Lec 9: 06/02/17 (Mon)

2.2 Randomization of a function on $\mathbb{T}^d$ (and $\mathbb{R}^d$)

$$f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in\cdot x}, \quad x \in \mathbb{T}^d.$$

· Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be a collection of independent mean-zero random variables with distributions of real and imaginary parts

assume indep.

$$\mu_n^{(r)}, \ \mu_n^{(i)}$$

$$\int e^{\gamma\cdot x} d\mu_n^{(r)}(x) \le e^{c\gamma^2}$$
$$\int e^{\gamma\cdot x} d\mu_n^{(i)}(x) \le e^{c\gamma^2}, \qquad \forall\, n \in \mathbb{Z}^d,\ \gamma \in \mathbb{R}$$

$\Leftarrow$ satisfied by standard $\mathbb{C}$-valued Gaussian r.v.'s, Bernouilli, (or uniform distri on $S^1 \subset \mathbb{C}$)

Rmk: In many situations, it suffices to assume (uniform) bddness of the k^{th} moment $\mathbb{E}[|g_n|^k] \le c < \infty$

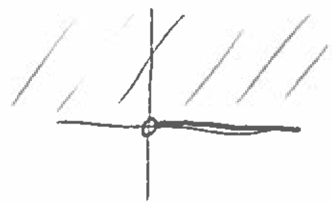
$\Rightarrow$ Randomization f^ω of f:

$$f^\omega(x) = \sum_{n\in\mathbb{Z}^d} g_n(\omega)\,\hat{f}(n)\,e^{in\cdot x}$$

Rmk: If f is real, i.e. $\hat{f}(-n) = \overline{\hat{f}(n)}$, $\forall n \in \mathbb{Z}^d$.

Let $I = \bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{Z}_+ \times \{0\}^{d-k-1}$ "=" $\mathbb{Z}^d/2$

$\Rightarrow \mathbb{Z}^d = I \cup \{-I\} \cup \{0\}$.



Then, introduce $\{g_n\}_{n\in I\cup\{0\}}$

and set $g_{-n} = \overline{g_n}$, $\forall n \in \mathbb{Z}^d$. ($g_0$ is real-valued.)

$\Rightarrow f^\omega(x) = \sum_{n\in\mathbb{Z}^d} g_n(\omega)\hat{f}(n)e^{in\cdot x}$ is real-valued.

<u>Moral:</u> If $f \in H^s \setminus H^{s+\varepsilon}$, then $f^\omega \in H^s \setminus H^{s+\varepsilon}$ a.s.

i.e. there is no gain in differentiability.

- If $f \in L^2$, then $f^\omega \in L^p$, $\forall\ 2 \le p < \infty$, a.s.

$\Rightarrow$ gain of integrability.

<u>Lemma 2.1:</u> (i) Given $f \in H^s(\mathbb{T}^d)$, let f^ω be the randomization defined above. Then, the following tail estimate holds:

$$P\left(\|f^\omega\|_{H^s} > \lambda\right) \le C \exp\left(-c\frac{\lambda^2}{\|f\|_{H^s}^2}\right) \to 0 \text{ as } \lambda \to \infty$$

for all $\lambda > 0$. In particular, $f^\omega \in H^s$, a.s.

(ii) Let $f \in L^2(\mathbb{T}^d)$. Then, for any finite $p \ge 2$, we have

$$P\left(\|f^\omega\|_{L^p} > \lambda\right) \le C \exp\left(-c\frac{\lambda^2}{\|f\|_{L^2}^2}\right).$$

for all $\lambda > 0$. In particular, $f^\omega \in L^p$, a.s.

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Lemma 2.2: Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be as above. Then, we have

$$\Big\| \sum_{n\in\mathbb{Z}^d} c_n g_n \Big\|_{L^p(\Omega)} \lesssim \sqrt{p}\, \| c_n \|_{\ell^2_n}.$$

Pf: We first prove

$$P\Big(\Big|\sum c_n g_n\Big| > \lambda \Big) \le C e^{-c\,\lambda^2/\|c_n\|^2_{\ell^2_n}}.$$

- $$\int e^{t\sum c_n g_n}\, dP \overset{\text{indep}}{=} \prod_n \int e^{t c_n g_n}\, dP$$

$$\le \prod_n e^{\alpha (t c_n)^2} = e^{\alpha^2 t^2 \sum c_n^2}.$$

$$\Rightarrow \quad P\Big(\sum c_n g_n > \lambda \Big) \le \frac{\mathbb{E}\big[e^{t\sum c_n g_n}\big]}{e^{t\lambda}} \le \frac{e^{\alpha^2 t^2 \sum c_n^2}}{e^{t\lambda}}$$

choose $t = \dfrac{\lambda}{2\alpha^2 \sum c_n^2}$

$$= e^{-\frac{\lambda^2}{4\alpha^2 \sum c_n^2}},$$

Similarly, $P\big(\sum c_n g_n < -\lambda \big) \le C e^{-c\,\lambda^2/\|c_n\|^2_{\ell^2_n}}$.

$$\Rightarrow \left\| \sum c_n g_n \right\|_{L^p(\Omega)}^p \le p \int_0^\infty \lambda^{p-1} e^{-c \lambda^2 / \|c_n\|_{\ell^2_n}^2} d\lambda.$$

$$\overset{\text{ch. of var}}{\lesssim} p \left(\sum c_n^2 \right)^{p/2} \underbrace{\int_0^\infty \lambda^{p-1} e^{-\lambda^2/2} d\lambda}_{\sim (p-1)!! = \frac{(2p)!}{2^p \cdot p!}}$$

$$\sim \left(p \sum c_n^2 \right)^{p/2}.$$

$\boxtimes$

Aside $\int_{\mathbb{R}} x^k e^{-x^2/2} dx$, k even.

$\Leftarrow$ IBP does not help. Use the moment generating function

$$\mathbb{E}[e^{tX}] \overset{\text{ch. f var}}{=} e^{\frac{1}{2}t^2} = \sum_{n=0}^\infty \frac{1}{n!\,2^n} t^{2n}$$

$$\mathbb{E}\left[\sum_{k=0}^\infty \frac{(tX)^k}{k!}\right] = \sum_{k=0}^\infty \frac{\mathbb{E}[X^k]}{k!} t^k. \quad \text{Compare the coefficients}$$

Pf of Lemma 2.1 : By Minkowski's integral inequality.

$$\forall\, p \geq 2 \quad \Big\| \underset{\substack{\| \\ \|\langle \nabla \rangle^s f^\omega\|_{L^2_x}}}{\|f^\omega\|_{H^s}} \Big\|_{L^p(\Omega)} \leq \Big\| \Big\| \sum_n \underbrace{\langle n \rangle^s \hat{f}(n)\, e^{in\cdot x}}_{C_n(x)} g_n \Big\|_{L^p(\Omega)} \Big\|_{L^2_x}$$

$$\overset{\text{Lem 2.2}}{\lesssim} \sqrt{p}\, \Big\| \underbrace{\| \langle n \rangle^s \hat{f}(n) \|_{\ell^2_n}}_{= \|f\|_{H^s}} \Big\|_{L^2_x(\mathbb{T}^d)}$$

$$\sim \sqrt{p}\, \|f\|_{H^s}.$$

By Chebyshev's ineq,

$$P\left(\|f^\omega\|_{H^s} > \lambda\right) < \left(\frac{C_0\, p^{1/2} \|f\|_{H^s}}{\lambda}\right)^p, \quad \forall\, p \geq 2.$$

Let $p = \left(\dfrac{\lambda}{C_0\, e \|f\|_{H^s}}\right)^2$. if $p \geq 2$, then we have

$$P\left(\|f^\omega\|_{H^s} > \lambda\right) < e^{-p} = e^{-c\, \lambda^2 / \|f\|^2_{H^s}}$$

If $p = \left(\frac{\lambda}{C_0 e \|f\|_{H^s}} \right)^2 \leq 2$, then

Choose C s.t. $C e^{-2} \geq 1$. Then, we have

$$P(\|f^\omega\|_{H^s} > \lambda) \leq 1 \leq C e^{-2} \leq C e^{-c\lambda^2 / \|f\|_{H^s}^2}$$

(Altenative way: Establish $\mathbb{E}\left[e^{\alpha \|f^\omega\|_{H^s}^2} \right] < \infty$

by Taylor expansion + $\| \|f^\omega\|_{H^s} \|_{L^p(\Omega)} \lesssim \sqrt{p} \|f\|_{H^s}$

(ii) Given $p \geq 2$, let $q \geq p$.

$$\| \|f^\omega\|_{L^p_x} \|_{L^q(\Omega)} \leq \| \| \sum_n \underbrace{\hat{f}(n) e^{in\cdot x}}_{c_n} \cdot g_n \|_{L^q(\Omega)} \|_{L^p_x}$$

$$\lesssim \sqrt{q} \| \underbrace{\| \hat{f}(n) \|_{\ell^2_n}}_{= \|f\|_{L^2}} \|_{L^p_x}$$

$$\sim \sqrt{q} \|f\|_{L^2}. \quad \forall q \geq p.$$

$\square$

<u>Lemma 2.3</u>: Suppose $f = \sum \hat{f}(n) e^{in\cdot x} \in H^s \setminus H^{s+\varepsilon}$ for some $\varepsilon > 0$.

Suppose $\exists\, c > 0$ s.t.

$$\limsup_{|n|\to\infty} \underbrace{P(\{|g_n| \le c\})}_{\mu_n(-c,\, c)} \le 1 - \delta < 1$$

(e.g. satisfied if g_n is i.i.d. and $g_n \not\equiv 0$)

Then, $\quad P(f^\omega \in H^{s+\varepsilon}) = 0$

<u>Pf</u>: For simplicity, assume g_n is real-valued.

$$\int e^{-\|f^\omega\|^2_{H^{s+\varepsilon}}}\, dP \overset{\text{indep}}{=} \prod_n \underbrace{\int}_{\int_{-c}^{c} + \int_{|g_n| \ge c}} e^{-\langle n\rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 |g_n|^2}\, d\mu_n$$

$$\le \prod_n \Big(\mu_n(-c,c) + \underbrace{e^{-c^2 \langle n\rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2}}_{=:\alpha_n} (1 - \mu_n(-c,c)) \Big)$$

$$= \prod_n \big(\mu_n(-c,c)(1-\alpha_n) + \alpha_n \big)$$

$$\le \prod_n \big((1-\delta)(1-\alpha_n) + \alpha_n \big) = \prod_n \big(1 - \delta(1-\alpha_n) \big) \overset{(**)}{=} 0$$

$$\Rightarrow \quad \|f^\omega\|_{H^{s+\varepsilon}} = \infty, \quad a.s.$$

• Proof of $**$: By assumption, $\sum \langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 = \infty$.

Then, we claim $\sum (1-\alpha_n) = \infty$

$$\sum (1-\alpha_n) = \sum \left(1 - e^{-c^2 \langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2}\right)$$

• If $\langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 \not\to 0$ as $|n| \to \infty$, then $\sum (1-\alpha_n) \geq \sum (1-\varepsilon) = \infty$.

• otherwise, $\exists N$ s.t. $c^2 \langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 \leq 1$, $\forall |n| \geq N$

$$\Rightarrow \sum_{|n|\geq N} (1-\alpha_n) \geq C \sum_{|n|\geq N} \langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 = \infty$$

(b/c $e^{-x} \leq 1 - C'x$ for $0 \leq x \leq 1$ (for some $C' \ll 1$)

Then,
$$\left(\prod_n (1-\delta(1-\alpha_n))\right)^{-1} = \prod_n \left(1 + \frac{\delta(1-\alpha_n)}{1-\delta(1-\alpha_n)}\right)$$
$$\leq \prod_n (1 + 2\delta(1-\alpha_n)) \quad \text{by choosing } \delta \leq \tfrac{1}{2}.$$
$$= \infty.$$

$\left(\prod_n (1 + \underset{>0}{a_n}) < \infty \iff \sum a_n < \infty.\right)$