

Lec 3 22/01/18 (Mon)

①

sec 1: LWP of NLS in $H^s(\mathbb{R}^d)$, $s > d/2$

$$(NLS) \quad i\partial_t u + \Delta u = \pm |u|^{p-1}u$$

First, consider the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

⇒ Take F.T. in x

$$\begin{cases} i\partial_t \hat{u}(\xi) - |\xi|^2 \hat{u}(\xi) = 0 \\ \hat{u}(t, \xi)|_{t=0} = \hat{u}_0(\xi) \end{cases}$$

$$\begin{aligned} \Delta &= \partial_1^2 + \cdots + \partial_d^2 \xrightarrow{\text{F.T.}} \hat{\Delta}(\xi) = -4\pi^2 (\xi_1^2 + \cdots + \xi_d^2) \\ &\downarrow \\ (2\pi i \xi_1)^2 &= -4\pi^2 |\xi|^2. \end{aligned}$$

omitted for simplicity

$$\text{For fixed } \xi, \quad \hat{u}(t, \xi) = e^{-it|\xi|^2} \hat{u}_0(\xi) \quad \textcircled{2}$$

$$\Rightarrow u(t, x) = \mathcal{F}^{-1}(e^{-it|\xi|^2} \hat{u}_0(\xi))(x)$$

$$=: (e^{it\Delta} u_0)(x) \quad \partial_t u = i\Delta u$$

- $S(t) = e^{it\Delta}$ = linear Schrödinger operator

Next, we consider the nonhomogeneous lin. Schrödinger eqn

$$\begin{cases} i\partial_t u + \Delta u = F(t, x), & F \text{ given (and "nice")} \\ u|_{t=0} = u_0 \end{cases}$$

F.T. in x

$$\Rightarrow \partial_t \hat{u}(\xi) + i|\xi|^2 \hat{u}(\xi) = -i \hat{F}(t, \xi), \quad \forall \xi \in \mathbb{R}^d.$$

$$\partial_t (e^{it|\xi|^2} \hat{u}(\xi)) = -i e^{it|\xi|^2} \hat{F}(t, \xi)$$

integrating factor

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Integrate from 0 to t.

$$e^{it|\tilde{z}|^2} \hat{u}(t, \tilde{z}) - \hat{u}_0(\tilde{z}) = -i \int_0^t e^{it'|\tilde{z}|^2} \hat{F}(t', \tilde{z}) dt'.$$

$$\Rightarrow \hat{u}(t, \tilde{z}) = \underbrace{e^{-it|\tilde{z}|^2} \hat{u}_0(\tilde{z})}_{\text{given}} - i \int_0^t \underbrace{e^{-i(t-t')|\tilde{z}|^2} \hat{F}(t', \tilde{z})}_{\text{unknown}} dt'.$$

Take \mathcal{F}^{-1}

$$\Rightarrow u(t) = \underbrace{s(t) u_0}_{\text{given}} - i \int_0^t s(t-t') F(t') dt'$$

• Back to NLS with nonlinearity $N(u, \bar{u}) = u^{p_1} \bar{u}^{p_2}$

$$p_1 + p_2 = p$$

We say u is a soln to (NLS) if $p_j \in \mathbb{N} \cup \{0\}$.

u satisfies the following Duhamel formulation:

$$\begin{aligned} u(t) &= s(t) \underbrace{u_0}_{\text{given}} - i \int_0^t s(t-t') \underbrace{N(u, \bar{u})(t')}_{\text{unknown}} dt' \\ &=: \Gamma_{u_0}(u)(t) \end{aligned}$$

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Goal: Given $u_0 \in H^s(\mathbb{R}^d)$, show that $\exists! u$

$$\text{s.t. } \underline{u} = \Gamma_{u_0}(\underline{u}) \text{ on } [-T, T], \quad T = T(u_0) > 0.$$

i.e. u is a fixed of Γ_{u_0} .

• Basic properties of $S(t)$

① $S(t)$ is unitary on $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$

$$\begin{aligned} \|S(t)f\|_{H^s} &= \left(\int \langle \xi \rangle^{2s} |e^{-it|\xi|^2} \hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &= \|f\|_{H^s} \end{aligned}$$

② $S(t)f \in C(\mathbb{R}_+; H_x^s(\mathbb{R}^d))$, $f \in H^s$

$$S(\cdot)f : t \mapsto S(t)f \in H^s$$

↑ H^s -valued function (in t)

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Fix $t \in \mathbb{R}$.

Semigroup property

$$\| S(t+h)f - S(t)f \|_{H^s} \quad S(t_1 + t_2) = S(t_1)S(t_2)$$

$$= \| S(\cancel{t}) (S(h) - 1)f \|_{H^s}$$

← write down on the Fourier side

$$\text{separate} \cdot |\tilde{\beta}| > N \text{ s.t. } \underbrace{\| P_{|\tilde{\beta}| > N} f \|_{H^s}}_{= \tilde{\mathcal{F}}^{-1}(1_{|\tilde{\beta}| > N} \hat{f}(\tilde{\beta}))} < \frac{\varepsilon}{4}$$

$\cdot |\tilde{\beta}| \leq N$, use mean value theorem

$$|(S(h) - 1)^1(\tilde{\beta})| = |e^{-ih|\tilde{\beta}|^2} - 1|$$

$$\leq N^2 |h|.$$

- Fix $u_0 \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$ ⑥

$$\| P_{u_0}(u) \|_{C_T H^s} \leq \| S(t) u_0 \|_{C_T H^s} + \left\| \int_0^t S(t-t') N(u, \bar{u})(t') dt' \right\|_{C_T H^s}$$

$(C_T H^s = C([-T, T]; H^s(\mathbb{R}^d)))$

$$\| u \|_{C_T H^s} = \| \| u(t) \|_{H^s} \|_{L_T^\infty} \rightarrow L^\infty([-T, T])$$

$$\leq \| u_0 \|_{H^s} + \int_0^T \| N(u, \bar{u}) \|_{C_T H^s} dt.$$

unitarity of $S(t)$ & Minkowski's integral ineq. incap of t

$$\leq \| u_0 \|_{H^s} + T \| u^{P_1} \bar{u}^{P_2} \|_{C_T H^s}$$

$$\leq \| u_0 \|_{H^s} + CT \| u \|_{C_T H^s}^{P_1} \| \bar{u} \|_{C_T H^s}^{P_2} \stackrel{\text{WANT}}{\leq} 2 \| u_0 \|_{H^s} = : R$$

$$= \| u \|_{C_T H^s}^P$$

For $u \in \overline{B}_R =$ closed ball of radius R in $C_T H^s$ (7)

$$\begin{aligned} \Rightarrow \| \Gamma_{u_0}(u) \|_{C_T H^s} &\leq \| u_0 \|_{H^s} + C T R^p \\ &\leq \frac{R}{2} + \underbrace{(2CTR^{p-1}) \cdot \frac{R}{2}}_{\leq 1 \text{ by choosing } T \leq (2CR^{p-1})^{-1}} = R \end{aligned}$$

$$\Rightarrow \Gamma_{u_0} : \overline{B}_R \hookrightarrow$$

$$\underset{u, v \in \overline{B}_R}{\| \Gamma_{u_0}(u) - \Gamma_{u_0}(v) \|_{C_T H^s}} \leq \int_0^T \| N(u) - N(v) \|_{C_T H^s} dt.$$

$$\begin{aligned} u^{p_1} \bar{u}^{p_2} - v^{p_1} v^{p_2} &= \cancel{(u-v)} u^{p_1-1} \bar{u}^{p_2} \\ &\quad + v \cancel{(u-v)} u^{p_1-2} \bar{u}^{p_2} \\ &\quad + \dots + v^{p_1} \bar{v}^{p_2-1} \cancel{(u-\bar{v})} \end{aligned}$$

Telescoping sum

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$$\leq \tilde{C}T \left(\sum_{j=0}^{p-1} \|u\|_{CTH^s}^{p-1-j} \|v\|_{CTH^s}^j \right) \|u-v\|_{CTH^s}.$$

Young's ineq: $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$
 $\hat{j}=1: \|u\|_{CTH^s}^{p-2} \|v\|_{CTH^s}$
 $\leq C \left(\|u\|_{CTH^s}^{\frac{(p-2)p}{p-2}} + \|v\|_{CTH^s}^{\frac{p-1}{p-1}} \right)^{\frac{p-2}{p-1} + \frac{1}{p-1}} = 1$

$$\leq \tilde{C}T \left(\|u\|_{CTH^s}^{p-1} + \|v\|_{CTH^s}^{p-1} \right) \|u-v\|_{CTH^s}$$

$$\leq \underbrace{\tilde{C}T R^{p-1}}_{< 1} \|u-v\|_{CTH^s}$$

by choosing $T = \frac{1}{2\tilde{C}}R^{p-1}$

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Summary: $\forall u, v \in \overline{B_R}$

$$\| \Gamma_{u_0}(u) \|_{C^s} \leq R$$

$$\| \Gamma_{u_0}(u) - \Gamma_{u_0}(v) \|_{C^s} \leq \frac{1}{2} \| u - v \|_{C^s},$$

by choosing $T = \min\left(\frac{1}{2CR^{p-1}}, \frac{1}{2\tilde{C}R^{p-1}}\right)$

$$\text{Recall that } R = 2 \| u_0 \|_{H^s}$$

$$\Rightarrow T = T(\| u_0 \|_{H^s}) > 0$$

• Banach fixed pt thm (Contraction mapping principle)
 $\overset{\uparrow}{T}$ depends only on the H^s -norm of u_0 .

A contraction on a closed ball in a complete metric space has a unique fixed pt.

$$T: \overline{B_R} \hookleftarrow \text{ and } \| T(u) - T(v) \| \leq \theta \| u - v \| \quad \theta < 1$$

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$$\Rightarrow \exists! u \in \overline{B_R} \text{ s.t. } u = \Gamma_{u_0}(u)$$

Namely, u is a soln to (NLS).

Remarks: ① $u \in C_T H^s$ (\Leftarrow requirement for LWP)

$$u(t) = \underbrace{S(t)u_0}_{\text{in } C_T H^s} - i \int_0^t S(t-t') N(u, \bar{u})(t') dt'$$

$$\begin{aligned} G(t+h) - G(t) &= \int_0^{t+h} S(t+h-t') \cdots dt' - \int_0^t S(t-t') \cdots dt' \\ &= \underbrace{\int_t^{t+h} S(t+h-t') \cdots dt'}_{\text{use shortness of } [t, t+h]} - \underbrace{\int_0^t (S(t+h-t') - S(t-t')) \cdots dt'}_{= S(t-t')(S(h)-1)} \end{aligned}$$

② At this point, u is unique only in $\overline{B_R} \subset C_T H^s$.

Show u is unique in $C_T H^s$ (possibly by shrinking $T = T(R)$ a bit.)

OR (i) Gronwall's ineq.

(ii) Bootstrap argument.