

- Back to the proof of Theorem (norm inflation at general data)

WANT $\|u_{0,n} - u_0\|_{H^s} < \frac{1}{n}$ but $\|u_{n(t_n)}\|_{H^s} > n$.

\Leftarrow suffices to prove, for $N = N(n) \gg 1$,

(i) $RA^{d/2}N^s \ll \frac{1}{n}$

$$\|\phi_n\|_{H^s} = \|u_{0,n} - u_0\|_{H^s}$$

(ii) $TR^2A^{2d} \ll 1$

(iii) $TR^3A^{2d} \cdot f(A) \gg n$

$$T\|S(t)\phi_n\|_{FL'}^2 \ll 1 \Rightarrow \text{LWP \& conv of power series}$$

(iv) $TR^3A^{2d} \cdot f(A) \gg T^2R^5A^{4d}f(A)$

(v) $T \ll N^{-2}$

$$\sum_{j=2}^{\infty} \|\sum_j(u_{0,n})\|_{H^s}$$

(vi) $RA^d \gg \|u_0\|_{FL'}, Rf(A) \gg \|u_0\|_{L^2}$

• Assume (i) - (vi).

(2)

$$\begin{aligned} & \|u_{0,m}\|_{\mathcal{FL}} \sim RA^d. \\ \xrightarrow{(ii)} & u_n \text{ exists on } [-T, T] \text{ and} \\ & \text{power series converges in } C_T \mathcal{FL}'_x. \end{aligned}$$

$$\begin{aligned} & \left\| \sum_{j=2}^{\infty} \square_j (u_{0,m})(T) \right\|_{H^s} \\ & \stackrel{\substack{\text{Lem 4} \\ \text{geo. series}}}{\sim} T^2 R^4 A^{2d} \left(R f(A) + \|u_0\|_{L^2} \right) \\ & \sim T^2 R^5 A^{2d} f(A) \end{aligned}$$

$$\begin{aligned} \Rightarrow & \|u_n(T)\|_{H^s} \geq \left\| \square_0 (\phi_m)(T) \right\|_{H^s} \\ & - \left\| \square_0 (u_{0,m})(T) \right\|_{H^s} \leftarrow \text{use ③ from Lec 18.} \quad \quad \quad \\ & - \left\| \square_1 (u_{0,m})(T) - \square_1 (\phi_m)(T) \right\|_{H^s} \leftarrow \text{use ④} \\ & - \left\| \sum_{j=2}^{\infty} \square_j (u_{0,m})(T) \right\|_{H^s} \end{aligned}$$

(3)

$$\begin{aligned} &\gtrsim \frac{\textcolor{red}{TR^3 A^{2d} f(A)}}{- (1 + RA^{d/2} N^s)} \\ &\quad - TR^2 A^{2d} \|u_0\|_{L^2} \\ &\quad - T^2 R^5 A^{4d} f(A) \\ &\sim \textcolor{brown}{TR^3 A^{2d} f(A)} \gg n \end{aligned}$$

(iii)

- Now, we verify (i) - (vi).

Case 1: $s < -\frac{d}{2}$ ($\Rightarrow f(A) = 1$)

$$A = N^{\frac{1}{d}(1-s)}, \quad R = N^{2s}, \quad T = N^{-2-3s}, \quad s \ll 1$$

$$\cdot RA^{\frac{d}{2}} N^s = N^{s + \frac{1}{2} + \frac{3}{2}s} \ll 1/n$$

$$\cdot TR^2 A^{2d} = N^{-s} \overset{s < 0}{\ll 1}$$

$$\cdot TR^3 A^{2d} = N^s \gg n.$$

Case 2 : $s = -\frac{d}{2}$ ($\Rightarrow f(A) \sim (\log A)^{\frac{d}{2}}$) ④

$$A = \frac{N^{\frac{1}{d}}}{(\log N)^{\frac{1}{16d}}}, \quad R = 1, \quad T = \frac{1}{N^2(\log N)^{\frac{1}{8}}}$$

- $RA^{\frac{d}{2}}N^s = N^{\underbrace{\frac{1}{2}(1-d)}_{\leq 0}}(\log N)^{-\frac{1}{32}} \ll 1/m$

- $TR^2A^{2d} = (\log N)^{-\frac{1}{4}} \ll 1.$

- $TR^3A^{2d}(\log A)^{\frac{1}{2}} \sim (\log N)^{-\frac{1}{4}}(\log N - \frac{1}{16}\log \log N)^{\frac{1}{2}} \sim (\log N)^{\frac{1}{4}} \gg m.$

Case 3 : $-\frac{d}{2} < s < 0$ ($\Rightarrow f(A) \sim A^{\frac{d}{2}+s}$) relevant only for $d \geq 2$.

$$A = N^{\frac{2}{d}-\delta}, \quad R = N^{-1-s+\frac{d}{2}\delta-\theta}, \quad T = N^{-2+2s+d\delta+\theta}$$

where $\delta \gg \theta > 0$ small s.t.

$$\underline{-2s > d\delta + \theta} \quad \text{and} \quad -s\delta > 2\theta$$

(5)

- $RA^{\frac{d}{2}}N^s = N^{-\theta} \ll \frac{1}{n}$
- $TR^2A^{2d} = N^{-\theta} \ll 1$. $\underbrace{\frac{-d+2}{d} \geq 0}_{\text{by } d \geq 2}$
- $TR^3A^{2d} \cdot A^{\frac{d}{2}+s} = N^{\left(\frac{-d+2}{d}\right)s - 2\theta - sf} \geq N^{-2\theta - sf} \gg n.$

◻

- Christ - Colliander - Tao '03 : NLS on \mathbb{R}^d .

$s < -\frac{d}{2}$ or $0 < s < \text{Scrit}$. \Rightarrow Left open
the range $-\frac{d}{2} \leq s < 0$.

\Leftarrow ODE argument.

$$i\partial_t u + |u|^2 u = 0 \xrightarrow{\text{ODE for fixed } x.} u(t, x) = e^{it|u_0(x)|^2} u_0(x)$$

$$\Rightarrow \|u(t)\|_{H^s} \sim_{u_0} t^s, \quad s > 0.$$

Combine this with scaling.

• Nonuniqueness of NLS below $L^2(\mathbb{T})$ Christ '05
arXiv.

⑥

$$(\text{WNLS}) \quad i\partial_t u + \partial_x^2 u + N(u) = 0 \quad \leftarrow \text{renormalized NLS}$$

$$N(u) = |u|^2 u - 2(f|u|^2 dx) u$$

$$= |u|^2 u - 2 \cdot \text{const.} u \quad \text{if } u \notin L_x^2$$

$$\widehat{N(u)}(n) = \sum_{\substack{n=n_1+n_2 \\ n \neq n_1, n_3}} \widehat{u}_{n_1} \overline{\widehat{u}_{n_2}} \widehat{u}_{n_3} - |\widehat{u}_n|^2 \widehat{u}_n$$

↑

$$n = n_1 + n_2$$

$$n \neq n_1, n_3$$

when $n_1 = n_2$ (and $n_3 = n$)

$$\sum_{n=n_1+n_2} \widehat{u}_{n_1} \overline{\widehat{u}_{n_2}} \widehat{u}_{n_3}$$

$$= (f|u|^2 dx) \widehat{u}_n$$

We need $u \in L_x^3$ ($\Leftarrow u \in H_x^{1/6}$)

to make sense of $N(u)$ as a spatial
space-time

as a product of
distribution u .
distribution.

\Rightarrow We need to consider a more generalized notion of solns.

Def 1: seq of Fourier cutoff operators $\{P_N\}_{N \in \mathbb{N}}$.

(7)

$$\widehat{P_N f}(n) = m_N(n) \widehat{f}(n).$$

$m_N : \mathbb{Z} \rightarrow \mathbb{C}$, finite support, wif bdd

$$\lim_{N \rightarrow \infty} m_N(n) = 1, \quad \forall n \in \mathbb{Z}$$

Def 2: $u \in \mathcal{D}'((0, \tau) \times \mathbb{T})$.

We say $N(u)$ exist and is equal to $v \in \mathcal{D}'_{T, x}$ if

$$\lim_{N \rightarrow \infty} \underbrace{N(P_N u)}_{} = v$$

always makes b/c $P_N u \in C_x^\infty$

for any seq $\{P_N\}_{N \in \mathbb{N}}$ of Fourier cutoff operators.
in the distributional sense.

Rmk: We define the nonlinearity $N(u)$ as the limit of
the nonlinearity for smoothed u .

Def 3: We say $u \in C_T H^s$ is a weak soln to (WNLS) ⑧
in the extended sense with $u|_{t=0} = u_0$
if $N(u)$ exists in the sense of Definition 2
and u satisfies (WNLS) in the distributional sense.

Thm: $s < 0$. $\exists u \in C_T H^s$, $u \not\equiv 0$ s.t.

u is a weak soln to (WNLS) in the extended sense

with $u|_{t=0} = 0$. Moreover, $S(t)N(u)$ exists in $C_T^{-1} H^s$ -norm.

$$\left(\begin{array}{l} \bullet F \in C_T^{-1} H^s \text{ if } \int_0^t F(\tau) d\tau \in C_T H^s \\ \|F\|_{C_T^{-1} H^s} = \max_{t \in [0, T]} \left\| \int_0^t F(\tau) d\tau \right\|_{H^s} \end{array} \right)$$

\Rightarrow non-uniqueness