

It remains to prove (iii) :

$$\left\| \int_0^t S(t-t') F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}^{q'}$$

$(q, r), (\tilde{q}, \tilde{r})$, Sch. admissible.

$$(q, r) = (\tilde{q}, \tilde{r})$$

char. func / indicator func

$$\begin{aligned} \left\| \int_0^t S(t-t') F(t') dt' \right\|_{L_t^q L_x^r} &= \left\| \int_R S(t-t') \underbrace{\mathbf{1}_{[0,t]}(t')}_{L_x^{r'}} F(t') dt' \right\|_{L_t^q L_x^r} \\ &\stackrel{\text{as before}}{\lesssim} \left\| \int_R \frac{1}{|t-t'|^{d(\frac{1}{2}-\frac{1}{r})}} \left\| \mathbf{1}_{[0,t]}(t') F(t') \right\|_{L_x^{r'}} dt' \right\|_{L_t^q} \\ &\stackrel{\text{H-L-S}}{\lesssim} \|F\|_{L_t^{q'} L_x^{r'}} \end{aligned}$$

$$\textcircled{a} \quad \left\| \int_{t' < t} S(t-t') F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \quad \textcircled{2}$$

$$\textcircled{b} \quad \left\| \int_{t' < t} S(t-t') F(t') dt' \right\|_{L_t^\infty L_x^2} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$$

$$\textcircled{a} \quad \mathbb{1}_{(-\infty, t]} = \mathbb{1}_{(-\infty, 0)} + \mathbb{1}_{[0, t]}$$

Pf of \textcircled{b}

$$\begin{aligned} (\text{LHS})^2 \\ \text{at fixed } t \end{aligned} = \left\langle \underbrace{\int_{t_1 < t} S(t-t_1) F(t_1) dt_1}_{S(t) S(-t_1)}, \underbrace{\int_{t_2 < t} S(t-t_2) F(t_2) dt_2}_{S(t) S(-t_2)} \right\rangle_{L_x^2}$$

$$\left(\int (S(t)f) \bar{g} dx = \int f \overline{S(t)g} dx \right)$$

$$= \int_{t_1 < t} \left\langle F(t_1), \int_{t_2 < t} S(t_1 - t_2) F(t_2) dt_2 \right\rangle_{L_x^2} dt_1,$$

$$\stackrel{\text{H\"older}}{\leq} \int_R \|F(t_1)\|_{L_x^{r'}} \left\| \int_{t_2 < t} S(t_1 - t_2) F(t_2) dt_2 \right\|_{L_x^r} dt_1,$$

$$\stackrel{\text{H\"older}}{\leq} \|F\|_{L_t^{q'} L_x^{r'}} \left\| \int_{t_2 < t} S(t-t_2) F(t_2) dt_2 \right\|_{L_t^q L_x^r} \stackrel{\textcircled{a}}{\lesssim} (\text{RHS})^2$$

□

- WTS (iii)

(3)

$$\Leftrightarrow \left| \iint_{t' < t} \langle S(t-t') F(t'), G(t) \rangle_{L_x^2} dt' dt \right|$$

④

$$\lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \|G\|_{L_t^{q'} L_x^{r'}}$$

where the implicit const is indep of G . (and F)

- Fix (\tilde{q}, \tilde{r}) .

a) \Rightarrow ④ holds for $q = \tilde{q}$, $r = \tilde{r}$.

b) \Rightarrow ④ holds for $q = \infty$, $r = 2$.

interpolation

\Rightarrow ④ holds for any $q \geq \tilde{q}$ (and $r = r(q)$.)

- For ④ with $q < \tilde{q}$, we use symmetry in ④.

④ with $q \geq \tilde{q}$ implies

$$\left| \int_{t'} \left\langle F(t'), \int_{t > t'} S(t-t') G(t) dt \right\rangle_{L_x^2} dt' \right| \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \|G\|_{L_t^{q'} L_x^{r'}}$$

view F as a duality var.

duality

$$\Leftrightarrow \left\| \int_{t>t'} s(t-t') G(t) dt \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \lesssim \|G\|_{L_t^{q'} L_x^{r'}}$$

(4)

By relabelling,

$$\left\| \int_{t'>t} s(t-t') F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} , \text{ for } q \leq \tilde{q}$$

Now, write $\mathbb{1}_{[0,t]} = \underbrace{\mathbb{1}_R(t)}_{\equiv 1} - \mathbb{1}_{(t,\infty)}$

$\Rightarrow \textcircled{*}$ holds for $q \leq \tilde{q}$

□

Rmk: Christ - Kiselev lemma (for inserting $\mathbb{1}_{[0,t]}$). See Tao's book.

$$\begin{aligned} \left\| \int_{\mathbb{R}} s(t-t') F(t') dt' \right\|_{L_t^q L_x^r} &= \left\| s(t) \int_{\mathbb{R}} s(-t') F(t') dt' \right\|_{L_t^q L_x^r} \\ &\stackrel{(i)}{\lesssim} \left\| \int_{\mathbb{R}} s(-t') F(t') dt' \right\|_{L_x^2} \stackrel{(ii)}{\lesssim} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \end{aligned}$$

(5)

Sec 5: LWP of NLS, part II

Ex 1: $d=1, p=3$ (cubic NLS on \mathbb{R})

$$S_{crit} = \frac{d}{2} - \frac{2}{p-1} = -\frac{1}{2}$$

Thm 1: Let $s \geq 0$. cubic NLS on \mathbb{R} is locally well-posed in $H^s(\mathbb{R})$

Pf: $s=0$. Let $u_0 \in L^2(\mathbb{R})$

$$(NLS) \Leftrightarrow u(t) = \Gamma_{u_0}(u)(t) := S(t)u_0 + i \int_0^t S(t-t')|u|^2 u(t') dt'$$

\uparrow
on $[-T, T]$

\uparrow $L^{\frac{8}{3}}_{[-T, T]}$

- Note: $(g, r) = (8, 4)$ is admiss.

$$\frac{2}{8} + \frac{1}{4} = \frac{1}{2}$$

- Set $X_T = \underbrace{C([-T, T]; L_x^2(\mathbb{R}))}_{= C_T L_x^2} \cap \underbrace{L_T^8 L_x^4}_{\downarrow L_T^8([-T, T]; L_x^4)} \quad \begin{matrix} X \cap Y \\ \leftarrow \|u\|_{X \cap Y} \\ = \|u\|_X + \|u\|_Y \end{matrix}$

$$\|\Gamma_{u_0}(u)\|_{X_T} \stackrel{\text{Str.}}{\leq} C_1 \|u_0\|_{L_x^2} + C_2 \underbrace{\||u|^2 u\|_{L_T^{8/7} L_x^{4/3}}}_{\substack{\text{H\"older int} \\ \leq C_3 T^{1/2} \||u|^2 u\|_{L_T^{8/3} L_x^{4/3}}}} \quad (8.4), \text{admis}$$

$$\frac{7}{8} = \frac{1}{2} + \frac{3}{8}$$

$$\underbrace{\leq \frac{3}{8} \|u\|_{L_T^8 L_x^4}^3}_{\substack{\text{H\"older} \\ \text{int.}, \infty}} \leq \|u\|_{X_T}^3$$

$$\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

For $u \in \overline{B}_R \subset X_T$,

$$\begin{aligned} \|\Gamma_{u_0}(u)\|_{X_T} &\leq C_1 \|u_0\|_{L_x^2} + \underbrace{C_3 T^{1/2} R^2 \cdot R}_{\leq 1/2} \\ &\leq 2 C_1 \|u_0\|_{L_x^2} =: R \quad \nwarrow \text{by choosing } T = T(R) \text{ suff. small} \end{aligned}$$

provided

$$\underline{C_3 T^{1/2} R^2 \leq 1/2}.$$

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- Similarly, for $u, v \in X_T$

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} \leq C_3 T^{1/2} \underbrace{\|(u^2 u - v^2 v)\|_{L_T^{8/3} L_x^{4/3}}}_{\text{telescoping sum}}$$

$$\begin{aligned} u\bar{u}u - v\bar{v}v &= (u-v)\bar{u}u \\ &\quad + v(\bar{u}-\bar{v})u \\ &\quad + v\bar{v}(u-v) \end{aligned}$$

Young's

$$\leq C_4 T^{1/2} \left(\|u\|_{X_T}^2 + \|v\|_{X_T}^2 \right) \|u - v\|_{X_T}$$

$$\text{Young's: } ab \leq \frac{ap}{p} + \frac{b^p}{p}$$

$$\begin{aligned} &\leq \underbrace{2C_4 T^{1/2} R^2}_{\leq 1/2} \end{aligned}$$

By choosing $T^{1/2} \sim R^{-2}$ i.e. $T \sim R^{-4}$, we conclude that

Γ_{u_0} is a contraction on $\overline{B_R} \subset X_T$, $R = 2C_1 \|u_0\|_{L_x^2}$

\Rightarrow Banach fixed pt thm $\Rightarrow \exists! u \in \overline{B_R} \subset X_T$ s.t. $u = \Gamma_{u_0}(u)$



Rmk : ① Local existence time $T = T(\|u_0\|_{L^2}) \sim \frac{1}{\|u_0\|_{L^2}^4}$ ⑧

\Rightarrow Using the L^2 -conservation, we conclude that
Cubic NLS on \mathbb{R} is global well-posed in $L^2(\mathbb{R})$.



② Uniqueness holds in $\overline{B_R} \subset C_T L_x^2 \cap L_T^\infty L_x^4$

\Rightarrow uniqueness in $C_T L_x^2 \cap L_T^\infty L_x^4$

conditional uniqueness.

(unconditional uniqueness \Leftarrow uniqueness in the entire $C_T H^s(\mathbb{R}^d)$)

③ should show continuity of u in t . (with values in L^2)
- omitted