

- No lecture on 29/01/18 (Mon)

Sec 2: Conservation laws, global well-posedness
and persistence of regularity

$$(NLS) \quad i\partial_t u + \Delta u = \pm |u|^{p-1}u. \quad \Rightarrow \quad \partial_t u = i\Delta u \mp i|u|^{p-1}u$$

- Conservation laws

Mass : $M(u) = \int_{\mathbb{R}^d} |u(x)|^2 dx$

If u is a soln to (NLS), then $M(u)(t) = M(u_0)$

$$\begin{aligned} \partial_t \int |u|^2 &= 2 \operatorname{Re} \int (\partial_t u) \bar{u} = 2 \operatorname{Re} i \int (\Delta u) \bar{u} \mp 2 \operatorname{Re} i \underbrace{\int |u|^{p+1}}_{=0} \\ &= -2 \operatorname{Re} i \int |\nabla u|^2 = 0. \end{aligned}$$

- For "rough" solns, we use well-posedness theory (continuous dependence) to prove the conservation of mass.

Given $u_0 \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$, take $\{u_{0,n}\}_{n=1}^\infty \subset H^\infty(\mathbb{R}^d)$

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s.t. $u_{0,n} \rightarrow u_0$ in $H^s(\mathbb{R}^d)$.

Denote by u_n , the soln to (NLS) with $u_n|_{t=0} = u_{0,n}$, i.e.

$$u_n(t) = S(t)u_{0,n} + i \int_0^t S(t-t') |u_n|^{p-1} u_n(t') dt'.$$

- $u(t) = S(t)u_0 + i \int_0^t S(t-t') |u|^{p-1} u(t') dt'$.

- For $n \gg 1$, we have $\|u_{0,n}\|_{H^s} \leq \|u_0\|_{H^s} + 1 =: R/2$
 $\xrightarrow{\text{LWP}}$ u_n and u exist on $[-T, T]$, $T = T(R) > 0$.

Also,

$$\begin{aligned} \|u_n - u\|_{CTH^s} &\leq \|u_{0,n} - u_0\|_{H^s} + \underbrace{CTR^{p-1}}_{\leq 1/2 \text{ by choosing } T=T(R) \text{ small}} \|u_n - u\|_{CTH^s} \\ &\Rightarrow \|u_n - u\|_{CTH^s} \lesssim \|u_{0,n} - u_0\|_{H^s} \end{aligned}$$

\Rightarrow continuous dependence, i.e.

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$\Phi = \text{soln map} : u_0 \in H^s \xrightarrow{\Phi(u_0)} u \in C_T H^s$
is continuous.

- Conservation of $M(u_n)$ and $u_n \rightarrow u$ in $C_T L^2$ implies the conservation of $M(u)$.

- Hamiltonian (= energy) : $H(u) = \frac{1}{2} \int |\nabla u|^2 dx \pm \frac{1}{p+1} \int |u|^{p+1} dx$

+ = defocusing case (repulsive)

- = focusing case (attractive)

- $H(u(t)) = H(u_0)$

- Momentum : $P(u) = \text{Im} \int u \nabla \bar{u} dx \stackrel{\text{IBP}}{=} -i \int u \nabla \bar{u} \in \mathbb{C}^d$.

- $P(u)(t) = P(u_0)$

• Consider the 1-d defocusing NLS, $s > \frac{1}{2}$. (4)

Let $u_0 \in H^s(\mathbb{R})$.

$$\Rightarrow \|u(t)\|_{H^1}^2 = \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 < \infty \text{ by Sobolev}$$

$$\lesssim M(u)|t) + H(u(t))$$

$$\stackrel{\text{cons}}{=} M(u_0) + H(u_0) =: C^2(u_0) < \infty$$

$$\|f\|_{L^q} \lesssim \|u\|_{H^s}^{-\frac{1}{2}}$$

$$\begin{cases} \frac{s}{d} = \frac{1}{2} - \frac{1}{q}, q < \infty \\ \frac{s}{d} > \frac{1}{2}, q = \infty \end{cases}$$

\Rightarrow Set $R = 2C(u_0)$ ($\geq 2\|u(t)\|_{H^1}$) and repeat the LWP argument.

$$\frac{\|u(T)\|_{H^1} \leq R/2}{\overbrace{R \mapsto R^{2T}}_{0 \leq T \sim R^{1-p}}} \Rightarrow \|u(jT)\|_{H^1} \leq R/2, \forall j \in \mathbb{Z}.$$

$\Rightarrow u$ exists globally in time.

Thm 1: $p \in 2N+1$. Then, 1-d def. NLS is globally well-posed
in $H^s(\mathbb{R})$.

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 $H^s(\mathbb{R}), s \geq 1$.

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Persistence of regularity: If $u_0 \in H^s(\mathbb{R})$, $s > 1$,
then $u_0 \in H^1(\mathbb{R}) \Rightarrow u \in C(\mathbb{R}; H^1(\mathbb{R}))$

Q: $u \in C(\mathbb{R}; H^s(\mathbb{R}))$?

Yes: product estimate: $s \geq 0$

$$(*) \quad \|fg\|_{H^s} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}$$

(\Leftarrow follows from the para product formula.

$$\begin{aligned} d=1: \quad \| |u|^{p-1} u \|_{H^s} &\lesssim \|u\|_{L^\infty}^{p-1} \|u\|_{H^s} \lesssim \|u\|_{H^1}^{p-1} \|u\|_{H^s} \\ &\lesssim R^{p-1} \|u\|_{H^s} \end{aligned}$$

$$\Rightarrow \|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + CR^{p-1} \int_0^t \|u(t')\|_{H^s} dt'$$

Gronwall $\Rightarrow \|u(t)\|_{H^s} \leq e^{CRt} \|u_0\|_{H^s} < \infty$ **

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$$\Rightarrow u(t) \in H^s(\mathbb{R})$$

& By LWP, $u \in C(\mathbb{R}; H^s(\mathbb{R}))$.

Rmk: ~~**~~ provides an exponential upper bound on the growth of a soln.

- It is of importance to improve this bound.

Bourgain, Staffilani, mid 90's : $\|u(t)\|_{H^s} \leq C(u_0) t^{\alpha(s-1)}$

Q: subpolynomial upper bound?

Q: Can we construct a soln s.t. the H^s -norm actually grows?

- $d=2$: defocusing cubic NLS ($p=3$)

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- LWP in $H^s(\mathbb{R}^2)$, $s > 1$.

Thm 2: The 2-d def cubic NLS is globally well-posed in $H^2(\mathbb{R}^2)$.

- Brezis-Gallouët inequality '80 : $s > d/2$

$$\|f\|_{L^\infty} \leq c_s \|f\|_{H^{d/2}} \left[\log \left(2 + \frac{\|f\|_{H^s}}{\|f\|_{H^{d/2}}} \right) \right]^{1/2}$$

(Sobolev's embedding thm : $\|f\|_{L^\infty} \lesssim \|f\|_{H^s}$, $s > d/2$)

Pf: $g = f / \|f\|_{H^{d/2}}$. $\Rightarrow \|g\|_{H^{d/2}} = 1$.

$$\text{WTS: } \|g\|_{L^\infty} \leq c_s \left(\log \left(2 + \|g\|_{H^s} \right) \right)^{1/2}.$$

$$\|g\|_{L^\infty} \leq \|\hat{g}\|_{L^1} = \int_{|\tilde{z}| \leq R} |\hat{g}(\tilde{z})| d\tilde{z} + \int_{|\tilde{z}| > R} |\hat{g}(\tilde{z})| d\tilde{z} \quad (8)$$

$$= \int_{|\tilde{z}| \leq R} \langle \tilde{z} \rangle^{d/2} |\hat{g}(\tilde{z})| \frac{1}{\langle \tilde{z} \rangle^{d/2}} d\tilde{z} + \int_{|\tilde{z}| > R} \langle \tilde{z} \rangle^s |\hat{g}(\tilde{z})| \frac{1}{\langle \tilde{z} \rangle^s} d\tilde{z}.$$

$$\stackrel{C-S}{\leq} \underbrace{\|g\|_{H^{d/2}}}_{\sim} \left(\log(2+R) \right)^{1/2} + \|g\|_{H^s} \underbrace{\left(\int_{|\tilde{z}| > R} \frac{1}{\langle \tilde{z} \rangle^{2s}} d\tilde{z} \right)^{1/2}}_{\sim R^{d/2-s}}$$

$$\text{Set } R = \|g\|_{H^s}^\theta ; \quad \theta = \frac{1}{s-d/2} > 0$$

$$\lesssim \left(\log(2 + \|g\|_{H^s}) \right)^{1/2}$$

□

$$\|u(t)\|_{H^2} \stackrel{*}{\leq} \|u_0\|_{H^2} + C \int_0^t \|u(t')\|_{H^2} \|u(t')\|_{L^\infty}^2 dt'. \quad (9)$$

(***)

$$\begin{aligned} &\stackrel{B-G}{\leq} \|u_0\|_{H^2} + C \int_0^t \|u(t')\|_{H^2} \left(1 + \log(1 + \|u(t')\|_{H^2}) \right) dt' \\ &=: F(t). \end{aligned}$$

depends on $\|u(t)\|_{H^1}$ appearing in B-G inequality.

$$\Rightarrow F'(t) = \|u(t)\|_{H^2} (1 + \log(1 + \|u(t)\|_{H^2}))$$

$$\leq F(t) (1 + \log(1 + F(t))).$$

$$\Rightarrow \frac{d}{dt} \log(1 + \log(1 + F(t))) \leq C$$

$$\Rightarrow \|u(t)\|_{H^2} \leq F(t) \leq e^{e^{ct}} C(u_0)$$

\Rightarrow Thm 2.