

$T \leq 1$

Prop 1: $s < 0$.

(2)

Suppose $u \in C^\infty([0, T] \times \mathbb{T})$ s.t.

$\forall n, \hat{u}(t, n) \rightarrow 0$ at ∞ -order as $t \rightarrow 0+$

Then, $\forall \varepsilon > 0, \exists v, F \in C^\infty([0, T] \times \mathbb{T})$

whose Fourier coeff. $\rightarrow 0$ at ∞ -order as $t \rightarrow 0+$

s.t. v is a soln to (NLS_F)

with bounds:

$$\|v - u\|_{C_T H^s} \leq \varepsilon$$

$$\|S(t)F\|_{C_T^{-1} H^s} \leq \varepsilon$$

Interaction representation: $S(-t)u(t)$

$$S(t)u(t) = \underbrace{u_0}_{=0} + i \int_0^t S(-t') N(u)(t') dt'$$

$$S(t)v_m(t) = \underbrace{v_m(0)}_{=0} + i \int_0^t S(-t') N(v_m)(t') dt' - i \underbrace{\int_0^t S(-t') F_m(t') dt'}_{=0}$$

WTS: $\underbrace{N(V_n)}_{S(t)} \rightarrow \underbrace{N(u)}_{S(t)} \text{ in } C_T^{-1} H^s$

and $u := \lim_{n \rightarrow \infty} V_n \neq 0 \text{ in } C_T H^s$

In terms of the Fourier coeff, we have

(NLS_F) $\Leftrightarrow i \partial_t \hat{u}_n - i n^2 \hat{u}_n + \sum_{\substack{n=n_1, n_2+n_3 \\ n \neq n_1, n_3}} \hat{u}_{n_1} \overline{\hat{u}_{n_2}} \hat{u}_{n_3} - |\hat{u}_n|^2 \hat{u}_n = \hat{F}_n(t)$

$y_n(t) = \mathcal{F}_x (S(t) u(t)) (n) = e^{i t n^2} \hat{u}(t, n)$

$\Leftrightarrow \partial_t y_n = i \sum_{*} e^{i \phi(\bar{n}) t} y_{n_1} \overline{y_{n_2}} y_{n_3} - i |y_n|^2 y_n - i e^{i t n^2} \hat{F}_n(t)$

$= NR(y) = \text{non-resonant}, = R(y) = \text{resonant part}$

$\phi(\bar{n}) = \phi(n_1, n_2, n_3, n) = n^2 - n_1^2 + n_2^2 - n_3^2$ of the nonlinearity

under $n = n_1 - n_2 + n_3 \Rightarrow 2(n - n_1)(n - n_3)$

$$\Leftrightarrow \frac{dy}{dt} = \underbrace{NR(y) + R(y)}_{=N(y)} + f,$$

$$f_n = -ie^{itn^2} \hat{F}_n(t)$$

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We say $x(t) = \{x_n(t)\}_{n \in \mathbb{Z}}$ has support in $S \subset \mathbb{Z}$
 if $x_n(t) \equiv 0, \forall n \notin S, \forall t \in [0, T]$

Prop 2 (\Rightarrow Prop 1)

$S < 0$. $x \in C^\infty([0, T])$, seq-valued
 finite supp

vanishes at ∞ -order as $t \rightarrow 0+$.

Then, $\forall \varepsilon > 0$

\exists seq-valued func $y, g \in C^\infty([0, T])$ with finite supp

s.t. $\partial_t y = N(y) + g, y(t) \rightarrow 0$ at ∞ -order as $t \rightarrow 0+$

$$\|y - x\|_{C_T L^2_S} \leq \varepsilon$$

$$\|g\|_{C_T^{-1} L^2_S} \leq \varepsilon$$

$$\|a\|_{L^2_S} = \|\hat{F}(a)\|_{HS} = \left(\sum \langle m \rangle^{2s} |a_n|^2 \right)^{1/2}$$

Moreover, $\forall M > 0$,

we can construct y and g s.t.

$y - x$
 g are supported on $[M, \infty)$

Pf of Prop 2:

let $f = \partial_t x - N(x)$

\uparrow finite support. $S = \{n_j : 1 \leq j \leq A\}$

We construct

$$\tilde{S} \subset \mathbb{Z} \cap [M, \infty)$$

• Pick $m_1 \geq M$ and set m'_1 by

$$2m_1 - m'_1 = n_1$$

Recall $h \times h \times h \rightarrow \text{low}$
 $N - 2N + N = 0$

\curvearrowright for norm inflation

$$m_1 - m'_1 + m_1 = n_1$$

choose $m_1 \gg 1$ s.t. $m'_1 \geq M$

• Pick $m_2 \gg m_1, m'_1$ and set m'_2

$$2m_2 - m'_2 = m_2$$

choose $m_2 \gg 1$ s.t. $m'_2 \geq M$ (also $m'_2 \gg m_1, m'_1$)

Repeat this process for $j=1, 2, \dots, A$

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$$\Rightarrow 2m_j - m'_j = n_j, \quad \forall 1 \leq j \leq A. \quad (\Rightarrow m'_j \approx 2m_j)$$

$$\tilde{S} = \{m_1, m'_1, \dots, m_A, m'_A\}$$

Additional constraints:

① $k, l, m \in \tilde{S}$ and $l \neq k, m$.

Then, $|k - l + m| \geq M$

unless $(k, l, m) = (m_j, m'_j, m_j)$ for some j .

② $k, l \in \tilde{S}$, $n \in \text{supp } x$.

Then, $|k - \underline{n} + l| \geq M$

Also, $|k - l + \underline{n}| \geq M$ unless $k = l$.

③ $k \in \tilde{S}$, $m, n \in \text{supp } x$

$$|k - m + n|, |m + k - n| \geq M$$

Construct $h \in C_T^\infty$, $h_m, m \in \tilde{S}$.

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(*) $i h_{m_j} \overline{h_{m_j'}} h_{m_j}(t) \equiv \frac{1}{2} e^{-i\phi(m_j, \overline{m_j'}, m_j, n_j)t} f_{m_j}(t)$

for $m_j \in S$, and set $h_m \equiv 0, \forall m \notin \tilde{S}$.

\Rightarrow Set $y = x + h$

$\begin{matrix} \uparrow & \uparrow \\ \text{low} & \text{high} \end{matrix}$

$R(x)(m) = -i|x_n|^2 x_n$

• disjoint supp: $R(x+h) = R(x) + R(h)$

• $\|h\|_{C_T L_s^2} \leq CM^s \leq \varepsilon$

\uparrow by choosing $M \gg 1$ ($s < 0$)

\downarrow

$\|y - x\|_{C_T L_s^2} \leq \varepsilon$

Set $g = \frac{dy}{dt} - N(y)$

$y = x+h$
 $\partial_t x = N(x) + f$

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$\Rightarrow g = f - \underline{NR(h)}$

+ $\frac{dh}{dt} \leq \varepsilon$ in $C_T^{-1} L^2$

- $R(h)$ supported on $\{|n| \geq M\} \rightarrow \|R(h)\|_{C_T L^2} \leq CM^s \leq \varepsilon$

- $(NR(x+h) - NR(x) - \underline{NR(h)})$

① & ④, $f - NR(h)$ sup on $\{|n| \geq M\}$

$\rightarrow \leq \varepsilon$ in $C_T L^2$

② & ③, sup on $\{|n| \geq M\}$

$\Rightarrow \|g\|_{C_T^{-1} L^2} \leq \varepsilon$



Construction of the soln:

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- $x^{(1)}$, seq valued, finite supp
→ 0 at ∞ -order as $t \rightarrow 0+$.

$$\text{AND } \|x_0^{(1)}\|_{C_T} \geq 1$$

Proceed inductively.

Apply Prop 2 with $x = x^{(n)}$

$$\Rightarrow x^{(n+1)} = y$$

$$h^{(n)} = y - x = x^{(n+1)} - x^{(n)}$$

$$f^{(n+1)} = \partial_t x^{(n+1)} - N(x^{(n+1)})$$

$\underbrace{\quad}_{g}$

$$\|h^{(n)}\|_{C_T L_S^2} \leq \varepsilon_n, \quad \|f^{(n+1)}\|_{C_T^{-1} L_S^2} \leq \varepsilon_n$$

$$\text{Set } \varepsilon_n \leq 2^{-n-1}.$$

$$\Rightarrow x = \lim_{n \rightarrow \infty} x^{(n)} \text{ in } C_T L_S^2, \quad x = x^{(1)} + \sum_{n=1}^{\infty} h^{(n)} \quad (10)$$

$$\text{but } \|x_0\|_{C_T} \geq \|x_0^{(1)}\|_{C_T} - \sum_{n=1}^{\infty} \|h^{(n)}\|_{C_T} \geq 1/2$$

i.e. $x \neq 0$.

Show ① $\lim_{N \rightarrow \infty} N(P_N x)$ exists in $C_T^{-1} L_S^2$

i.e. $N(x)$ exists in the weak sense

② $N(P_N x^{(n)})$ Cauchy in $C_T^{-1} L_S^2$

$$\Rightarrow N(x^{(n)}) \rightarrow \underbrace{N(x)}_{\text{defined in the weak sense}} \text{ in } C_T^{-1} L_S^2$$

defined in the weak sense

$$x^{(n)}(t) = 0 + \int_0^t N(x^{(n)})(t') dt' + \underbrace{\int_0^t f^{(n)}(t') dt'}_{\downarrow 0}$$

$$\downarrow$$

$$x(t) = 0 + \int_0^t \underline{\underline{N(x)}}(t') dt'$$

\downarrow
0

$$\text{Let } u(t) = S(t) \mathcal{F}^{-1}(x(t))$$

(11)

\Rightarrow u is a weak soln to (WNLS) in the extended sense
with $u|_{t=0} = 0$ but $u \neq 0$.