

Lec 16 19/03/18 (Mon)

①

Sec 8: Ill-posedness of NLS in low regularities

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^{p-1}u = 0 \\ u|_{t=0} = u_0 \in H^s(M), \end{cases} \quad M = \mathbb{R}^d \text{ or } \mathbb{T}^d$$

$$\mathbb{T} = \mathbb{R}/\mathbb{Z}$$

• Bad behaviors for $s \leq s_0$ (or $s < s_0$)

① failure of the nonlinear estimate.

lin est: $\left\| \int_0^t S(t-t') |u|^{p-1} u(x) dt' \right\|_{X_T^s} \lesssim \| |u|^{p-1} u \|_{N_T^s}$

$$X_T^s = \text{solution space} \subset C_T H_x^s$$

nonlin: $\| |u|^{p-1} u \|_{N_T^s} \lesssim \| u \|_{X_T^s}^p \leftarrow \text{This fails for } s < s_0$

ex: 2-d cubic NLS on \mathbb{R}^2 .

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nonlin esti: $\| \langle \nabla \rangle^s (|u|^2 u) \|_{L_{T,x}^{4/3}} \lesssim \| \langle \nabla \rangle^s u \|_{L_{T,x}^4}^3$, $s \geq 0$
 \Leftarrow fails for $s < 0$

(2) failure of C^k -smoothness of the soln map:

$$\Phi: u_0 \in H^s \longmapsto u \in C_T H^s$$

• If $p \in 2\mathbb{N}+1$, the nonlin $|u|^{p-1}u$ is algebraic.

$\Rightarrow \Phi$ is analytic if we can solve the fixed pt problem

$\Gamma_{u_0}(u) = u$ by the standard contraction argument.

• failure of C^k -smoothness does not show ill-posedness but says that we can not use a contraction argument.

(for example, one would need to use a more robust energy method (say, in the context of the short-time Fourier restriction norm method.) ③

③ failure of uniform continuity of the soln map.
(on bounded sets in H^s . i.e. local unif conti)

- same comment as in ②
- mild ill-posedness

④ failure of continuity of the soln map

- ill-posed.

⑤ failure of uniqueness or existence

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Back to

(2) failure of C^3 -smoothness of cubic NLS on $H^s(\mathbb{T}^d)$, $s < 0$.

$$(*) \quad \begin{cases} i \partial_t u + \Delta u \pm |u|^2 u = 0 \\ u|_{t=0} = \underline{\underline{f}} \phi \end{cases}$$

for some $\phi \in H^s(\mathbb{T}^d)$
(smooth)

• $u(t, x; f) = \text{soln to } (*) \text{ with parameter } f \in \mathbb{R}$.

Note: $u(t, x; 0) \equiv 0$.

• Let $\Phi(t) : u_0 \in H^s \mapsto u(t) \in H^s$.

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Suppose $\Phi(t)$ is C^k -smooth for some small $t > 0$.
(Note: $\Phi(t)$ is well-defined on smooth functions.)

By the smoothness around the zero func,

$$\begin{aligned} \Phi(t)(\underbrace{u_0(\delta)}_{=\delta\phi}) &= \Phi(t)(\underbrace{u_0(\delta=0)}_{=0}) + \nabla\Phi(t)(0) \cdot \delta\phi \\ &\quad + \frac{1}{2} \nabla^2\Phi(t)(0)(\delta\phi, \delta\phi) + \dots \end{aligned}$$

$$\Rightarrow \left\| \frac{d^k}{d\delta^k} \Phi(t)(\delta\phi) \Big|_{\delta=0} \right\|_{H^s} \lesssim \underbrace{\| \nabla^k \Phi(t)(0) \|_{(H^s)^{\otimes k} \rightarrow H^s}}_{\leq C < \infty} \| \phi \|_{H^s}^k$$

Namely,

$$(**) \quad \left\| \partial_s^k \Phi(t)(\delta\phi) \Big|_{\delta=0} \right\|_{H^s} \lesssim \| \phi \|_{H^s}^k$$

↑
Chain rule

$$\partial_s^k \Phi(t)(\delta\phi) = \nabla^k \Phi(t)(\delta\phi)(\underbrace{\phi, \dots, \phi}_{k\text{-times}})$$

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$$u(t) = \int S(t) \phi \pm i \int_0^t S(t-t') \underbrace{|u|^2 u(t)} dt'$$

depends on δ (higher order in δ)

$$\frac{\partial u}{\partial \delta} \Big|_{\delta=0} = S(t) \phi$$

$$\frac{\partial^2 u}{\partial \delta^2} \Big|_{\delta=0} = \pm i \int_0^t S(t-t') \frac{\partial^2 (|u|^2 u(t))}{\partial \delta^2} \Big|_{\delta=0} dt'$$

Apply the product rule: $u_\delta^2 \bar{u}$, $u_{\delta\delta} \bar{u} u$, ...

- contains at least one u or \bar{u} but $u(\delta=0) \equiv 0$.

$$= 0.$$

$$\frac{\partial^3 u}{\partial \delta^3} \Big|_{\delta=0} \sim \int_0^t S(t-t') u_\delta^2 \bar{u}_\delta \Big|_{\delta=0} dt'$$

$$= \int_0^t S(t-t') |S(t') \phi|^2 S(t') \phi dt'$$

Given $N \in \mathbb{N}$, let $\phi = N^{-s} e^{iN \cdot x_1}$ ← supp on $N e_1 = (N, 0, \dots, 0)$ ⑦

$$\Rightarrow \|\phi\|_{H^s} \sim 1$$

$$\Rightarrow S(t)\phi = N^{-s} e^{iN \cdot x_1 - iN^2 t}$$

$$\cdot |S(t')\phi|^2 S(t)\phi = N^{-3s} e^{iN \cdot x_1 - iN^2 t'}$$

$$\Rightarrow S(t-t') (|S(t)\phi|^2 S(t)\phi) = \underbrace{N^{-3s} e^{iN \cdot x_1 - iN^2 t}}_{\text{indep of } t'}$$

$$\Rightarrow \left. \frac{\partial^3 u(t)}{\partial \delta^3} \right|_{\delta=0} \sim t N^{-3s} e^{iN \cdot x_1 - iN^2 t}$$

$$\Rightarrow \left\| \left. \frac{\partial^3 u(t)}{\partial \delta^3} \right|_{\delta=0} \right\|_{H^s} \sim t N^{-2s} \text{ but } \|\phi\|_{H^s} \sim 1$$

\Rightarrow **(**)** can not hold for $s < 0$.

(This argument first appeared in the KdV, mKdV context by Bourgain '97.)

③ failure of local unit continuity

⑧

Construct a family of pairs of smooth initial data (and solns) s.t.

Given any small $t > 0$ and $\varepsilon > 0$,
 $\exists u_{0,\varepsilon}$ and $v_{0,\varepsilon}$ s.t.

$$\|u_{0,\varepsilon} - v_{0,\varepsilon}\|_{H^s} < \varepsilon \text{ but } \|u_\varepsilon(t) - v_\varepsilon(t)\|_{H^s} \gtrsim 1.$$

On \mathbb{R} : Kenig-Ponce-Vega '01: focusing

family of soliton solutions with parameters.

• Christ-Colliander-Tao '03: defocusing

family of approximate solutions

On \mathbb{T}^d : Burg - Gérard - Tzvetkov '02 (cubic NLS)

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\Leftarrow enough to consider the $d=1$ case.

(For general $d \geq 1$, set $u(x) = u(x_1, 0, \dots, 0)$)

Consider

$$u_{N,a}(t, x) = a e^{i(Nx - N^2t \pm |a|^2 t)}$$

for $a \in \mathbb{C}$ and $N \in \mathbb{N}$.

- $u_{N,a}$ is a soln to the cubic NLS.
- Choose $a = N^{-s} \alpha$ and $a' = N^{-s} \alpha'$.

$$\|u_{N,a}(0) - u_{N,a'}(0)\|_{H^s} \sim |\alpha - \alpha'| \quad (\xrightarrow{\text{WANT}} 0)$$

• For small $t_0 > 0$,

$$\begin{aligned} \|u_{N,a}(t_0) - u_{N,a'}(t_0)\|_{H^s} &\sim \left| \alpha e^{\pm i N^{-2s} |\alpha|^2 t_0} - \alpha' e^{\pm i N^{-2s} |\alpha'|^2 t_0} \right| \\ &= \left| \alpha - \alpha' e^{\pm i N^{-2s} (|\alpha'|^2 - |\alpha|^2) t_0} \right| \sim |\alpha| + |\alpha'| \sim 1. \end{aligned}$$

$= \bar{c} + o(1)$ by choose $N = N(t_0, \alpha, \alpha') \gg 1$

Choose $|\alpha|, |\alpha'| \sim 1$

$$|\alpha - \alpha'| \sim \varepsilon.$$

$$\left(\begin{array}{l} \text{Note: } |\alpha'|^2 - |\alpha|^2 = (|\alpha'| - |\alpha|)(|\alpha'| + |\alpha|) \sim \varepsilon. \\ \Rightarrow N \sim \varepsilon^{-\frac{1}{2s}} \rightarrow \infty \text{ as } \varepsilon \rightarrow 0. \end{array} \right.$$

$$\Rightarrow \|u_{N,\alpha}(t) - u_{N,\alpha'}(t)\|_{H^s} \begin{cases} < \varepsilon, & t=0 \\ \sim 1, & t=t_0 > 0. \end{cases}$$