H^s} < \frac{1}{n} \quad \text{but} \quad \|u_n(t_n)\|_{H^s} > n.$$

Main idea: $s < 0$

Exploit $\underset{N}{\text{high}} \times \underset{2N}{\text{high}} \times \underset{N}{\text{high}} \rightarrow \underset{0}{\text{low interaction}}$

$$\mathcal{F}_x(u_1 \bar{u}_2 u_3)(\xi) = \sum_{\tilde{\xi} = \tilde{\xi}_1 - \tilde{\xi}_2 + \tilde{\xi}_3} \widehat{u}_1(\tilde{\xi}_1) \overline{\widehat{u}_2(\tilde{\xi}_2)} \widehat{u}_3(\tilde{\xi}_3)$$

$$N = N(n) \gg 1$$

Given $n \in N$, let $N = N(n) \gg 1$ (5)

$$\hat{\Phi}_n(\vec{z}) = R \left(\mathbb{1}_{Ne_1 + Q_A}(\vec{z}) + \mathbb{1}_{2Ne_1 + Q_A}(\vec{z}) \right)$$



$$Q_A = \left[-\frac{A}{2}, \frac{A}{2} \right)^d, A \ll N$$

Choose R, A large s.t.

① $\frac{RA^d}{2} \gg \|u_0\|_{\mathcal{F}L'}$

$$\|\phi_n\|_{\mathcal{F}L'}$$

- $\|\phi_n\|_{H^s} \sim RA^{d/2} N^s$

Set $u_{0,n} = u_0 + \phi_n$. and study the corresponding soln u_n

with $u_n|_{t=0} = u_{0,n}$

• Power series expansion of u_n

(6)

$$u_n = \sum_{j=0}^{\infty} \square_j (u_{0,n})$$

- Lemma 2 guarantees absolute & unif convergence

on $[-T, T]$, if $T \lesssim (\|u_0\|_{FL^1} + RA^d)^{-2} \stackrel{\textcircled{1}}{\sim} (RA^d)^{-2}$.

(2) $\|u_{0,n} - u_0\|_{H^s} = \|\phi_n\|_{H^s} \sim RA^{d/2} N^s$

(3) $\|\underbrace{\phi_n(u_{0,n})}_{\substack{\text{lim soln } S(t)(u_0 + \phi_n)}}\|_{H^s} \leq \|u_0\|_{H^s} + \|\phi_n\|_{H^s} \stackrel{\textcircled{2}}{\sim} u_0 + RA^{d/2} N^s$

(4) $\|\square_1(u_{0,n})(t) - \underbrace{\square_1(\phi_n)(t)}_{\substack{\text{Lem3 with } p=2}}\|_{H^s} \leq \|\dots\|_{L^2}$ We'll show this is large.

$$\lesssim t \|u_0\|_{L^2} \underbrace{(\|u_0\|_{FL^1}^2 + \|\phi_n\|_{FL^1}^2)}_{\stackrel{\textcircled{3}}{\sim} R^2 A^{2d}}$$

$$\lesssim t \|u_0\|_{L^2} R^2 A^{2d}.$$

Lemma 4: $s < 0$

(7)

$$\| \mathbb{D}_j(u_{0,m})(t) \|_{H^s} \leq C^j t^j (RA^d)^{2j} \left(Rf(A) + \| u_0 \|_{L^2} \right)$$

$$f(A) = \begin{cases} 1, & s < -d/2 \\ (\log A)^{1/2}, & s = -d/2 \\ A^{d/2+s}, & s > -d/2 \end{cases}$$

Pf: $\text{supp } \hat{\phi}_n = \text{two disjoint cubes of vol } A^d$.

$\Psi_{\phi_n}(T) = (2^j+1) - \text{fold product of } S(t)\phi_n \text{ and its c.c.}$
 \uparrow
 $\tilde{m} T(j)$

$\Rightarrow \text{supp } \mathcal{F}_x(\Psi_{\phi_n}(T)) \subset (\text{at most}) 2^{2j+1} \text{ cubes of vol } A^d$.

With $s < 0$ ($\langle z \rangle^s$ is decreasing)

$$\| \langle z \rangle^s \|_{L^2_{\frac{1}{3}}(\text{supp } \mathcal{F}_x(\Psi_{\phi_n}(T)))} \leq \| \langle z \rangle^s \|_{L^2_{\frac{1}{3}}(C^j Q_A)}$$

$$\lesssim \begin{cases} 1, & s < -d/2 \\ C^j (\log A)^{1/2}, & s = -d/2 \\ C^j A^{d/2+s}, & s > -d/2 \end{cases}$$

"shift the support toward the origin while keeping the vol!"

(8)

$$\begin{aligned}
 \Rightarrow \| \square_j(\phi_n)(t) \|_{H^s} &\leq \sum_{T \in T(j)} \| \Psi_{\phi_n}(T)(t) \|_{H^s} \\
 &\stackrel{\text{H\"older}}{\leq} \sum_{T \in T(j)} \| \langle \xi \rangle^s \|_{L_x^2(\text{supp } \mathcal{F}_x(\Psi_{\phi_n}(T)))} \| \Psi_{\phi_n}(T)(t) \|_{FL^\infty} \\
 &\stackrel{\text{Lemmas 1 \& 2}}{\leq} C^j t^j \underbrace{(RA^d)^{2j-1} (RA^{d/2})^2 \cdot f(A)}_{= (RA^d)^{2j} \cdot R}
 \end{aligned}$$

$$\begin{aligned}
 \cdot \| \square_j(u_0 + \phi_n)(t) - \square_j(\phi_n)(t) \|_{H^s} &\leq \| \dots \|_{L^2} \\
 &\leq C^j t^j \| u_0 \|_{L^2} \underbrace{\left(\| u_0 \|_{FL'}^{2j} + \| \phi_n \|_{FL'}^{2j} \right)}_{\sim (RA^d)^{2j}}
 \end{aligned}$$

◻

Prop 5: $0 < t \ll N^{-2}$

(9)

$$\left\| \mathbb{E}_t (\Phi_n)(t) \right\|_{H^s} \gtrsim t R^3 A^{2d} f(A)$$

$$I[S(t)\phi_n, S(t)\phi_n, S(t)\phi_n]$$

Pf: $\mathbb{E}_t (\Phi_n)(t) = \int_0^t S(t-t') (S(t') \phi \overline{S(t)} \phi S(t') \phi) dt'$

$$F_x(\mathbb{E}_t (\Phi_n)(t))(\bar{z}) = e^{-it|\bar{z}|^2} \int_{\bar{z} = \bar{z}_1 - \bar{z}_2 + \bar{z}_3}^0 e^{+it'(\underbrace{|\bar{z}|^2 - |\bar{z}_1|^2 + |\bar{z}_2|^2 - |\bar{z}_3|^2}_{M(\bar{z})})} d\bar{t}'$$

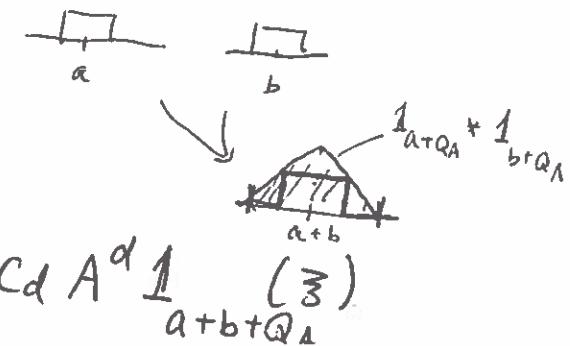
$$\cdot |\bar{z}_j| \lesssim N \Rightarrow |\bar{z}| \lesssim N$$

$$\Rightarrow t' M(\bar{z}) \ll 1 \quad \text{for } 0 < t' \ll N^{-2}.$$

$$\Rightarrow \operatorname{Re} \int_0^t e^{it' M(\bar{z})} dt' \geq \frac{1}{2} t.$$

$$\cdot |F_x(\mathbb{E}_t (\Phi_n)(t))(\bar{z})| \gtrsim t R^3 A^{2d} \mathbb{1}_{Q_A}(\bar{z})$$

$$\uparrow \mathbb{1}_{a+Q_A} * \mathbb{1}_{b \in Q_A}(\bar{z}) \geq C d A^d \mathbb{1}_{a+b+Q_A}(\bar{z})$$



$$\text{With } \|\langle z \rangle^s\|_{L^{\frac{2}{3}}(Q_A)} \sim f(A),$$

(10)

we obtain the desired lower bound.



- Back to the proof of Theorem (norm inflation at general data)

WANT $\|u_{0,n} - u_0\|_{H^s} < \frac{1}{n}$ but $\|u_n(t_n)\|_{H^s} > n$.

\Leftarrow suffices to prove, for $N = N(n) \gg 1$,

(i) $RA^{d/2}N^s \ll \frac{1}{n}$

$$\|\phi_n\|_{H^s} = \|u_{0,n} - u_0\|_{H^s}$$

(ii) $TR^2A^{2d} \ll 1$

(iii) $TR^3A^{2d} \cdot f(A) \gg n$

$$T\|S(t)\phi_n\|_{FL'}^2 \ll 1 \Rightarrow \text{LWP \& conv of power series}$$

(iv) $TR^3A^{2d} \cdot f(AT) \gg T^2R^5A^{4d}f(A)$

(v) $T \ll N^{-2}$

$$\sum_{j=2}^{\infty} \|\sum_j(u_{0,n})\|_{H^s}$$

(vi) $RA^d \gg \|u_0\|_{FL'}, Rf(A) \gg \|u_0\|_{L^2}$

• Assume (i) - (vi).

(2)

$$\begin{aligned} & \|u_{0,m}\|_{\mathcal{FL}} \sim RA^d. \\ \xrightarrow{(ii)} & u_n \text{ exists on } [-T, T] \text{ and} \\ & \text{power series converges in } C_T \mathcal{FL}'_x. \end{aligned}$$

$$\begin{aligned} & \left\| \sum_{j=2}^{\infty} \square_j (u_{0,m})(T) \right\|_{H^s} \\ & \stackrel{\substack{\text{Lem 4} \\ \text{geo. series}}}{\sim} T^2 R^4 A^{2d} \left(R f(A) + \|u_0\|_{L^2} \right) \\ & \sim T^2 R^5 A^{2d} f(A) \end{aligned}$$

$$\begin{aligned} \Rightarrow \|u_n(T)\|_{H^s} & \geq \left\| \square_0 (\phi_m)(T) \right\|_{H^s} \\ & - \left\| \square_0 (u_{0,m})(T) \right\|_{H^s} \leftarrow \text{use ③ from Lec 18.} \\ & - \left\| \square_1 (u_{0,m})(T) - \square_1 (\phi_m)(T) \right\|_{H^s} \leftarrow \text{use ④} \\ & - \left\| \sum_{j=2}^{\infty} \square_j (u_{0,m})(T) \right\|_{H^s} \end{aligned}$$

(3)

$$\begin{aligned} &\gtrsim \frac{\text{TR}^3 A^{2d} f(A)}{- (1 + RA^{d/2} N^s)} \\ &\quad - \text{TR}^2 A^{2d} \|u_0\|_{L^2} \\ &\quad - T^2 R^5 A^{4d} f(A) \\ &\sim \text{TR}^3 A^{2d} f(A) \gg n \end{aligned}$$

(iii)

- Now, we verify (i) - (vi).

Case 1: $s < -\frac{d}{2}$ ($\Rightarrow f(A) = 1$)

$$A = N^{\frac{1}{d}(1-s)}, \quad R = N^{2s}, \quad T = N^{-2-3s}, \quad s \ll 1$$

$$\cdot RA^{\frac{d}{2}} N^s = N^{s + \frac{1}{2} + \frac{3}{2}s} \ll 1/n$$

$$\cdot \text{TR}^2 A^{2d} = N^{-s} \overset{s < 0}{\ll 1}$$

$$\cdot \text{TR}^3 A^{2d} = N^s \gg n.$$

Case 2 : $s = -\frac{d}{2}$ ($\Rightarrow f(A) \sim (\log A)^{\frac{d}{2}}$) ④

$$A = \frac{N^{\frac{1}{d}}}{(\log N)^{\frac{1}{16d}}}, \quad R = 1, \quad T = \frac{1}{N^2(\log N)^{\frac{1}{8}}}$$

- $RA^{\frac{d}{2}}N^s = N^{\underbrace{\frac{1}{2}(1-d)}_{\leq 0}}(\log N)^{-\frac{1}{32}} \ll 1/m$

- $TR^2A^{2d} = (\log N)^{-\frac{1}{4}} \ll 1.$

- $TR^3A^{2d}(\log A)^{\frac{1}{2}} \sim (\log N)^{-\frac{1}{4}}(\log N - \frac{1}{16}\log \log N)^{\frac{1}{2}} \sim (\log N)^{\frac{1}{4}} \gg m.$

Case 3 : $-\frac{d}{2} < s < 0$ ($\Rightarrow f(A) \sim A^{\frac{d}{2}+s}$) relevant only for $d \geq 2$.

$$A = N^{\frac{2}{d}-\delta}, \quad R = N^{-1-s+\frac{d}{2}\delta-\theta}, \quad T = N^{-2+2s+d\delta+\theta}$$

where $\delta \gg \theta > 0$ small s.t.

$$\underline{-2s > d\delta + \theta} \quad \text{and} \quad -s\delta > 2\theta$$

(5)

- $\cdot RA^{\frac{d}{2}}N^s = N^{-\theta} \ll \frac{1}{n}$
- $\cdot TR^2A^{2d} = N^{-\theta} \ll 1$. $\underbrace{\geq 0}_{\text{by } d \geq 2}$
- $\cdot TR^3A^{2d} \cdot A^{\frac{d}{2}+s} = N^{\left(\frac{-d+2}{d}\right)s - 2\theta - sf} > 0$
 $\geq N^{-2\theta - sf} \gg n.$

◻

- Christ - Colliander - Tao '03 : NLS on \mathbb{R}^d .

$s < -\frac{d}{2}$ or $0 < s < \text{Scrit}$. \Rightarrow Left open
the range $-\frac{d}{2} \leq s < 0$.

\Leftarrow ODE argument.

$$i\partial_t u + |u|^2 u = 0 \xrightarrow{\text{ODE for fixed } x.} u(t, x) = e^{it|u_0(x)|^2} u_0(x)$$

$$\Rightarrow \|u(t)\|_{H^s} \sim_{u_0} t^s, \quad s > 0.$$

Combine this with scaling.

• Nonuniqueness of NLS below $L^2(\mathbb{T})$ Christ '05
arXiv.

(6)

$$(\text{WNLS}) \quad i\partial_t u + \partial_x^2 u + N(u) = 0 \quad \leftarrow \text{renormalized NLS}$$

$$N(u) = |u|^2 u - 2(f|u|^2 dx) u$$

$$= |u|^2 u - 2 \cdot \text{cs} \cdot u \quad \text{if } u \notin L_x^2$$

$$\widehat{N(u)}(n) = \sum_{\substack{n=n_1+n_2 \\ n \neq n_1, n_3}} \widehat{u}_{n_1} \overline{\widehat{u}_{n_2}} \widehat{u}_{n_3} - |\widehat{u}_n|^2 \widehat{u}_n$$

↑

$$n = n_1 + n_2$$

$$n \neq n_1, n_3$$

when $n_1 = n_2$ (and $n_3 = n$)

$$\sum_{n=n_1+n_2} \widehat{u}_{n_1} \overline{\widehat{u}_{n_2}} \widehat{u}_{n_3}$$

$$= (f|u|^2 dx) \widehat{u}_n$$

We need $u \in L_x^3$ ($\Leftarrow u \in H_x^{1/6}$)

to make sense of $N(u)$ as a spatial
space-time distribution.

as a product of
distribution u .

\Rightarrow We need to consider a more generalized notion of solns.

Def 1: seq of Fourier cutoff operators $\{P_N\}_{N \in \mathbb{N}}$. (7)

$$\widehat{P_N f}(n) = m_N(n) \widehat{f}(n).$$

$m_N : \mathbb{Z} \rightarrow \mathbb{C}$, finite support, wif bdd

$$\lim_{N \rightarrow \infty} m_N(n) = 1, \quad \forall n \in \mathbb{Z}$$

Def 2: $u \in \mathcal{D}'((0, \tau) \times \mathbb{T})$.

We say $N(u)$ exist and is equal to $v \in \mathcal{D}'_{T, x}$ if

$$\lim_{N \rightarrow \infty} \underbrace{N(P_N u)}_{} = v$$

always makes b/c $P_N u \in C_x^\infty$

for any seq $\{P_N\}_{N \in \mathbb{N}}$ of Fourier cutoff operators.
in the distributional sense.

Rmk: We define the nonlinearity $N(u)$ as the limit of
the nonlinearity for smoothed u .

Def 3: We say $u \in C_T H^s$ is a weak soln to (WNLS) ⑧
in the extended sense with $u|_{t=0} = u_0$
if $N(u)$ exists in the sense of Definition 2
and u satisfies (WNLS) in the distributional sense.

Thm: $s < 0$. $\exists u \in C_T H^s$, $u \not\equiv 0$ s.t.
 u is a weak soln to (WNLS) in the extended sense
with $u|_{t=0} = 0$. Moreover, $S(t)N(u)$ exists in $C_T^{-1} H^s$ -norm.

$$\left(\begin{array}{l} \bullet F \in C_T^{-1} H^s \text{ if } \int_0^t F(\tau) d\tau \in C_T H^s \\ \|F\|_{C_T^{-1} H^s} = \max_{t \in [0, T]} \left\| \int_0^t F(\tau) d\tau \right\|_{H^s} \end{array} \right)$$

\Rightarrow non-uniqueness

- This non-uniqueness result works with any data in $\mathcal{F}L^1$.
- The soln constructed in Thm does not belong to the $X^{s,b}$ -space
 $X^{s,b}$ -space \leftarrow Fourier restriction norm method (Bourgain '93)
(Miyaji-Tsutsumi '17)
- applies to a wide class of equations in negative Sobolev spaces.
- Scheffer '93, Shnirelman '97 on 2-d Euler.

Idea: Study the non-homog. problem:

$$\begin{aligned}
 (\text{NLS}_F) \quad & i\partial_t v + \partial_x^3 v + N(v) = F \quad \text{on } \mathbb{T} \\
 & \parallel \\
 & (|v|^2 - 2f|v|^2)v
 \end{aligned}$$

$T \leq 1$

Prop 1: $s < 0$.

(2)

Suppose $u \in C^\infty([0, T] \times \mathbb{T})$ s.t.

$\forall n, \hat{u}(t, n) \rightarrow 0$ at ∞ -order as $t \rightarrow 0^+$

Then, $\forall \varepsilon > 0, \exists v, F \in C^\infty([0, T] \times \mathbb{T})$

whose Fourier coeff. $\rightarrow 0$ at ∞ -order as $t \rightarrow 0^+$

s.t. v is a soln to (NLS_F)

with bounds:

$$\|v - u\|_{C_T H^s} \leq \varepsilon$$

$$\|S(t)F\|_{C_T^{-1} H^s} \leq \varepsilon$$

Interaction representation: $S(-t)u(t)$

$$S(t)u(t) = \underbrace{u_0}_{\substack{\parallel \\ 0}} + i \int_0^t S(-t') N(u)(t') dt'$$

$$S(t)v_n(t) = \underbrace{v_{n(0)}}_{\substack{\parallel \\ 0}} + i \int_0^t S(-t') N(v_n)(t') dt' - i \underbrace{\int_0^t S(-t') F_n(t') dt'}_{\substack{\parallel \\ 0}}$$

(3)

WTS: $\begin{array}{c} N(v_n) \rightarrow N(u) \text{ in } C_T^{-1} H^s \\ \downarrow S(t) \quad \downarrow S(t) \end{array}$

and $u := \lim_{n \rightarrow \infty} v_n \neq 0 \text{ in } C_T H^s$

In terms of the Fourier coeff., we have

$$(NLS_F) \Leftrightarrow i\partial_t \widehat{u}_n - im^2 \widehat{u}_n + \sum_{\substack{n = n_1 - n_2 + n_3 \\ n \neq n_1, n_3}} \widehat{u}_{n_1} \overline{\widehat{u}_{n_2}} \widehat{u}_{n_3} - |\widehat{u}_n|^2 \widehat{u}_n = \widehat{F}_n(t)$$

$$y_m(t) = \mathcal{F}_x(SFT u(t))(m) = e^{itn^2} \widehat{u}(t, n)$$

$$\Leftrightarrow \partial_t y_n = i \underbrace{\sum * e^{i\phi(\bar{n})t} y_{n_1} \overline{y_{n_2}} y_{n_3}}_{= NR(y)} - i |y_n|^2 y_n - ie^{itn^2} \widehat{F}_n(t)$$

$$\phi(\bar{n}) = \phi(n_1, n_2, n_3, n) = n^2 - n_1^2 + n_2^2 - n_3^2 \quad \text{of the nonlinearity}$$

under $n = n_1 - n_2 + n_3$ $\Rightarrow 2(m - n_1)(m - n_3)$

$$\Leftrightarrow \frac{dy}{dt} = \underbrace{NR(y) + R(y)}_{=N(y)} + f, \quad f_n = -i e^{itn^2} \hat{F}_n(t)$$

We say $x(t) = \{x_n(t)\}_{n \in \mathbb{Z}}$ has support in $S \subset \mathbb{Z}$
if $x_n(t) \equiv 0, \forall n \notin S$.
 $\forall t \in [0, T]$

Prop 2 (\Rightarrow Prop 1)

$S < 0$. $x \in C^\infty([0, T])$, seq-valued
finite supp
vanishes at ∞ -order as $t \rightarrow 0+$.

Then, $\forall \varepsilon > 0$

\exists seq-valued func $y, g \in C^\infty([0, T])$ with finite supp.

s.t. $\partial_t y = N(y) + g$, $y(t) \rightarrow 0$ at ∞ -order as $t \rightarrow 0+$.

$$\|y - x\|_{C_T l_S^2} \leq \varepsilon$$

$$\|g\|_{C_T^{-1} l_S^2} \leq \varepsilon.$$

$$\|a\|_{l_S^2} = \|F(a)\|_{HS} = \left(\sum_m |a_m|^2 \right)^{1/2}$$

(5)

Moreover, $\exists M > 0$,

we can construct y and g s.t.

$y - x$

g are supported on $[M, \infty)$

Pf of Prop 2:

let $f = \partial_t x - N(x)$

\downarrow finite support. $S = \{n_j : 1 \leq j \leq A\}$

We construct

$$\tilde{S} \subset \mathbb{Z} \cap [M, \infty)$$

Recall $h \times h \times h \rightarrow \text{low}$

- Pick $m_1 \geq M$ and set m'_1 by

$$N - 2N + N = 0$$

for norm inflation.

$$2m_1 - m'_1 = n_1 \quad \rightarrow$$

$$m_1 - m'_1 + m_1 = n_1$$

choose $m_1 \gg 1$ s.t. $m'_1 \geq M$

- Pick $m_2 \gg m_1, m'_1$ and set m'_2

$$2m_2 - m'_2 = n_2$$

choose $m_2 \gg 1$ s.t. $m'_2 \geq M$ (also $m'_2 \gg m_1, m'_1$)

Repeat this process for $j=1, 2, \dots, A$

(6)

$$\Rightarrow 2m_j - m'_j = n_j, \quad \forall 1 \leq j \leq A. (\Rightarrow m'_j \approx 2m_j)$$

$$\tilde{S} = \{m_1, m'_1, \dots, m_A, m'_A\}$$

Additional constraints:

① $k, l, m \in \mathbb{F}$ and $l \neq k, m$.

Then, $|k - l + m| \geq M$

unless $(k, l, m) = (m_j, m'_j, m_j)$ for some j .

② $k, l \in \tilde{S}$, $m \in \text{supp } x$.

Then, $|k - \underline{m} + l| \geq M$

Also, $|k - l + \underline{n}| \geq M$ unless $k = l$.

③ $k \in \tilde{S}$, $m, n \in \text{supp } x$

$$|k - m + n|, |m + k - n| \geq M$$

Construct $h \in C_T^\infty$, $h_m, m \in \tilde{S}$. (7)

$$\textcircled{*} \quad i h_{m_j} \overline{h_{m'_j}} h_{m_j}(t) \equiv \frac{1}{2} e^{-i\phi(m_j, \bar{m}'_j, m_j, n_j)t} f_{m_j}(t)$$

for $m_j \in S$, and set $h_m \equiv 0$, $\forall m \notin \tilde{S}$.

$$\Rightarrow \text{Set } y = x + h$$

\uparrow low \uparrow high

$$R(x)(n) = -i |x_n|^2 x_n$$

- disjoint supp : $R(x+h) = R(x) + R(h)$

- $\|h\|_{C_T l_s^2} \leq CM^s \leq \varepsilon$

\downarrow by choosing $M \gg 1$ ($s < 0$)

$$\underline{\|y-x\|_{C_T l_s^2} \leq \varepsilon}$$

⑧

$$\text{Set } g = \frac{dy}{dt} - N(y)$$

$$y = x + h$$

$$\frac{dy}{dt} = N(x) + f.$$

$$\Rightarrow g = f - NR(h)$$

$$+ \left| \frac{dh}{dt} \right| \leq \varepsilon \text{ in } C_T^{-1} l_s^2$$

$$- |R(h)| \text{ supported on } \{|n| \geq M\} \rightarrow \|R(h)\|_{C_T l_s^2} \leq CM^s \leq \varepsilon$$

$$- (NR(x+h) - NR(x) - NR(h))$$

① & ④, $f - NR(h)$ supp on $\{|n| \geq M\}$

$$\rightarrow \leq \varepsilon \text{ in } C_T l_s^2$$

② & ③, supp on $\{|n| \geq M\}$

$$\Rightarrow \|g\|_{C_T^{-1} l_s^2} \leq \varepsilon.$$



(9)

• Construction of the soln:

- $x^{(1)}$, seq valued, finite supp
 $\rightarrow 0$ at ∞ -order as $t \rightarrow 0+$.

AND $\|x_0^{(1)}\|_{C_T} \geq 1$

Proceed inductively.

Apply Prop 2 with $x = x^{(n)}$

$$\Rightarrow x^{(n+1)} = y$$

$$h^{(n)} = y - x = x^{(n+1)} - x^{(n)}$$

$$\underbrace{f^{(n+1)}}_g = \partial_t x^{(n+1)} - N(x^{(n+1)})$$

$$\|h^{(n)}\|_{C_T l_s^2} \leq \varepsilon_n, \quad \|f^{(n+1)}\|_{C_T^{-1} l_s^2} \leq \varepsilon_n$$

$$\text{Set } \varepsilon_n \leq 2^{-n-1}.$$

$$\Rightarrow x = \lim_{n \rightarrow \infty} x^{(n)} \text{ in } C_T l_s^2, \quad x = x^{(1)} + \sum_{n=1}^{\infty} h^{(n)}$$

(10)

$$\text{but } \|x_0\|_{C_T} \geq \|x_0^{(1)}\|_{C_T} - \sum_{n=1}^{\infty} \|h^{(n)}\|_{C_T} \geq 1/2$$

i.e. $x \neq 0.$

Show ① $\lim_{N \rightarrow \infty} N(P_N x)$ exists in $C_T^{-1} l_s^2$

i.e. $N(x)$ exists in the weak sense

② $N(P_N x^{(n)})$ Cauchy in $C_T^{-1} l_s^2$

$$\Rightarrow N(x^{(n)}) \rightarrow \underbrace{N(x)}_{\text{weak}} \text{ in } C_T^{-1} l_s^2$$

defined in the weak sense

$$x^{(n)}(t) = 0 + \int_0^t N(x^{(n)})(t') dt' + \underbrace{\int_0^t f^{(n)}(t') dt'}_{\downarrow}$$

$$x(t) = 0 + \int_0^t \underline{N(x)(t')} dt'$$

\downarrow
0

(11)

$$\text{Let } u(t) = S(t) \mathcal{F}^{-1}(x(t))$$

$\Rightarrow u$ is a weak soln to (WNLS) in the extended sense
with $u|_{t=0} = 0$ but $u \neq 0$.