

Sec 3: Scaling heuristics

- Scaling symmetry (dilation symmetry)

If u is a soln to (NLS) with $u|_{t=0} = u_0$,

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x^d,$$

then set

$$u^\lambda(t, x) = \frac{1}{\lambda^a} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

$$u_0^\lambda(x) = \frac{1}{\lambda^a} u_0\left(\frac{x}{\lambda}\right)$$

$$\Rightarrow a+2 = ap \Rightarrow a = \frac{2}{p-1}$$

$$\Rightarrow u^\lambda(t, x) = \frac{1}{\lambda^{2/p-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \text{ is also a soln to (NLS)}$$

with the scaled initial data $u^\lambda|_{t=0} = u_0^\lambda$.

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This scaling symmetry induces the so-called scaling-critical Sobolev index $s_c = s_{\text{crit.}}$

$$\|f^\lambda\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|f\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$$

$$\cdot \|f^\lambda\|_{\dot{H}^s} = \left(\int |\xi|^{2s} |\hat{f}^\lambda(\xi)|^2 d\xi \right)^{1/2}$$

$$\left(f^\lambda(x) = \frac{1}{\lambda^{d/p-1}} f\left(\frac{x}{\lambda}\right) \Rightarrow \widehat{f^\lambda}(\xi) = \lambda^{d-\frac{2}{p-1}} \widehat{f}(\lambda \xi) \right)$$

$$= \left(\int |\lambda \xi|^{2s} |\widehat{f}(\lambda \xi)|^2 d(\lambda^d \xi) \right)^{1/2} \lambda^{d-\frac{2}{p-1}-s-\frac{d}{2}}$$

$$= \lambda^{\frac{d}{2}-\frac{2}{p-1}-s} \|f\|_{\dot{H}^s}, \quad \forall \lambda > 0.$$

$$\Rightarrow \boxed{s_c = \frac{d}{2} - \frac{2}{p-1}} \quad (< \frac{d}{2})$$

Given $u_0 \in H^s(\mathbb{R}^d)$, the Cauchy problem (NLS) is ③

- subcritical (w.r.t. scaling) if $s > s_c = s_c(d, p)$
⇒ expect good behavior, LWP, etc.

- critical if $s = s_c$.

delicate balance between

lin. dispersion and nonlinear concentration

- supercritical if $s < s_c$

⇒ expect ill-posedness

- subcritical case: $\|u_0^\lambda\|_{\dot{H}^s} = \lambda^{\frac{s_c-s}{2}} \|u_0\|_{\dot{H}^s}$

$$u \text{ on } [0, T] \longleftrightarrow u^\lambda \text{ on } [0, \lambda^2 T] \quad \lambda \gg 1$$
$$u_0 \gg u_0^\lambda \text{ (in } \dot{H}^s \text{)}$$

small data ⇒ soln lives longer.

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• supercritical case

$$u \text{ on } [0, T] \longleftrightarrow u^1 \text{ on } [0, \lambda^2 T]$$

$$u_0 \ll u_0^1 \text{ (in } H^s\text{)}$$

\Rightarrow larger initial data , longer time of existence

\Leftarrow Too good to be true.

• critical case: $s = s_c$

need more info than the H^s -norm of initial data.

Other symmetries: time translation: $t \rightarrow t + t_0$

spatial translation: $x \rightarrow x + x_0$

$u \mapsto e^{i\theta} u$, $\theta \in \mathbb{R}$ $\xrightarrow{\text{Noether}}$ mass conservation

Galilean symmetry: $u(t, x) \mapsto e^{i\frac{V}{2} \cdot x} e^{-i\frac{|V|^2}{4}t} u(t, x + vt)$

time reversal: $u(t, x) \mapsto \overline{u(-t, x)}$

induces another critical regularity $s_c^\infty = 0$.

- If we prove LWP by a fixed pt argument
(contraction)

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for $|u|^{p+1}u$, $p \in 2N + 1$ (algebraic), then

$$\Phi : u_0 \in H^s \mapsto u \in C_T H^s$$

is analytic.

So, if we know that the soln map is not smooth, then this means that we can not prove LWP by a fixed pt argument. (but does not say that it's ill-posed.)

- We will discuss more about ill-posedness later in the course.

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Sec 4 : Strichartz estimate

① dispersive estimate

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases} \Rightarrow u(t) = \mathcal{F}^{-1}(e^{-4\pi^2 i t |\xi|^2} \hat{u}_0(\xi)) = S(t) u_0$$

$$\begin{aligned} \Rightarrow S(t) u_0(x) &= \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} e^{-4\pi^2 i t |\xi|^2} \hat{u}_0(\xi) d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot (x - 2\pi \xi t)} \hat{u}_0(\xi) d\xi \end{aligned}$$

If \hat{u}_0 is "localized" around $\xi = \xi_0 \in \mathbb{R}^d$,

$$\Rightarrow 2\pi \xi (\sim 2\pi \xi_0) = \text{phase velocity}.$$

Consider $u_0 = e^{2\pi i x \cdot \xi_0}$ ← plane wave at freq ξ_0 . (7)

$$\Rightarrow s(t) u_0(x) = e^{2\pi i x \cdot \xi_0 - 4\pi^2 i t |\xi_0|^2}$$

$$= e^{2\pi i \xi_0 \cdot \left(\frac{x - 2\pi \xi_0 t}{\cancel{t}} \right)}$$

phase velocity = velocity for the propagation
of oscillation (= how phase changes)

Next, let's consider the following spatially localized wave:

$$v_0 = \underbrace{e^{-\frac{|x|^2}{4\sigma^2}}}_{\text{spatial localization}} e^{2\pi i x \cdot \xi_0}$$

$$\Rightarrow \hat{v}_0(\xi) = (4\pi\sigma^2)^{d/2} e^{-4\pi^2 \sigma^2 |\xi - \xi_0|^2}$$

↳ follows from FACT: ① $g(x) = e^{-\pi|x|^2} \Rightarrow \hat{g}(\xi) = e^{-\pi|\xi|^2}$
 ② $g_\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \Rightarrow \hat{g}_\varepsilon(\xi) = \varepsilon^d \hat{g}(\varepsilon \xi)$

Pf of ① : Let $F(s) = \int_{\mathbb{R}} e^{-\pi(x+is)^2} dx, s \in \mathbb{R}$ ⑧

$$\Rightarrow \frac{d}{ds} F(s) = \int_{\mathbb{R}} -2\pi i (x+is) e^{-\pi(x+is)^2} dx \\ = \int_{\mathbb{R}} i \frac{d}{dx} (e^{-\pi(x+is)^2}) dx = 0 \Rightarrow F(s) \equiv \text{const.}$$

$$\bullet \mathcal{F}(e^{-\pi|x|^2})(\vec{z}) = \int_{\mathbb{R}^d} e^{-\pi|x|^2} e^{-2\pi i x \cdot \vec{z}} dx = \prod_{j=1}^d \int_{\mathbb{R}} e^{-\pi(x_j + i z_j)^2} e^{\pi(i z_j)^2} dx_j \\ = \underbrace{\left(\int_{\mathbb{R}} e^{-\pi y^2} dy \right)^d}_{=1} e^{-\pi|\vec{z}|^2}$$

$$S(t) V_0 = (4\pi \sigma^2)^{d/2} \int e^{2\pi i x \cdot \vec{z} - 4\pi i t |\vec{z}|^2 - 4\pi^2 \sigma^2 |\vec{z} - \vec{z}_0|^2} d\vec{z}$$

$$\text{phase} = \frac{-4\pi^2(\sigma^2 + it)|\vec{z}|^2 + 2\pi \vec{z} \cdot (ix + 4\pi \sigma^2 \vec{z}_0) - 4\pi^2 \sigma^2 |\vec{z}_0|^2}{\text{complete square}}$$

$$= -4\pi^2(\sigma^2 + it) \left| \vec{z} - \frac{ix + 4\pi \sigma^2 \vec{z}_0}{4\pi(\sigma^2 + it)} \right|^2 \\ + \frac{|ix + 4\pi \sigma^2 \vec{z}_0|^2}{4(\sigma^2 + it)} - 4\pi^2 \sigma^2 |\vec{z}_0|^2$$

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Note that $\frac{|ix + 4\pi\sigma^2 \bar{z}_0|^2}{4(\sigma^2 + it)} - 4\pi^2 \sigma^{-2} |\bar{z}_0|^2$

$$= \frac{1}{\sigma^2 + it} \left(-\frac{|x|^2}{4} + 2\pi i \sigma^2 x \cdot \bar{z}_0 + 4\pi^2 \sigma^4 |\bar{z}_0|^2 \right) - 4\pi^2 \sigma^{-2} |\bar{z}_0|^2$$

$$= 2\pi i x \cdot \bar{z}_0 + \frac{1}{\sigma^2 + it} \left(-\frac{|x|^2}{4} + 2\pi i x \cdot \bar{z}_0 - 4\pi^2 \sigma^2 it |\bar{z}_0|^2 \right)$$

Complete square

$$= 2\pi i x \cdot \bar{z}_0 + \frac{1}{\sigma^2 + it} \left\{ \left(-\frac{1}{4} \right) |x - 4\pi t \bar{z}_0|^2 - 4\pi i t (\sigma^2 + it) |\bar{z}_0|^2 \right\}$$

$$= 2\pi i x \cdot \bar{z}_0 - 4\pi i t |\bar{z}_0|^2 - \frac{|x - 4\pi t \bar{z}_0|^2}{4(\sigma^2 + it)} \quad \leftarrow \text{indep of } \bar{z}$$

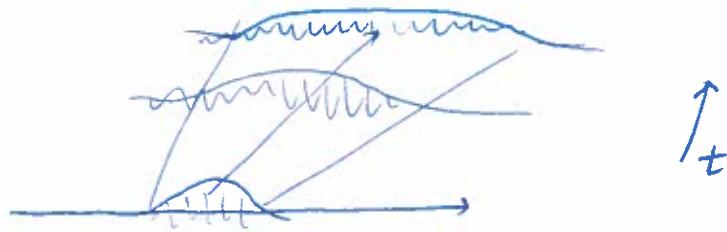
$$\Rightarrow S(t) V_0 = e^{2\pi i x \cdot \bar{z}_0 - 4\pi i t |\bar{z}_0|^2 - \frac{|x - 4\pi t \bar{z}_0|^2}{4(\sigma^2 + it)}}$$

$$\times (4\pi \sigma^2)^{d/2} \int e^{-4\pi^2(\sigma^2 + it) |z - \frac{ix + 4\pi \sigma^2 \bar{z}_0}{4\pi(\sigma^2 + it)}|^2} dz$$

$$= \underbrace{\left(\frac{\sigma^2}{\sigma^2 + it} \right)^{d/2}}_{\star} e^{2\pi i x \cdot \bar{z}_0 - 4\pi i t |\bar{z}_0|^2 - \frac{|x - 4\pi t \bar{z}_0|^2}{4(\sigma^2 + it)}}$$

localized wave travel at speed $4\pi \bar{z}_0$ = group velocity

- width of the wave packet increase since $\operatorname{Re} \frac{1}{4(\sigma^2 + it)} < \frac{1}{4\sigma^2}$ (10)
- amplitude of the wave packet decreases $\left(\frac{\sigma^2}{\sigma^2 + it} \right)^{1/2} \rightarrow 0$ as $t \rightarrow \infty$



Dispersive relation: lin. Schrödinger equation

$$i \partial_t u + \Delta u = 0$$

space-time F.T.

$$\Rightarrow -2\pi \tau \hat{u}(\tau, \vec{x}) - 4\pi^2 |\vec{x}|^2 \hat{u}(\tau, \vec{x}) = 0$$

$$\Rightarrow -2\pi (\tau + 2\pi |\vec{x}|^2) \hat{u}(\tau, \vec{x}) = 0.$$

i.e. The space-time Fourier transform $\hat{u}(\tau, \vec{x})$ of a soln to the linear Schrödinger eqn is a distribution supported on the paraboloid: $\tau + 2\pi |\vec{x}|^2 = 0$.

\Rightarrow Dispersive relation for Schrödinger egn: $\omega(\vec{z}) = 2\pi|\vec{z}|^2$

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• phase velocity: $\omega(\vec{z}) \frac{\vec{z}}{|\vec{z}|^2} = 2\pi\vec{z}$

• group velocity: $\nabla \omega(\vec{z}) = 4\pi\vec{z}$

Aside: * on page ⑨: Let $G(z) = \int_R e^{-z|x|^2} dx$, $z = a+ib$
with $a > 0$.

For $a > 0$, we have

$$G(a) = \int e^{-a|x|^2} dx \quad a^{1/2}x = \pi^{1/2}y$$

$$= \sqrt{\frac{\pi}{a}} \underbrace{\int e^{-\pi|y|^2} dy}_{=} = \sqrt{\frac{\pi}{a}}$$

Hence, two analytic functions $G(z)$ and $H(z) = \sqrt{\frac{\pi}{z}}$ on $\{Re z > 0\}$
agree on the positive real axis.

$$\Rightarrow G(z) = \int_R e^{-z|x|^2} dx = \sqrt{\frac{\pi}{z}} \text{ on } \{Re z > 0\}$$

\Rightarrow * on p. ⑨

The limiting case is given by the following

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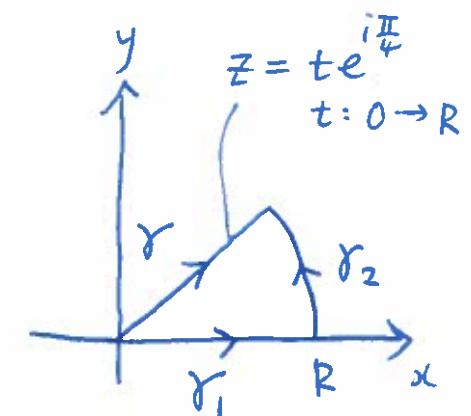
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Fresnel integral: $\int_{-\infty}^{\infty} e^{-ix^2} dx = \sqrt{\frac{\pi}{i}}$

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\gamma=R(R)} e^{-z^2} dz = e^{i\frac{\pi}{4}} \int_0^{\infty} e^{-it^2} dt \\ & \quad || \\ & \quad = \frac{\sqrt{i}}{2} \int_R^{\infty} e^{-ix^2} dx \end{aligned}$$

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\gamma_1} e^{-z^2} dz + \underbrace{\lim_{R \rightarrow \infty} \int_{\gamma_2} e^{-z^2} dz}_{=0} \\ & \quad = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \end{aligned}$$

\Rightarrow **



$$\begin{aligned}
 \Rightarrow K_t(x) &= \mathcal{F}^{-1}(e^{-4\pi^2 it |\xi|^2})(x) \\
 &= \int_{\mathbb{R}^d} e^{-4\pi^2 it |\xi|^2} e^{2\pi i x \cdot \xi} d\xi \\
 &= \prod_{j=1}^d \left(e^{-x_j^2/4it} \int_{\mathbb{R}} e^{-4\pi^2 it (\xi_j - \frac{x_j}{4\pi t})^2} d\xi_j \right) \\
 &= \frac{1}{(4\pi i t)^{d/2}} e^{-|x|^2/4it} = \frac{1}{2\pi} \sqrt{\frac{\pi}{it}} = \sqrt{\frac{1}{4\pi i t}}
 \end{aligned}$$

$\xrightarrow{\text{so as } t \rightarrow 0}$

$$\begin{aligned}
 \Rightarrow S(t)u_0(x) &= K_t * u_0(x) \\
 &= \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4it} u_0(y) dy
 \end{aligned}$$

Lemma (Dispersive estimate):

$$\|S(t)f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{d/2} \|f\|_{L_x^1(\mathbb{R}^d)}$$