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Lec 1 15/01/18 (Mon)

- Nonlinear dispersive PDEs

examples:

① Nonlinear Schrödinger equations (NLS)

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

$$\partial_t = \frac{\partial}{\partial t}, \quad \Delta = \sum_{j=1}^d \partial_j^2 = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$$

$$u: (t, x) \in \mathbb{R} \times \mathbb{R}^d \rightarrow u(t, x) \in \mathbb{C}, \quad p > 1$$

② Nonlinear wave equations (NLW)

$$-\partial_t^2 u + \Delta u = \pm |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d$$

u,  $\mathbb{R}$ -valued

③ generalized Korteweg-de Vries equations (gKdV)

$$\partial_t u + \partial_x^3 u = \pm \partial_x(u^p), \quad (t, x) \in \mathbb{R} \times \mathbb{R}$$

u,  $\mathbb{R}$ -valued,  $p=2$ , KdV  
 $p=3$ , modified KdV (mKdV)

## Questions:

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- Well-posedness?

We say (NLS) is locally well-posed in  $B^s(\mathbb{R}^d)$

if given any  $u_0 \in B^s(\mathbb{R}^d)$ ,  $\exists!$  soln  $u$  to (NLS)  
on  $[-T, T]$ ,  $T = T(u_0) > 0$ , with  $u|_{t=0} = u_0$ .

- For NLW, initial data  $(u, \partial_t u)|_{t=0} = (u_0, u_1) \in B^s \times B^{s-1}$ .
- If solutions exist for short times, then
  - Does the solution exist globally in time?  
 $\Rightarrow$  if so, we say (NLS) is globally well-posed  
(if we can take  $T = \infty \Leftarrow T \gg 1$  arbitrarily large)
  - Otherwise, the solution may cease to exist at some time.  
 $\Leftarrow$  finite time blowup (formation of singularity)

- If GWP holds, then behavior of global-in-time solns? ③
  - scattering:  $u(t) = \text{soln to (NLS)}$   
 behaves asymptotically (as  $t \rightarrow \pm\infty$ )  
 like a linear solution  $w$ :  $i\partial_t w + \Delta w = 0$   
 (but with different data)  
 $\Rightarrow \|u(t)\| \rightarrow 0$  in some appropriate norm.
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- In some situations (i.e. if "s is too small"), we have ill-posedness results for these equations

Back to LWP: existence of unique solutions for short times  
 with stability (continuity of the soln map):

$$\Phi : u_0 \in B^s(\mathbb{R}^d) \longmapsto u \in C([-T, T]; B^s(\mathbb{R}^d))$$

is continuous for  $T = T(u_0) > 0$  small

If ill-posed, then what is the nature of the ill-posedness? ④

- discontinuity of the soln map (often  $u_0 = 0$ )
  - non-uniqueness
  - non-existence

## Sec 0: Background materials

- Lebesgue spaces:  $L^p(X)$ ,  $X = \mathbb{R}^d$

$$L^p(X) = \left\{ f \text{ on } X : \|f\|_{L^p(X)} \stackrel{\text{def}}{=} \left( \int_X |f(x)|^p dx \right)^{1/p} < \infty \right\}$$

- $\| \cdot \|_{L^p}$  is a norm
  - $L^p(X)$  is a normed vector space ( $p \geq 1$ )
  - Also complete  $\Rightarrow$  Banach space
  - $p=2$ : Hilbert space (Banach & inner product)

$$\langle f, g \rangle_{L^2} = \int_X f \bar{g} dx. \quad (\text{or } \Re \int f \bar{g} dx)$$

•  $(X, dx), (Y, dy)$ , measure spaces

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$$\|f\|_{L_y^q L_x^r} = \left\| \underbrace{\|\cdot\|}_{\text{on } X} \right\|_{L_x^r} \| \cdot \|_{L_y^q}$$

$$= \left( \int_Y \left( \int_X |f(x, y)|^r dx \right)^{q/r} dy \right)^{1/q}.$$

$L^q(Y; L^r(X))$

function of  $x$

$$C(\mathbb{R}_+; L^p(X)) = \left\{ f(t, x) : t \mapsto \underbrace{f(t)}_{\text{This map is continuous}} \in L^p(X) \right\}$$

$$(\mathbb{R}, |\cdot|_{\mathbb{R}}) \mapsto (X, \|\cdot\|_{L^p(X)})$$

## Key inequalities:

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$q$  is the Hölder conjugate of  $p$ .

① Hölder inequality:  $\frac{1}{p} + \frac{1}{q} = 1$ .

$p = q = 2$   
Cauchy-Schwarz

$$\|fg\|_{L^r(X)} \leq \|f\|_{L^p(X)} \|g\|_{L^q(X)}$$

In general,  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ :  $\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$ .

② Interpolation of  $L^p$ -spaces (log-convexity of  $L^p$ -norms)

$$0 < p < q \leq \infty, \quad \theta \in [0, 1].$$

$$\|f\|_{L^r} \leq \|f\|_{L^p}^\theta \|f\|_{L^q}^{1-\theta}, \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

$$\left( \begin{array}{l} \text{• } p = \infty: \\ \|f\|_{L^\infty(X)} = \underset{x \in X}{\text{ess sup}} |f(x)| \end{array} \right.$$

$$p \leq r \leq q$$

③ Minkowski's integral inequality :  $1 \leq p \leq q \leq \infty$ . (7)

$$\left\| \|f\|_{L^p(X)} \right\|_{L^q(Y)} \leq \left\| \|f\|_{L^q(Y)} \right\|_{L^p(X)}$$

usual Minkowski's ineq :  $\left\| \sum_{j \in \mathbb{Z}} f_j \right\|_{L^q(Y)} \leq \sum_{j \in \mathbb{Z}} \|f_j\|_{L^q(Y)}$   
 < Take  $L^p(X) = \ell'(\mathbb{Z})$

④ Young's inequality:

convolution of  $f$  and  $g$  on  $\mathbb{R}^d$ .

$$\begin{aligned} f * g(x) &= \int_{\mathbb{R}^d} f(x-y) g(y) dy \\ &= \int_{\mathbb{R}^d} f(y) g(x-y) dy. \end{aligned}$$

Young's :  $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ .  $1 \leq p, q, r \leq \infty$ .

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

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⑤ Riesz - Thorin interpolation theorem

$$1 \leq p_j, q_j \leq \infty, \quad j = 0, 1$$

$T$ , linear operator defined on  $L^{p_j}(R^d) \rightarrow L^{q_j}(R^n)$

s.t.

$$\|Tf\|_{L^{q_j}} \leq A_j \|f\|_{L^{p_j}}, \quad j = 0, 1.$$

Then,

$$\|Tf\|_{L^q} \leq A_0^\theta A_1^{1-\theta} \|f\|_{L^p}, \quad \theta \in [0, 1].$$

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$$

• Fourier transform on  $\mathbb{R}^d$

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$f$  on  $\mathbb{R}^d$ . The Fourier transform of  $f$  is given by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^d$$

Here,  $x \cdot \xi = \sum_{j=1}^d x_j \xi_j$ .

• well defined for  $f \in L^1(\mathbb{R}^d)$

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^d} |f(x)| dx = \|f\|_{L^1(\mathbb{R}^d)} < \infty.$$

• Schwartz class  $\underset{\uparrow}{\mathcal{S}(\mathbb{R}^d)} = C^\infty(\mathbb{R}^d) + \text{fast decay}$   
 "rapidly decreasing".

$$\|f\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty$$

• multi-index notation:  $\alpha \in (\mathbb{N} \cup \{0\})^d$      $\alpha = (\alpha_1, \dots, \alpha_d)$

$$x \in \mathbb{R}^d, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$$

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} \quad \partial_j = \frac{\partial}{\partial x_j}, \quad |\alpha| = \sum_{j=1}^d \alpha_j$$

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Inverse Fourier transform:

$$\mathcal{F}^{-1}(f)(x) = \check{f}(x)$$

$$\stackrel{\text{def}}{=} \hat{f}(-x) = \int_{\mathbb{R}^d} f(\xi) e^{\frac{2\pi i}{\lambda} x \cdot \xi} d\xi$$

different definitions :  $\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int f(x) e^{-ix \cdot \xi} dx$ ,  $\check{f} = \frac{1}{(2\pi)^{d/2}} \hat{f}(\xi)$

$$\hat{f}(\xi) = L \cdot \int \dots , \check{f} = \frac{1}{(2\pi)^{d/2}} \hat{f}(\xi)$$

Basic properties:

- $\|f\|_{L^2} = \|\hat{f}\|_{L^2} = \|\check{f}\|_{L^2}$  Plancherel's identity

- $\mathcal{F} : L^2 \rightarrow L^2$ , bijection  
 $f \rightarrow \hat{f}$

- $(\hat{f})^\vee = f = (\check{f})^\wedge$

- Parseval  $\int f(x) \overline{g(x)} dx = \int \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi$

$$\langle f, g \rangle_{L^2_x} = \langle \hat{f}, \hat{g} \rangle_{L^2_\xi}$$

Hausdorff - Young's inequality:  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $p \geq 2$  (11)

$$\|\hat{f}\|_{L^p} \leq \|f\|_{L^{p'}}$$

$$\Leftarrow \begin{aligned} \|\hat{f}\|_{L^\infty} &\leq \|f\|_{L^1} \\ \|\hat{f}\|_{L^2} &= \|f\|_{L^2} \end{aligned} \quad \& \text{ Riesz - Thorin interpolation}$$