

We showed "mild ill-posedness" of NLS in $H^s(\mathbb{T}^d)$, $s < 0$ (failure of C^3 -smoothness / local unif. continuity).

- Why is $s = 0$ important for NLS?

scaling critical regularity : $S_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1}$

cubic ($p=3$) : $S_{\text{crit}} = \frac{d}{2} - 1$ ($= 0 \Leftrightarrow d=2$)

- Galilean invariance : u is a soln to NLS on \mathbb{R}^d

$$\Rightarrow u^\beta(t, x) = e^{i\frac{\beta}{2}x} e^{-i\frac{\|\beta\|^2}{4}t} u(t, \underline{x + \beta t}) \text{ is also a soln.}$$

\uparrow shift by $\frac{\beta}{2}$ on the Fourier side

- Galilean invariance makes sense on \mathbb{T}^d if $\beta \in 2\mathbb{Z}^d$.

$$\Rightarrow \hat{U}^\beta(t, \bar{z}) = e^{-i\frac{3}{4}\beta t^2} e^{i\bar{z}\beta t} \hat{u}(t, \bar{z} - \frac{\beta}{2}) \quad (2)$$

$$|\hat{U}^\beta(t, \bar{z})| = |\hat{u}(t, \bar{z} - \frac{\beta}{2})|$$

$\Rightarrow L_x^2$ -norm is invariant under Galilean symmetry.

i.e. $s_{\text{crit}}^\infty = 0$ is another critical regularity for NLS
(associated with Galilean sym.)

Also, the Fourier-Lebesgue space $\mathcal{FL}^{s,p}$

$$\|f\|_{\mathcal{FL}^{s,p}} = \|\langle \bar{z} \rangle^s \hat{f}(\bar{z})\|_{L_p^s} \quad (H^s = \mathcal{FL}^{s,2})$$

is invariant under Galilean symmetry when $s=0$.

- $d=1$: cubic NLS

$$S_{\text{crit}} = -\frac{1}{2} < 0 = S_{\text{crit}}^{\infty}$$

- $s < 0$: failure of uniform continuity.

LWP on \mathbb{R} is open. (only existence : Christ - Colliander - Tao '08
Koch - Tataru '07, '12.)

- On \mathbb{T} : cubic NLS is ill-posed in $H^s(\mathbb{T})$, $s < 0$.

• CCT '03 : discontinuity of soln map in $H^s(\mathbb{T})$, $s < 0$

• Molinet '09 : discontinuity in $L_x^2(\mathbb{T})$ endowed with weak topology into the space of distributions $((C^\infty(\mathbb{T}))')$

⇒ discontinuity in $H^s(\mathbb{T})$, $s < 0$.

• Guo - Oh '16 : non-existence of solns for $u_0 \notin L^2(\mathbb{T})$

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On \mathbb{T} , the "correct" eqn to study outside $L^2(\mathbb{T})$ is

(WNLS)



$$i\partial_t u + \partial_x^2 u \pm (\|u\|^2 - \underline{2\int |u|^2 dx}) u = 0.$$

Wick ordered NLS: renormalized NLS

- If u solves (NLS), then $\underbrace{e^{-2it\int f|u|^2 dx}}_{=: G(u)} u$ solves (WNLS).
 $u \in C(\mathbb{R}; L^2(\mathbb{T}))$

- G is invertible on $C(\mathbb{R}; L^2(\mathbb{T}))$.

But G does not make sense outside $L^2(\mathbb{T})$ (for initial data)

- It turned out that (WNLS) is a better eqn to study outside $L^2(\mathbb{T})$ and a good a priori bound on solns to (WNLS) combined with G^{-1} gives the non-existence result for the original NLS in $H^s(\mathbb{T})$, $s < 0$.

$$(WNLS) : i\partial_t u + \partial_x^2 u + (|u|^2 - 2 \cdot \infty) u = 0. \quad (5)$$

Back to

④ failure of continuity of soln map for the cubic NLS
on $M = \mathbb{R}^d$ or \mathbb{T}^d .

In order to show that the soln map is NOT conti at u_0 with the H^s -topology, it suffices to construct, for each $\varepsilon > 0$,
soln u_ε to (NLS) and $t_\varepsilon \in (0, \varepsilon)$ s.t.

$$\|u_\varepsilon(0) - u_0\|_{H^s(M)} < \varepsilon \text{ but } \|u_\varepsilon(t_\varepsilon) - u(t_\varepsilon)\|_{H^s} \gtrsim 1.$$

soln s.t. $u|_{t=0} = u_0$

In general, we take $u_0 = 0$
 $\Rightarrow u \equiv 0$.

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Norm inflation (CCT '03): Given $\varepsilon > 0$,

\exists soln u_ε to (NLS) and $t_\varepsilon \in (0, \varepsilon)$ s.t.

$$\|u_\varepsilon(0)\|_{H^s} < \varepsilon \quad \text{but} \quad \|u_\varepsilon(t_\varepsilon)\|_{H^s} > \frac{1}{\varepsilon}$$

- Norm inflation \Rightarrow discontinuity at $u_0 = 0$.
- Norm inflation at general initial data: $d \geq 1$, $M = \mathbb{R}^d$ or \mathbb{T}^d .
Suppose that $s \in \mathbb{R}$ satisfies

(i) $s \leq -\frac{1}{2} = \underline{s}$ when $d = 1$ (includes s)

(ii) $s < 0$ when $d \geq 2$. (When $d = 2$, $\underline{s} = 0$)

Fix $u_0 \in H^s(M)$. Then, given $\varepsilon > 0$, \exists soln u_ε to the cubic NLS
(or WNLS) and $t_\varepsilon \in (0, \varepsilon)$ s.t smooth

$$\|u_\varepsilon(0) - u_0\|_{H^s} < \varepsilon \quad \text{but} \quad \|u_\varepsilon(t_\varepsilon)\|_{H^s} > \frac{1}{\varepsilon}$$

Kishimoto
(preprint)
Oh'17.

- Power series expansion indexed by trees

cubic NLS : $u(t) = S(t) u_0 \pm i \underbrace{\int_0^t S(t-t') |u|^2 u(t') dt'}_{= I[u]} = I[u, u, u]$

$$I[u_1, u_2, u_3] = \pm i \int_0^t S(t-t') u_1 \bar{u}_2 u_3(t') dt'.$$

We proved LWP in $H^s(M)$, $s > \frac{d}{2}$.

A similar argument yields LWP in the Wiener algebra

$$\mathcal{FL}'(M) = \mathcal{FL}^{0,1}(M) \quad \|f\|_{\mathcal{FL}^p} = \|\hat{f}^{(3)}\|_{L_3^p}$$

$$\uparrow \quad (\quad \overset{''}{A}(M) \quad \quad \quad \mathcal{FL}^p = \mathcal{FL}^{0,p} \text{ i.e. } s=0.$$

algebra.

- local existence time $T \sim \|u_0\|_{\mathcal{FL}'}^{-2} > 0$.

⑧

\Rightarrow Picard iteration converges.

$$P_0(\phi) = S(t)\phi$$

$$P_j(\phi) = S(t)\phi + I[P_{j-1}(\phi)], j \geq 1$$

\uparrow converges

- Tree (= ternary tree)



- 0 or 3 children

\uparrow \uparrow
non-terminal node

terminal node

$T(j)$ = collection of trees T of j^{th} generation

$$|T| = 3^{j+1}$$

\downarrow
 j parental nodes

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Fix $\phi \in \mathcal{FL}'$,

Given $T \in T(j)$, $j \geq \mathbb{Z}_{\geq 0}$, associate a multilin operator ($\bar{\in} \Phi$) by

- replace a non-terminal node by the Duhamel integral operator $\mathcal{I}[u_1, u_2, u_3]$, u_j = three children.
- replace a terminal node by the linear soln $s(t)\phi$.

Denote the map by $\Sigma : \bigcup_{j=0}^{\infty} T(j) \rightarrow \mathcal{D}'([-\tau, \tau] \times M)$

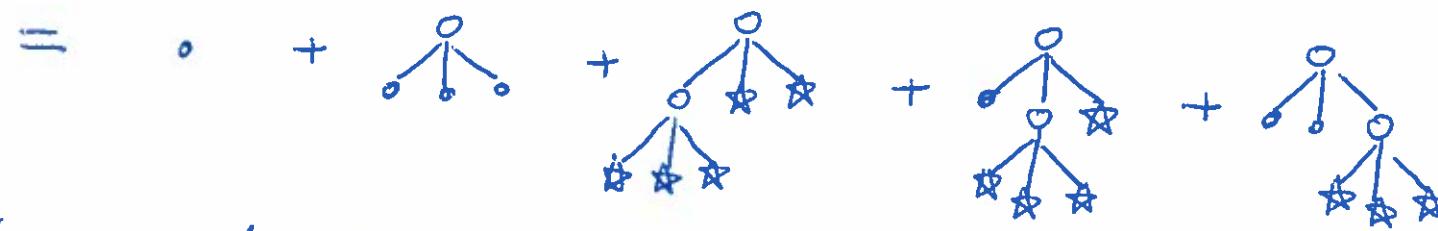
\circ = non-terminal node

\cdot = terminal node

\star = soln. u

$$(NLS) \iff \Phi = \circ + \underbrace{\begin{array}{c} \circ \\ | \\ \star \star \star \end{array}}_{=} = \circ + \begin{array}{c} \circ \\ | \\ \star \end{array} + \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \star \star \star \end{array}$$

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\Leftarrow repeat this process indefinitely

$$U = \sum_{j=0}^{\infty} E_j(\phi) = \sum_{j=0}^{\infty} \sum_{T \in \Pi(j)} \Psi_{\phi}(T) \leftarrow \begin{array}{l} \text{Power series} \\ \text{expansion} \end{array}$$