

Sec 6: More on estimates

6.1 Dispersive estimate for the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = f \end{cases}$$

$$\Rightarrow S(t)f(x) = e^{it\Delta}f(x) = \frac{1}{(4\pi i t)^d} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} f(y) dy$$

$$\Rightarrow \|S(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L_x^1}, \quad t \neq 0.$$

K\_t(x-y)

Goal: Prove the dispersive estimate WITHOUT the explicit formula.

• 1-d case:  $S(t)f = K_t * f$

$$K_t = \mathcal{F}^{-1}(e^{-it|\xi|^2}) = \int_{\mathbb{R}} e^{-it|\xi|^2 + ix\xi} d\xi$$

$$\underline{\text{Claim}}: \|K_t(x)\|_{L_x^\infty} \lesssim \frac{1}{|t|^{1/2}}, \quad (t \neq 0)$$

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Then, by Young's ineq.

$$\begin{aligned}\|S(t)f\|_{L_x^\infty} &= \|K_t * f\|_{L_x^\infty} & \frac{1}{\infty} + 1 &= \frac{1}{\infty} + 1 \\ &\leq \|K_t\|_{L_x^\infty} \|f\|_{L_x'} \\ &\lesssim \frac{1}{|t|^{1/2}} \|f\|_{L_x'}.\end{aligned}$$

Pf of Claim: Assume  $t > 0$

$$K_t(x) = \frac{1}{\sqrt{t}} \int_R e^{-i\zeta^2} + i \frac{x}{\sqrt{t}} \zeta d\zeta$$

$$= \frac{1}{\sqrt{t}} K_1 \left( \frac{x}{\sqrt{t}} \right)^y$$

$$\Rightarrow \|K_t\|_{L_x^\infty} = \frac{1}{\sqrt{t}} \|K_1\|_{L_y^\infty} \quad (t \text{ is fixed}).$$

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## Tool : Method of stationary phase

$$K_1(x) = \int_{\mathbb{R}} e^{-i\zeta^2 + ix\zeta} d\zeta \\ = \int_{\mathbb{R}} e^{-i\phi(\zeta)} d\zeta \quad \leftarrow \text{oscillatory integral}$$

where  $\phi(\zeta) = \zeta^2 - x\zeta \quad (x \text{ is fixed.})$

Idea : Integration by parts.

$$e^{-i\phi(\zeta)} = \frac{\partial_{\zeta} e^{-i\phi(\zeta)}}{-i\phi'(\zeta)} \quad \phi'(\zeta) = 2\zeta - x$$

← If  $\phi'(\zeta)$  is not small, good.

$$\int_{\mathbb{R}} e^{-i\phi(\zeta)} d\zeta = \int_{\mathbb{R}} \partial_{\zeta} e^{-i\phi(\zeta)} \cdot \frac{1}{-i\phi'(\zeta)} d\zeta$$

IBP

$$= \int_{\mathbb{R}} e^{-i\phi(\zeta)} \left( \frac{1}{i\phi'(\zeta)} \right)' d\zeta$$

Let  $\Psi \in C^\infty(\mathbb{R}; [0, 1])$  s.t.  $\Psi(\xi) = \begin{cases} 1, & |\xi| \leq 1 \\ 0, & |\xi| \geq 2 \end{cases}$  ④



$$\Rightarrow K_1(x) = \int_{\mathbb{R}} e^{-i\Phi(\xi)} \underbrace{\Psi(2\xi-x)}_{\Phi(\xi)} d\xi$$

$$+ \int_{\mathbb{R}} e^{-i\Phi(\xi)} (1-\Psi)(2\xi-x) d\xi =: I(x) + II(x)$$


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$$\circ |I(x)| \leq \int |\Psi(\underbrace{2\xi-x}_{-2 \leq \xi \leq 2})| d\xi \lesssim 1.$$

$$\begin{aligned} \cdot II(x) &\stackrel{IBP}{=} \int e^{-i\Phi(\xi)} \partial_{\xi} \left( \frac{(1-\Psi)(2\xi-x)}{i\Phi'(\xi)} \right) d\xi \\ &= \int e^{-i\Phi(\xi)} \left\{ -2 \frac{(1-\Psi)(2\xi-x)}{i(2\xi-x)^2} \right\} d\xi \quad (\Phi'(\xi) = |2\xi-x| \geq 1) \\ &\quad + \int e^{-i\Phi(\xi)} \frac{(1-\Psi)'(2\xi-x)}{i\Phi'(\xi)} d\xi \quad (1-\Psi)' = -\Psi' \\ &\quad \text{supported on } [-2, -1] \cup [1, 2] \end{aligned}$$

$$\Rightarrow |\mathbb{I}(x)| \leq c \int_{|2\bar{z}-x| \geq 1} \frac{1}{(2\bar{z}-x)^2} d\bar{z}$$

$$+ \int_{2\bar{z}-x \in \text{supp } \Psi'} |(1-\psi)'(2\bar{z}-x)| d\bar{z} \lesssim 1.$$

□

### 6.2 Glimpse on oscillatory integrals

$$I(\lambda) = \int_a^b e^{i\lambda \Phi(x)} \psi(x) dx.$$

phase  $\Phi(x)$ , real-valued

$\psi(x)$ , complex-valued, smooth, with cpt support.

Lemma:  $\text{supp } \psi \subset \underbrace{(a, b)}$

$\Phi'(x) \neq 0$  for all  $x \in [a, b]$ .  
cpt subset

Then,  $I(\lambda) = \Theta(\lambda^{-N})$  as  $\lambda \rightarrow \infty$ .  $\forall N \in \mathbb{N}$ .

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$$\left( \begin{array}{l} \text{Big } O \text{ notation, } f = \Theta(g^*) \text{ if } \lim \frac{|f|}{g} \leq c \\ \text{Little } o \text{ notation, } f = o(g) \text{ if } \lim \frac{|f|}{g} = 0. \end{array} \right.$$

Pf: Let  $Df(x) = \frac{1}{i\lambda\phi'(x)} \frac{df}{dx}$

$$\Rightarrow D(e^{i\lambda\phi}) = e^{i\lambda\phi}$$

$$\int (Df) g dx = \int f D^T g dx$$

Transpose:  $D^T f = -\frac{d}{dx} \left( \frac{f}{i\lambda\phi'(x)} \right)$

$$\Rightarrow I(\lambda) = \int_a^b \underbrace{e^{i\lambda\phi}}_{D^N(e^{i\lambda\phi})} \psi dx \stackrel{\substack{\text{IBP} \\ N \text{ times}}}{=} \int_a^b e^{i\lambda\phi} (D^T)^N \psi dx$$

↑ No bdry term  $\psi$  has cpt supp in  $(a, b)$

$$|\phi'(x)| \geq c \text{ on } [a, b]$$

$$\Rightarrow |I(\lambda)| \lesssim_{N, \psi, \phi} \lambda^{-N}$$

□

Prop (van der Corput)  $\phi$ , real-valued, smooth on  $(a, b)$  ⑦

Suppose  $\exists k \in \mathbb{N}$  st.  $|\phi^{(k)}(x)| \geq 1$  for all  $[a, b]$

Then,  $\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq C_k \underline{\lambda^{-1/k}}$ , indep of  $a, b$ .

provided (i)  $k \geq 2$ ,

much worse decay due to non-vanishing

or (ii)  $\phi'(x)$  is monotonic when  $k=1$ .

Pf: (ii)  $k=1$

$$\begin{aligned} \int_a^b e^{i\lambda\phi} dx &= \int_a^b D(e^{i\lambda\phi}) \cdot 1 dx \\ &= \underbrace{\int_a^b e^{i\lambda\phi} D^T(1) dx}_{+} + \underbrace{\frac{e^{i\lambda\phi}}{i\lambda\phi'} \Big|_a^b}_{1 \cdot 1 \leq 2/\lambda} \end{aligned}$$

$$\hookrightarrow \left| \int_a^b e^{i\lambda\phi} D^T(1) dx \right| \leq 2/\lambda.$$

$\phi'$  monotonic

$$= \frac{1}{\lambda} \left| \int_a^b e^{i\lambda\phi} \frac{d}{dx} \left( \frac{1}{\phi'} \right) dx \right| \leq \frac{1}{\lambda} \int_a^b \left| \frac{d}{dx} \left( \frac{1}{\phi'} \right) \right| dx.$$

$\Rightarrow \frac{1}{\phi'} \text{ monotonic}$

$$\Rightarrow \frac{1}{\lambda} \left| \int_a^b \frac{d}{dx} \left( \frac{1}{\phi'} \right) dx \right| \stackrel{\text{ETC}}{\leq} \frac{1}{\lambda} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \lesssim 1/\lambda.$$

(i)  $k \geq 2$ : We proceed by induction on  $k$ .

(F)

Suppose that the result holds for  $k$ .

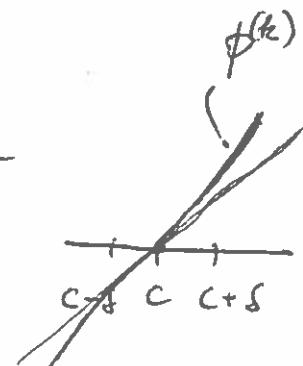
WLOG, assume  $\phi^{(k+1)}(x) \geq 1$ ,  $\forall x \in [a, b]$

- Let  $x = c$  be the (unique) point in  $[a, b]$

s.t.  $|\phi^{(k)}(x)|$  attains its min.

- If  $\phi^{(k)}(c) = 0$ , then  $|\phi^{(k)}(x)| \geq \delta$  on  $(c-\delta, c+\delta)^c$

write  $\int_a^b = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b$



$$\left| \frac{\phi^{(k)}}{\delta} \right| \geq 1$$

$$i \lambda \phi = i \lambda \delta \cdot \frac{\phi}{\delta}$$

By inductive hypothesis,

$$\left| \int_a^{c-\delta} + \int_{c+\delta}^b e^{i \lambda \phi} dx \right| \leq c_k (\lambda \delta)^{-1/k}$$

On the other hand,

$$\left| \int_{c-\delta}^{c+\delta} \dots \right| \leq 2\delta$$

equate them

$$\Rightarrow \delta \sim (\lambda \delta)^{-1/k} \Leftrightarrow \delta \sim \lambda^{-1/k+1}$$

(When  $k+1=2$ ,  
 $\phi^{(k+1)} \geq 1 \Rightarrow \phi'$  is monotonic.)

If  $\Phi^{(k)}(c) \neq 0$ , then  $c = a$  (or  $c = b$ ) ⑨

write  $\int_a^b = \int_a^{a+\delta} + \int_{a+\delta}^b$  and proceed as before



Cor: Same assumption as in the previous prop.

$$\left| \int_a^b e^{i\lambda \Phi(x)} \psi(x) dx \right| \leq C_k \lambda^{-1/k} \left[ |\psi(b)| + \int_a^b |\psi'(x)| dx \right]$$

Pf: Let  $F(x) = \int_a^x e^{i\lambda \Phi(y)} dy$

$\Rightarrow$  By Prop,  $|F(x)| \leq C_k \lambda^{-1/k}$

$$\Rightarrow \int_a^b e^{i\lambda \Phi(x)} \psi dx \stackrel{\text{IBP}}{=} F(b)\psi(b) - \cancel{F(a)\psi(a)} - \int_a^b \cancel{F(x)} \psi'(x) dx$$

$\stackrel{0}{\longrightarrow}$

