Lec 8: 01 / 02 / 17 (Wed)

Chap 2: Probabilistic well-posedness of dispersive PDEs

· Nonlinear wave equations on $\mathbb{T}^d$ (or $\mathbb{R}^d$)

$$\text{(NLW)} \qquad \underbrace{(-\partial_t^2 + \Delta)}_{= \Box} u = \pm |u|^{p-1} u$$

2.1 Review on deterministic theory of NLW.

· Consider the non-homog linear wave equation

$$\begin{cases} \Box u = \pm F \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \end{cases}$$

Duhamel

$$\Longleftrightarrow \quad u(t) = \underbrace{\cos(t|\nabla|)\, u_0 + \frac{\sin t|\nabla|}{|\nabla|} u_1}_{S(t)(u_0, u_1)} \mp \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} F(t')\, dt'$$

With $v = \partial_t u$,

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \mp \begin{pmatrix} 0 \\ F \end{pmatrix}$$

Basic tool: Strichartz estimates: $d \geq 2$

$s \geq 0$, $2 \leq q \leq \infty$, $2 \leq r < \infty$, $(q, r, d) \neq (2, \infty, 3)$

Scaling condition: $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s$

admissibility condition: $\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}$. ($\Leftarrow$ Knapp's counter example)

- Suppose (q, r), s-admissible

  $(\tilde{q}, \tilde{r})$, $(1-s)$-admissible $\quad 0 \leq s \leq 1$

Then, we have

On $\mathbb{R}^d$

$$\boxed{\begin{aligned} &\|(u, \partial_t u)\|_{L^\infty_t \dot{\mathcal{H}}^s} + \|u\|_{L^q_t L^r_x} \\ &\quad \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^s} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \end{aligned}}$$

$\dot{\mathcal{H}}^s = \dot{H}^s \times \dot{H}^{s-1}$

$\quad \Downarrow$

(u_0, u_1)

Rmk: Thanks to $|\nabla|^{-1}$ in the Duhamel term, Sobolev inquality is also very effective.

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ex: Cubic NLW on $\mathbb{R}^3$:

$$-\partial_t^2 u + \Delta u = \pm u^3.$$

- LWP in $\dot{\mathcal{H}}^1(\mathbb{R}^3)$

$$\Gamma_{(u_0,u_1)}(u) = S(t)(u_0, u_1) \mp \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} u^3(t')\,dt'.$$

Sobolev: $\|f\|_{L^6_x(\mathbb{R}^3)} \lesssim \|f\|_{\dot H^1(\mathbb{R}^3)}$ $\qquad \underbrace{\frac{1}{3}}_{s/d} = \frac{1}{2} - \frac{1}{6}$

$$\left(2^* = \frac{2d}{d-2}\right)$$

$$\|\Gamma u\|_{L^\infty_T \dot H^1} \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^1} \quad \Leftarrow \text{By unitarity of } e^{\pm it|\nabla|},$$

$$+ \int_0^T \underbrace{\|u^3\|_{L^\infty_T L^2_x}}\,dt',$$

$$= \|u\|^3_{L^\infty_T L^6_x} \lesssim \|u\|^3_{L^\infty_T \dot H^1}$$

$$\Rightarrow \quad \|\Gamma u\|_{L^\infty_T \dot H^1} \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^1} + T \|u\|^3_{L^\infty_T \dot H^1}$$

$$\|\Gamma u - \Gamma v\|_{L^\infty_T \dot H^1} \lesssim T \left(\|u\|^2_{L^\infty_T \dot H^1} + \|v\|^2_{L^\infty_T \dot H^1} \right) \|u - v\|_{L^\infty_T \dot H^1}.$$

$\Rightarrow$ By the fixed pt argument, cubic NLW on $\mathbb{R}^3$ is LWP in $\dot{\mathcal{H}}^1(\mathbb{R}^3)$

- The same proof works on $\mathbb{T}^3$.

Q: How low can we go?

scaling / dilation symmetry : $u(x,t)$ soln to NLW (with u^p)

$$\Rightarrow \quad u_\lambda(x,t) = \frac{1}{\lambda^{2/(p-1)}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right) \text{ is also a soln.}$$

We say that s is a critical Sobolev index if

$$\|f_\lambda\|_{\dot H^s(\mathbb{R}^d)} = \|f\|_{\dot H^s(\mathbb{R}^d)}, \quad f_\lambda(x) = \frac{1}{\lambda^{2/(p-1)}} f\left(\frac{x}{\lambda}\right)$$

$$\boxed{s_{crit} = \frac{d}{2} - \frac{2}{p-1}}$$

$d = 3$, $p = 3$; $s_{crit} = \frac{1}{2}$

Moral: Given $(u_0, u_1) \in \dot{\mathcal{H}}^s(\mathbb{R}^d)$,

(i) $s > s_{crit}$: subcritical $\Rightarrow$ (expect) well-posedness

(ii) $s = s_{crit}$: critical $\Rightarrow$ (expect) well-posedness but more delicated

(iii) $s < s_{crit}$: supercritical $\Rightarrow$ bad behavior (ill-posedness)

Norm inflation: $\forall \varepsilon > 0$, $\exists$ smooth soln u_ε and $t_\varepsilon \in (0, \varepsilon)$

s.t. ① $\| (u_\varepsilon(0), \partial_t u_\varepsilon(0)) \|_{\dot{\mathcal{H}}^s} < \varepsilon$

$s < s_{crit}$

but ② $\| (u_\varepsilon(t_\varepsilon), \partial_t u_\varepsilon(t_\varepsilon)) \|_{\dot{\mathcal{H}}^s} > \varepsilon^{-1}$.

$\Rightarrow$ failure of continuity of the soln map $\Phi : (u_0, u_1) \longmapsto (u, \partial_t u)$ (at $(0,0)$)

$$\dot{\mathcal{H}}^s \longrightarrow C_t \dot{\mathcal{H}}^s$$

Christ – Colliander – Tao '03 arXiv

• LWP of cubic NLW in $\dot{\mathcal{H}}^{\frac{1}{2}}(\mathbb{R}^3)$

↑ lowest possible regularity.

Use Strichartz with

$$s = \tfrac{1}{2}, \quad (q, r) = (4, 4) = (\hat{q}, \tilde{r})$$

$$\frac{1}{4} + \frac{3}{4} = \frac{3}{2} - \frac{1}{2}$$

$$\frac{1}{4} + \frac{3-1}{2\cdot 4} = \frac{1}{2} \le \frac{3-1}{4}$$

$$\Rightarrow \quad \|\Gamma u\|_{\boxed{L^\infty_T \dot{H}^{1/2} \cap L^4_T L^4_x}} \quad = X^{1/2}$$

$$(*) \qquad \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^{1/2}} + \underbrace{\|u^3\|_{L^{4/3}_T L^{4/3}_x}}_{= \|u\|^3_{L^4_{T,x}}} \le \|u\|^3_{L^\infty_T \dot{H}^{1/2} \cap L^4_{T,x}}$$

$$\le \eta^2 \|u\|_{X^{1/2}}$$

No T^θ !!

⇐ manifestation of criticality.

Can carry out a contraction mapping principle in B_η

where $\eta = 2C \|(u_0, u_1)\|_{\dot{\mathcal{H}}^1} \ll 1$

↑ const on lin soln

Rmk: can take $T=\infty$ if initial data is suff. small.

(small data global well-posedness)

For large data, note that

$$\underbrace{\| S(t)(u_0,u_1)\|_{L^4_{T,x}}}_{\left(\le \|(u_0,u_1)\|_{\dot H^{1/2}} < \infty.\right)} \to 0 \text{ as } T\to 0 \quad (\text{by DCT})$$

By Strichartz estimate

$$\|\Gamma u\|_{L^4_{T,x}} \le \underbrace{\| S(t)(u_0,u_1)\|_{L^4_{T,x}}}_{\to 0} + C\|u\|^3_{L^4_{T,x}}.$$

$\Rightarrow$ contraction on B_η in $L^4_{T,x}$

$$\eta = 2\| S(t)(u_0,u_1)\|_{L^4_{T,x}} \ll 1.$$

$\Rightarrow$ By ⊛, $u\in C_T\dot H^{1/2}$.

Rmk: ① The same LWP holds on $\mathbb{T}^3$

② Local existence time $T = T((u_0, u_1))$

(i.e. more info than $\|(u_0, u_1)\|_{\mathcal{H}^{1/2}}$ is needed.)

Let $s < 1/2$. Consider $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3) \setminus \mathcal{H}^{1/2}(\mathbb{T}^3)$

Q: Can we construct a soln u with $(u, \partial_t u)\big|_{t=0} = (u_0, u_1)$?

Idea: "Randomize" (u_0, u_1)

f on $\mathbb{T}^d$. $\qquad f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in\cdot x}$

Randomization of f:

$$f^{\omega}(x) = \sum_{n\in\mathbb{Z}^d} \underline{\underline{g_n(\omega)}} \hat{f}(n) e^{in\cdot x}$$