

• mass-critical case: $S_{\text{crit}} = 0$, $p = 1 + \frac{4}{d}$

$$\text{Let } J(u) = \frac{\left(\int |\nabla u|^2 \right) \left(\int |u|^{2/d} \right)^{2/d}}{\int |u|^{2+4/d}}, \quad u \neq 0$$

Prop: (i) $\min_{\substack{u \in H^1 \\ u \neq 0}} J(u)$ is attained at

$$\lambda_0^{d/2} Q(\lambda_0 x + x_0) e^{i\varphi_0}, \quad (\lambda_0, x_0, \varphi_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$$

\uparrow unique ground state

In particular, we have the following sharp Bragliardo-Nirenberg ineq:

$$\int |u|^{2+4/d} \leq \underline{J(Q)}^{-1} \int |\nabla u|^2 \left(\int |u|^{2/d} \right)^{2/d}$$

optimal const

$$(\Leftarrow J(Q) \leq J(u))$$

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(ii) "Rigidity". Let $u \in H^1$ s.t.

$$\int |u|^2 = \int Q^2, \quad H(u) = 0 \quad (= H(Q)) \quad \begin{cases} \text{For mass-subcrit,} \\ H(Q) < 0 \end{cases}$$

Then, $u(x) = A_0^{d/2} Q(\lambda_0 x + x_0) e^{ir_0}$

for some $(\lambda_0, x_0, r_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$.

Rmk: $\nabla u \in H^1$

$$H(u) \geq \frac{1}{2} \int |\nabla u|^2 \left(1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}} \right)^{4/d} \right) dx$$

Let $u_0 \in H^1$ s.t. $\|u_0\|_{L^2} < \|Q\|_{L^2}$

$$H(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{2+4/d} \int |u|^{2+4/d}$$

By mass conservation, $H(u) \geq \int |\nabla u|^2 dx$

↑
indep of t . (but depends on $\|u_0\|_{L^2}$)

$\Rightarrow H(u) + M(u) \gtrsim \|u\|_{H^1}^2$

\Rightarrow GWP in $H^1(\mathbb{R}^d)$, provided that $\|u_0\|_{L^2} \neq \|Q\|_{L^2}$ (3)
 (also scattering)

Note: mass-subcrit NLS is GWP in $H^1(\mathbb{R}^d)$
 (regardless of $\|u_0\|_{L^2}$)

• Q: $\|u_0\|_{L^2} = \|Q\|_{L^2}$?

- soliton: $U(t) = e^{it} Q$ is a global non-scattering soln.
- pseudo-conformal symmetry ($\text{Scrit} = 0$)

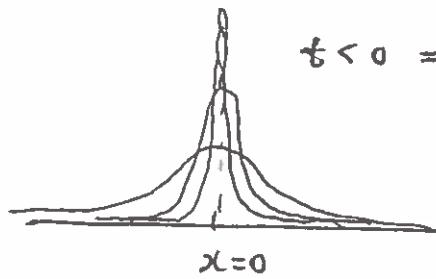
$$U(t, x) \xrightarrow{\quad} V(t, x) = \frac{1}{|t|^{d/2}} U\left(-\frac{1}{t}, \frac{x}{t}\right) e^{i \frac{|x|^2}{4t}}, \quad (t \neq 0)$$

Apply PC symmetry to $e^{it} Q$

$$\Rightarrow Q^*(t, x) = \frac{1}{|t|^{d/2}} Q\left(\frac{x}{t}\right) e^{-i \frac{|x|^2}{4t} + \frac{i}{t}}$$

\uparrow

sln to (NLS) for $t < 0$.



$$t < 0 \Rightarrow t = 0^-$$

(4)

Q^* blows up at time $t=0$ (starting at $t=-1$)

- $\|Q^*(t)\|^2 \rightarrow \|Q\|_{L^2}^2 \delta_{x=0}$ as $t \nearrow 0$

- $\|\nabla Q^*(t)\|_{L^2} \sim \frac{1}{|t|} \Leftarrow \text{blowup speed}$ Q^* , not stable

$\Leftarrow Q^*$ is the minimal mass blowup soln

- unique (Merle '93)

(if $M(u) = M(Q)$ and u blows up in a finite time,
then $u = Q^*$ (up symmetry))

- Other finite time blowup solns?

$$M(Q) < M(u_0) < M(Q) + \varepsilon \quad \text{Merle-Raphaël '00's} \sim$$

"log log" blowup soln $\sim \sqrt{\frac{\log \log (T^*-t)}{T^*-t}}$
 \Leftarrow stable

Rmk: slowest possible blowup speed $\gtrsim \frac{1}{\sqrt{T^*-t}}$ (by scaling) (5)

7.2 Virial identity & Morawetz estimate

viriel = "force"

$$i\partial_t u + \Delta u = \lambda |u|^{p-1}u, \quad \lambda = \pm 1$$

$$(\partial_t u = i\Delta u - i\lambda |u|^{p-1}u)$$

$\lambda = 1$, defocusing

$\lambda = -1$, focusing

Virial potential

$$V_a(t) = \int a(x) |u|^2 dx$$

↑
soln

$a(x)$ "nice" func

$$a(x) = |x|^2$$

$$a(x) = |x|$$

Compute $\partial_t^2 V_a(t)$.

See Lecnote from '16

$$\Rightarrow \partial_t^2 V_a(t) = 4 \int \operatorname{Re} (\partial_k u \partial_j \bar{u}) \partial_k \partial_j a$$

$$+ 2\lambda \frac{p-1}{p+1} \int |u|^{p+1} \Delta a$$

use eqn.

$$- \int |u|^2 \Delta^2 a$$

Einstein's summation notation:
sum over repeated indices

*

ex: Virial identity. : $a(x) = |x|^2 = \sum_{j=1}^d x_j^2$. $\Delta a = 2d$ (6)
 $\Delta^2 a = 0$

• $V(t) = \int |x|^2 |u(t, x)|^2 dx$ $\partial_k \partial_j a = 2 \delta_{jk}$ Kronecker delta

(= variance if we view $|u(t)|^2 dx$ as a prob meas.)

$$\begin{aligned}\Rightarrow \partial_t^2 V(t) &= 8 \int |\nabla u|^2 + 4d \lambda \frac{p-1}{p+1} \int |u|^{p+1} \\ &= 16 H(u) + \frac{4Ad}{p+1} \left(p - \underbrace{\left(1 + \frac{4}{d} \right)}_{\text{mass-critical power}} \right) \int |u|^{p+1}\end{aligned}$$

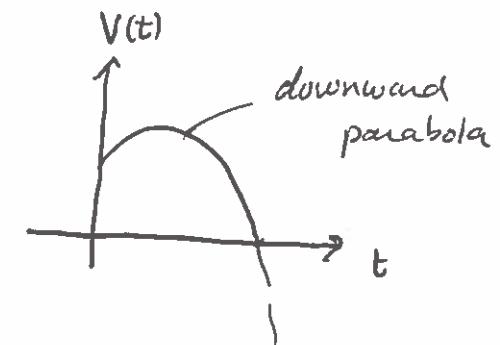
• mass-crit case : $\partial_t^2 V(t) = 16 H(u)$. \leftarrow conserved

If $H(u_0) < 0$, then $\partial_t^2 V(t) = 16 H(u_0) < 0$

• $V(t^*) < 0$ for some $t^* > 0$.

but $V(t) = \int |x|^2 |u(t, x)|^2 dx \geq 0$

$\Rightarrow u$ must blow up before time t^* (Glassey's argument,
Zakharov's



A similar argument works for mass-supercritical case.

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($\lambda = -1$: focusing)

Ex: Morawetz estimate : $a(x) = |x|$, $\partial_j a = \frac{x_j}{|x|}$

$$\Rightarrow \partial_j^2 a = \frac{1}{|x|} - \frac{x_j}{|x|^2} \cdot \frac{x_j}{|x|} \Rightarrow \Delta a(x) = \frac{d-1}{|x|}$$

$$\Rightarrow \dots \Rightarrow \Delta^2 a = -\frac{(d-1)(d-3)}{|x|^3} \leq 0 \text{ if } d \geq 3.$$

($\underline{d=3}$: $\frac{1}{|x|}$ = fundamental soln to $-\Delta \Rightarrow \Delta^2 a = -8\pi f$.

$$\begin{aligned} \Rightarrow \partial_t^2 V_a(t) &= \boxed{4 \int \frac{|k u|^2}{|x|} dx} \geq 0 \\ &\quad + 2 \lambda \frac{p-1}{p+1} \int \frac{|u|^{p+1}}{|x|} dx \\ &\quad - \underbrace{\int |u|^2 \Delta^2 a}_{\geq 0} \end{aligned}$$

$$\begin{aligned} |k u|^2 &= |\nabla u|^2 - \underbrace{\left| \frac{x}{|x|} \cdot \nabla u \right|^2}_{\text{angular component of grad.}} \\ &\quad \text{radial component} \end{aligned}$$

$$\Delta^2 a = -8\pi f \text{ when } d=3$$

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On the other hand,

$$\partial_t^2 V_\alpha(t) = \partial_t \cdot 2 \operatorname{Im} \int \nabla u \cdot \frac{x}{|x|} \bar{u}$$

$$\begin{aligned} \underline{\lambda=1 \text{ defocusing}} \quad & \int_{t_1}^{t_2} \int \frac{|u|^{p+1}}{|x|} dx dt \lesssim \sup_{t=t_1, t_2} \left| \operatorname{Im} \int \nabla u \cdot \frac{x}{|x|} \bar{u} dx \right| \\ & \lesssim \sup_{t_0, t_1} \|u(t)\|_{\dot{H}^{1/2}}^2 \\ \text{or} \quad & \lesssim_{C-S} M(u_0)^{1/2} H(u_0)^{1/2} \end{aligned}$$

\Rightarrow Send $t_1 \rightarrow -\infty$

$t_2 \rightarrow +\infty$

$$\begin{aligned} \underline{\text{Morawetz}}: \int_{R_t} \int_{R_x^q} \frac{|u(t, x)|^{p+1}}{|x|} dx dt & \lesssim \sup_{t+} \|u(t)\|_{\dot{H}^{1/2}}^2 \\ \text{or} \quad & M(u_0)^{1/2} H(u_0)^{1/2} \end{aligned}$$

defocusing
case $\lambda=1$

useful for radial case

- By repeating the derivation centered at y . ($d=3$) (9)

$$\text{(*)} \quad \partial_t \operatorname{Im} \int \nabla u(x) \cdot \frac{x-y}{|x-y|} \bar{u}(x) dx = 2 \int \frac{|t \nabla_y u(x)|^2}{|x-y|} dx + 2 \frac{1}{p+1} \int \frac{|u(x)|^{p+1}}{|x-y|} dx \\ + 4\pi |u(y)|^2$$

- Multiply (*) by $|u(y)|^2$ and $\int \cdot dy$.

$$\Rightarrow \int_{\mathbb{R}^+} \int_{\mathbb{R}_y^3} |u(y)|^4 dy dt \lesssim \sup_t \|u(t)\|_{L^2}^2 \|u(t)\|_{H^{\frac{1}{2}}}^2 \\ \text{or } M(u_0)^{3/2} H(u_0)$$

Interaction Morawetz estimate (Colliander - Keel - Staffilani - Takaoka - Tao med '00)