

- No class next week
- 1-d cubic NLS:  $i\partial_t u + \Delta u = \pm |u|^2 u$ ,  $x \in \mathbb{R}$

We prove LWP in  $L^2(\mathbb{R})$ .  $\stackrel{\uparrow}{\Rightarrow}$  GWP in  $L^2(\mathbb{R})$ .

mass conservation

What if  $u_0 \in H^s(\mathbb{R})$  for some  $s > 0$ ?

If  $s \in \mathbb{N}$ , we can use Leibniz rule:  $\partial_x(fg) = \partial_x f \cdot g + f \cdot \partial_x g$ .

For general  $s > 0$ , we need the fractional Leibniz rule:

$s \in (0, 1]$ ,  $1 < r, p_1, p_2, q_1, q_2 < \infty$  s.t.

Then,

$$\| |\nabla|^s (fg) \|_{L_x^r} \lesssim \| |\nabla|^s f \|_{L^{p_1}} \| g \|_{L^{q_1}} + \| f \|_{L^{p_2}} \| |\nabla|^s g \|_{L^{q_2}}$$

$$\left\{ \begin{array}{l} \frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}, \quad j = 1, 2. \\ |\nabla|^s \end{array} \right.$$

$$\underline{\text{Moral}}: |\nabla|^s(fg) \approx (|\nabla|^s f)g + f(|\nabla|^s g) \quad (2)$$

- $s=0$ : Hölder
- $s=1$ : "Leibniz rule" & Hölder

↑ really need  $s \in 2\mathbb{N}$ :  $|\nabla|^{2s} = (-\Delta)^{s/2}$

$$\textcircled{*} \Rightarrow \|\langle \nabla \rangle^s (fg)\|_{L_x^r} \lesssim \|\langle \nabla \rangle^s f\|_{L_x^{p_1}} \|g\|_{L_x^{q_1}} + \|f\|_{L_x^{p_2}} \|\langle \nabla \rangle^s g\|_{L_x^{q_2}}$$

$$\text{b/c } \langle \xi \rangle^s \sim 1 + |\xi|^s$$

- LWP of cubic NLS in  $H^s(\mathbb{R})$ ,  $s \geq 0$

Given  $T > 0$ , set  $X_T^s = C_T H^s \cap L_T^\infty W_x^{s,4}$

$$\|\Gamma_{u_0}(u)\|_{X_T^s} = \|\langle \nabla \rangle^s \Gamma_{u_0}(u)\|_{X_T^0} \stackrel{\text{str.}}{\lesssim} \|\langle \nabla \rangle^s u_0\|_{L_x^2} + \|\langle \nabla \rangle^s (|u|^2 u)\|_{L_T^{8/3} L_x^{4/3}}$$

$$\lesssim \|u_0\|_{H^s} + T^{1/2} \|\langle \nabla \rangle^s (|u|^2 u)\|_{L_T^{8/3} L_x^{4/3}}$$

$$\textcircled{*} \& \text{Hölder int} \quad \lesssim \|u_0\|_{H^s} + T^{1/2} \underbrace{\|u\|_{L_T^\infty W_x^{s,4}}^3}_{\lesssim \|u\|_{X_T^s}^3} \lesssim \|u\|_{X_T^s}^3$$

By a similar computation,

③

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T^s} \lesssim T^{1/2} \left( \|u\|_{X_T^s}^2 + \|v\|_{X_T^s}^2 \right) \|u - v\|_{X_T^s}$$

$\Rightarrow$  For  $T = T(\|u_0\|_{H^s}) > 0$  sufficiently small,

$\Gamma_{u_0}$  is a contraction  $\overline{B_R} \subset X_T^s$  where  $R \sim \|u_0\|_{H^s}$

$\Rightarrow$  1-d cubic NLS is LWP in  $H^s(\mathbb{R})$ ,  $s \geq 0$

$\Rightarrow$  GWP in  $H^s(\mathbb{R})$ , ( $s \geq 0$ ) (mass conservation & persistence of reg.)  
 improvement  $\leftarrow$  need to prove

$$\begin{aligned} & \|\langle v \rangle^s (|u|^2 u)\|_{L_T^{8/3} L_x^{4/3}} \\ & \lesssim \|u\|_{L_T^8 W_x^{s, 4}} \|u\|_{L_T^2 L_x^4}^2 \end{aligned}$$

$$\cdot \frac{d=2}{p=3} : \quad S_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1} = 1 - 1 = 0. \quad (4)$$

$L^2$ -critical / mass-critical.

Thm 2: Cubic NLS on  $\mathbb{R}^2$  is locally well-posed in  $L^2(\mathbb{R}^2)$  and is also globally well-posed in  $L^2(\mathbb{R}^2)$  with small initial data. ( $\|u_0\|_{L^2}$  is suff. small)

$$(q, r) = (4, 4) \text{ is Schrödinger admissible: } \frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

$$X_T = X_T^\circ = G L^2 \cap L_T^4 L_x^4.$$

$$\| \Gamma_{u_0}(u) \|_{X_T} \leq \| S(t) u_0 \|_{X_T} + C_2 \| |u|^2 u \|_{L_T^{4/3} L_x^{4/3}}$$

$$\leq \underline{\underline{C_1 \|u_0\|_{L^2}}} + C_2 \|u\|_{L_T^4 L_x^4}^3$$

No power of  $T$ .

(\*\*)

(5)

$$\| \Gamma_{u_0}(u) - \Gamma_{u_0}(v) \|_{X_T} \leq C_2 \| |u|^2 u - |v|^2 v \|_{L_T^{4/3} L_x^{4/3}}$$

$$\leq C_3 \left( \|u\|_{L_T^4 L_x^4}^2 + \|v\|_{L_T^4 L_x^4}^2 \right) \|u - v\|_{L_T^4 L_x^4}$$

We need this to be less than 1.

Set  $R = 2C_1 \|u_0\|_{L^2}$  and let  $u \in \overline{B}_R \subset X_T$ .

$$\Rightarrow \cdot \| \Gamma_{u_0}(u) \|_{X_T} \leq \frac{1}{2} R + C_2 R^3 \leq R \text{ if } R \ll 1. \quad (C_2 R^2 \leq \frac{1}{2})$$

$$\begin{aligned} \cdot \| \Gamma_{u_0}(u) - \Gamma_{u_0}(v) \|_{X_T} &\leq 2C_3 R^2 \|u - v\|_{X_T} \\ &\leq \frac{1}{2} \|u - v\|_{X_T} \text{ if } R \ll 1 \end{aligned}$$

$\Leftarrow$  These estimates hold true for  $T = \infty$ .

$\Rightarrow$  GWP for small initial data in  $L^2(\mathbb{R}^2)$

(by running a fixed pt argument in  $\overline{B}_R \subset X_\infty$ .)

(6)

• What about large  $L^2$ -data?

$$\|S(t)u_0\|_{L^4(\mathbb{R}_t; L_x^4)} \stackrel{\text{Str.}}{\approx} \|u_0\|_{L^2}$$

DCT

$$\Rightarrow \lim_{T \rightarrow 0} \frac{\|S(t)u_0\|_{L_T^4 L_x^4}}{L_T^4} = 0.$$

but  $\|S(t)u_0\|_{L_T^\infty L_x^2} (= \|u_0\|_{L_x^2}) \rightarrow 0$ . as  $T \rightarrow 0$ .

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$$\|\Gamma u_0(u)\|_{L_T^4 L_x^4} \leq \|S(t)u_0\|_{L_T^4 L_x^4} + C_2 \| |u|^2 u \|_{L_T^{4/3} L_x^{4/3}}$$

We do NOT apply the Strichartz esti.

Fix  $\gamma > 0$  small. Then, given  $u_0 \in L^2(\mathbb{R}^2)$ ,  $\exists T = T(u_0) > 0$  small s.t.

$$\|S(t)u_0\|_{L_T^4 L_x^4} \leq \frac{1}{2} \gamma.$$

(7)

Let  $u \in \overline{B_y} \subset L_T^4 L_x^4$ .

$$\Rightarrow \|\Gamma_{u_0}(u)\|_{L_T^4 L_x^4} \leq \frac{1}{2} y + C_2 y^3 \leq y$$

Also,

$$\begin{aligned} \|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{L_T^4 L_x^4} &\leq C_3 \left( \|u\|_{L_T^4 L_x^4}^2 + \|v\|_{L_T^4 L_x^4}^2 \right) \|u - v\|_{L_T^4 L_x^4} \\ &\leq \underbrace{2C_3 y^2}_{\leq 1/2} \|u - v\|_{L_T^4 L_x^4}, \quad \forall u, v \in \overline{B_y} \end{aligned}$$

$\Rightarrow$  Contraction mapping principle

$\Rightarrow \exists! u \in \overline{B_y} \subset L_T^4 L_x^4$ , solve to NLS. ( $u = \Gamma_{u_0}(u)$ )

By ~~(\*)~~,  $\|u\|_{C_T L^2} = \|\Gamma_{u_0}(u)\|_{C_T L^2} \leq C_1 \|u_0\|_{L^2} + C_2 y^3 < \infty$

$\Rightarrow u \in C_T L^2 \cap L_T^4 L_x^4$

- $T = T(u_0)$  depends on "the profile of  $u_0$ "  
 $\Leftarrow$  critical nature of the problem.

- We could simply run a contraction argument in

$$\overline{A_{R,\gamma}} = \{ u \in \overline{B_R} \subset X_T, \text{ and } u \in \overline{B_\gamma} \subset L_T^4 L_x^4 \}$$

$$R \sim \|u_0\|_{L^2}, \quad \gamma \ll 1,$$

endowed with the  $X_T$ -norm.

$$\textcircled{**} \Rightarrow \|\Pi_{u_0}(u)\|_{X_T} \leq C_1 \|u_0\|_{L^2} + C_2 \gamma^3$$

$$\leq 2C_1 \|u_0\|_{L^2} = R$$

$$\textcircled{***} \Rightarrow \|\Pi_{u_0}(u) - \Pi_{u_0}(v)\|_{X_T} \leq \underbrace{2C_3 \gamma^2}_{\leq \frac{1}{2}} \|u - v\|_{X_T} \Rightarrow \Pi_{u_0} : \overline{A_{R,\gamma}} \hookleftarrow$$

Also,  $\|\Pi_{u_0}(u)\|_{L_T^4 L_x^4} \stackrel{\leq \frac{1}{2}}{\leq} \frac{1}{2} \gamma + \gamma^3 \leq \gamma$ .