

Sec 7 : Global-in-time behavior of solns to NLS

Strichartz est

\Rightarrow LWP of NLS on \mathbb{R}^d if $s \geq \max(\frac{1}{2}, 0)$

(if $p \in 2N + 1$. Otherwise, need extra cond.)

Q1: Does the soln exists globally in time (Global well-posedness)
or does it cease to exist at some finite time?
(finite time blowup soln)

Q2: If u exists globally in time, then
what is the behavior of the soln u as $t \rightarrow \pm\infty$?

• Scattering: "asymptotic linear behavior" $\|u(t)\|_{L^\infty_x} \xrightarrow[t \rightarrow \pm\infty]{} 0$

$$\exists u_\pm \in H^s(\mathbb{R}^d) \text{ s.t. } \lim_{t \rightarrow \pm\infty} \underbrace{\|u(t) - S(t)u_\pm\|_{H^s}}_{\text{lin soln}} = 0$$

$$\lim_{t \rightarrow \pm\infty} u(t) = S(t)u_0$$

- non-scattering soln such as solitons:

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$$u(t) = e^{it} \underbrace{Q(x)}_{\text{indep of time}}$$



In particular, $\|u(t)\|_{L_x^\infty} = \|Q\|_{L_x^\infty} \not\rightarrow 0$.

Conjecture: soliton resolution conjecture.

For "generic" initial data, $u(t)$ decouples into a sum of solitons + radiation (= scattering part) as $t \rightarrow \pm\infty$.

still open: except for "integrable equations" such as KdV and NLW (radial, Kenig-Merle et al. '12~).

- We discussed the conservation of

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$$\text{Mass} : M(u) = \int |u|^2 dx$$

$$\text{Momentum} : P(u) = \text{Im} \int \bar{u} \nabla u \quad \leftarrow \text{not sign definite}$$

Hamiltonian / Energy :

$$H(u) = \frac{1}{2} \int |\nabla u|^2 dx \pm \frac{1}{p+1} \int |u|^{p+1} dx$$

$$(\Leftarrow i \partial_t u + \Delta u = \pm |u|^{p-1} u)$$

- \pm ... defocusing case / repulsive case

- \pm ... focusing case / attractive case

ex: 1-d cubic NLS, $\text{scrit} = -\frac{1}{2}$ ④

LWP in $L^2(\mathbb{R})$ in the subcritical sense ($T \sim \|u_0\|_{L^2}^{-\theta}$)
 mass cons
 \Rightarrow GWP in $L^2(\mathbb{R})$

but not on \mathbb{R}^2 : $\text{scrit} = 0$
 T was given s.t. $\|\delta(t)u_0\|_{L_T^4 L_x^4} \ll 1$.
 GWP in $L^2(\mathbb{R}^2)$ holds true but the proof is much more complicated. Dodson '12?

• 3-d cubic NLS (defocusing): $\text{scrit} = \frac{1}{2}$

HW: LWP in $H^1(\mathbb{R}^3)$ in the subcritical sense ($s=1 > \frac{1}{2}$)

$$\|u(t)\|_{H^1}^2 \leq \int |u|^2 dx + \int |\nabla u|^2 dx + \frac{2}{4} \int |u|^4 dx$$

$$= M(u(t)) + 2H(u(t)) \underset{\text{'cons.'}}{\circlearrowleft} M(u_0) + 2H(u_0) < \infty$$

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\Rightarrow GWP in $H^1(\mathbb{R}^3)$. \leftarrow We'll prove scattering.

- NOT true in the focusing case
 - GWP (and scattering) in $H^{1/2}(\mathbb{R}^3)$ is open.
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• We say NLS is

$$S_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1}$$

$$\begin{cases} \text{mass-critical if } S_{\text{crit}} = 0. & p = 1 + \frac{4}{d} \quad \begin{matrix} 1-d, \text{quintic} \\ 2-d, \text{cubic} \end{matrix} \\ \text{mass-subcritical if } S_{\text{crit}} < 0 & (\text{ex: 1-d cubic NLS}) \\ \text{mass-supercritical if } S_{\text{crit}} > 0. & \end{cases}$$

$$\begin{cases} \text{energy-critical if } S_{\text{crit}} = 1 & \frac{d \geq 3}{p = 1 + \frac{4}{d-2}} : \begin{matrix} 3-d: \text{quintic} \\ 4-d: \text{cubic} \end{matrix} \\ \text{energy-subcritical if } S_{\text{crit}} < 1 & (\text{ex: 3-d cubic}) \\ \text{energy-supercritical if } S_{\text{crit}} > 1 & \end{cases}$$

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- energy-critical defocusing NLS

GWPs & scattering: Bourgain '99, CKSTT '08, Visan

- energy-supercritical defocusing NLS

LWP in $\dot{H}^{\text{crit}}(\mathbb{R}^d)$ but GWP is open even for smooth solns.

(analogous to Navier-Stokes eqn: $\dot{H}^{1/2}(\mathbb{R}^3)$)

but only "energy" $\int |u|^2 dx$ is conserved

too weak to control

$\dot{H}^{1/2}$ -norm.

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Solitons & finite time blowup solutions

Consider the focusing NLS:

$$i\partial_t u + \Delta u = -|u|^{p-1}u.$$

Solitons (solitary wave solution)

$$u(t, x) = e^{it} \underbrace{\phi(x)}_{\text{profile}}$$

- Such u solves (NLS) iff ϕ solves the following elliptic PDE: ⑦

$$\textcircled{*} \quad \Delta \phi - \phi + |\phi|^{p-1} \phi = 0, \quad \phi \in H^1(\mathbb{R}^d)$$

FACT: $d=1$: all solns to $\textcircled{*}$ are translates of

Raphaël's
clay lec note.

$$Q(x) = \left(\frac{p+1}{2 \cosh^2(p-1)x} \right)^{\frac{p-1}{2}}$$



$d \geq 2$: \exists seq $\{Q_n\}_{n \geq 0}$ of real radial solns to $\textcircled{*}$

with increasing L^2 -norms s.t.

$Q_n(r)$ vanishes n times on \mathbb{R}_+

• Q_0 , radially sym, pos (ground state)

uniqueness: $\phi > 0, \phi \in H^1$

radial, C^2 , exp decaying (Gidas - Ni - Nirenberg '79
Kwong '89)

existence:

Berestycki - Lions - Peletier
'81

- shooting method (on ODE, $r > 0$)



- Ground states play an important role in elliptic, parabolic, dispersive PDEs
variational problem
functional inequality, etc.

- mass - subcritical : $p < 1 + \frac{4}{d}$. $\text{Scrit} < 0$

NLS scaling : $Q^{\lambda}(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$

Prop: (variational characterization of Q). $d \geq 1$, $1 < p < 1 + \frac{4}{d}$
 $M > 0$ fixed. Then, the minimization problem

$$\min_{\|u\|_{L^2} = M} H(u)$$

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has min attained at

$$Q^{\lambda(M)}(\cdot - x_0) e^{i\gamma_0} \text{ for all } x_0 \in \mathbb{R}^d, \gamma_0 \in \mathbb{R}$$

↑ rescale of Q s.t. $\|Q^{\lambda(M)}\|_{L^2} = M$.

($\text{Scrit} < 0$.

Scaling preserves H^{scrit} -norm but not L^2 -norm

- minimization problem

← Lagrange problem : $\frac{d}{d\varepsilon} H(u + \varepsilon v) \Big|_{\varepsilon=0} \leftarrow \text{Gâteaux deriv.}$

↳ Euler-Lagrange eqn : $\Delta\phi - \lambda\phi + |\phi|^{p-1}\phi = 0$

↑ Lagrange multiplier.