Lec 7: 30 / 01 / 17 (Mon)

Lemma 1.6: (Approximation lemma) $s < 1/2$

$u_0 \in H^s$ with $\|u_0\|_{H^s} \leq K$

Suppose soln u_N to (FNLS) with $u_N|_{t=0} = u_0$ satisfies

$$\|u_N(t)\|_{H^s} \leq K, \quad |t| \leq T.$$

Then, $\exists!$ soln u to (NLS) on $[-T, T]$ with $u|_{t=0} = u_0$.

Moreover, we have

$$\|u(t) - P_{\leq N} u_N(t)\|_{H^{s_1}} \leq C_0 e^{C_1 (1+K)^{C_2} T} K \underbrace{N^{s_1 - s}}_{\to 0}$$

for $s_1 < s$. (for suff. large $N \in \mathbb{N}$.)

Pf: (FNLS) & (NLS) with $u_N|_{t=0} = u|_{t=0} = u_0$ are
sketch
locally well-posed on $[-\delta, \delta]$, $\delta \sim (1+K)^{-\gamma}$, indep of N.

Infact, choose this as $\delta \sim (1+2K)^{-\gamma}$

Let $v_N = P_{\leq N} u_N$.

$$\|u - v_N\|_{X^{s_1}([0,\delta])} \lesssim \|\underbrace{\underbrace{u_0 - P_{\le N} u_0}_{= P_{>N} u_0}\|_{H^{s_1}}}_{\begin{subarray}{l} \le N^{s_1 - s}\|u_0\|_{H^s} \\ \le N^{s_1 - s} K \end{subarray}} + \text{nonlin term}$$

Write $|u|^{p-1}u = (Id - P_{\le N})(|u|^{p-1}u) + P_{\le N}(|u|^{p-1}u)$

① $\underbrace{(Id - P_{\le N})(|u|^{p-1}u)}_{\lesssim \delta^\theta N^{s_1 - s} K^p.}$

↑ LWP of u in $X^s([0,\delta])$

$\|u\|_{X^s([0,\delta])} \le CK.$

$+$ ② $P_{\le N}(|u|^{p-1}u - |v_N|^{p-1}v_N)$

$\lesssim \delta^\theta K^{p-1} \|u - v_N\|_{X^{s_1}([0,\delta])}$

$$\Rightarrow \quad \|u - v_N\|_{X^{s_1}([0,\delta])} \leq C K N^{s_1 - s} + \frac{1}{2} \|u - v_N\|_{X^{s_1}([0,\delta])}$$

$$\left(\Leftarrow \ \delta^\theta K^{p-1} \ll 1 \right.$$

$$\Rightarrow \quad \|u - v_N\|_{X^{s_1}([0,\delta])} \leq 2 C K N^{s_1 - s}$$

- Now iterate the argument $\sim T/\delta$ many times.

$$\|u - v_N\|_{X^{s_1}([\delta, 2\delta])} \lesssim \underbrace{\|u(\delta) - v_N(\delta)\|_{H^{s_1}}}_{\lesssim K N^{s_1 - s}} + \underbrace{\text{nonlin term}}_{\text{can be handled exactly as before.}}$$

$$\left(X^{s_1} \subset C_t H^{s_1} \right.$$

etc.

$$\Rightarrow \quad \|u(t) - v_N(t)\|_{H^{s_1}} \lesssim \underbrace{e^{c \frac{T}{\delta}}}_{\hookrightarrow\, e^{c(1+K)^\gamma T}} K N^{s_1 - s}$$

$\boxed{\diagup}$

Prop 1.7 (Almost a.s. GWP) $s < 1/2$

Given $T, \varepsilon > 0$, $\exists\ \Omega_{T,\varepsilon} \subset H^s(\mathbb{T})$ s.t

(i) $\mu(\Omega^c_{T,\varepsilon}) < \varepsilon$

(ii) For $u_0 \in \Omega_{T,\varepsilon}$, $\exists !$ soln u to (NLS) on $[-T, T]$

s.t.
$$\| u(t) \|_{H^s} \lesssim \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \quad |t| \le T.$$

Pf: Let $\Omega_N(T, \varepsilon)$ be as in Key Prop.

$$\Rightarrow \quad \| \Phi_N(t)(u_0) \|_{H^s} \le CK \quad \text{for } |t| \le T \text{ and } u_0 \in \Omega_N$$

- By Lemma 1.6, $\exists\ N_1 \gg 1$ s.t.
$$\| u(t) - u_N(t) \|_{H^{s_1}} \ll 1, \qquad |t| \le T$$
for $N \ge N_1$
$$\Rightarrow \quad \| u(t) \|_{H^{s_1}} \lesssim K \sim \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \qquad |t| \le T.$$

- Also
$$\mu(\Omega_N^c) \overset{C-S}{\le} \underset{\left(e^{\frac{1}{p+1}\int |P_{\le N} u|^{p+1}}\right)}{\| R_N \|_{L^2(d\rho_1)}} \left(\int_{\Omega_N^c} \underbrace{\mathbb{1}_{\{\|u\|_{L^2} \le r\}}}_{\le \mathbb{1}_{\{\|P_{\le N} u\|_{L^2} \le r\}}} d\rho_1 \right)^{1/2}$$

$$\left(\mathbb{1}_{\{\|P_{\leq N} u\|_{L^2} \leq r\}} \, d\rho_1 \; \substack{\ll \\ \gg} \; \mu_N \right.$$

$$\overset{C-S}{\lesssim} \left(\mu_N (\Omega_N^c) \right)^{\frac{1}{4}} < \varepsilon.$$

$\boxed{/}$

Thm 1.8: NLS is globally well-posed almost surely with respect to the Gibbs measure μ.

Pf: Fix $\varepsilon > 0$. Let $T_j = 2^j$, $\varepsilon_j = \varepsilon / 2^j$.

$\Rightarrow$ Construct $\Omega_j = \Omega_{T_j, \varepsilon_j}$

Let $\Omega_\varepsilon = \bigcap_{j=1}^{\infty} \Omega_j$

(i) $\mu(\Omega_\varepsilon^c) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$

(ii) If $u_0 \in \Omega_\varepsilon$, then the soln u to (NLS) with $u|_{t=0} = u_0$ exists globally in time.

- Let $\Sigma = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$.

(i) $\mu(\Sigma^c) \le \inf_{\varepsilon>0} \varepsilon = 0.$

(ii) If $u_0 \in \Sigma$, then the soln u to (NLS) with $u|_{t=0} = u_0$ exists globally in time.

$\boxtimes$

<u>Rmk:</u> We have the following probabilistic bound: $s < 1/2$

$$\| u(t) \|_{H^s} \lesssim C(u_0^\omega) \left(\log(1+|t|) \right)^{1/2}, \quad \forall\, t \in \mathbb{R}.$$

<u>Thm 1.9:</u> The Gibbs measure μ is invariant under the flow of NLS.

<u>Pf:</u> By time reversibility of $\Phi(t)$, it suffices to show

(*) $$\mu(A) \le \mu(\Phi(t) A)$$

for all measurable set $A \subset H^s$ and $t \in \mathbb{R}$

(If u solves NLS, so does $\bar{u}(-t)$.)

By inner regularity,

$$\mu(A) = \sup_{\substack{F \subset A \\ \text{closed in } H^s}} \mu(F).$$

i.e. $\exists \{F_n\}$ closed sets in H^s s.t.

$$F_n \subset A \quad \text{and} \quad \mu(A) = \lim_{n\to\infty} \mu(F_n)$$

- Claim: Suffices to prove (*) for closed sets.

$$\Leftarrow \quad \mu(A) = \lim_{n\to\infty} \mu(F_n)$$
$$\leq \overline{\lim_{n\to\infty}} \mu(\Phi(t) F_n)$$
$$\leq \mu(\Phi(t) A)$$

$F_n \subset A$ and uniqueness.

- Given a closed set $F \subset H^s$,

let $K_n = \{ u \in F : \|u\|_{H^\sigma} \leq n \}$, $\quad s < \sigma < \frac{1}{2}$.

Then, K_n is compact in H^s. (Rellich's lemma)

$\Rightarrow$ Suffices to prove ⊛ for compact sets.

$$\mu(F) = \lim_{n\to\infty} \mu(K_n)$$

Lem 1.1. & Prop 1.2

- Let K be a cpt set in H^s.

Since $\mu_N \rightharpoonup \mu$ & Portmanteau theorem,

we have

(1) $$\mu(\Phi(t)K + \overline{B_\varepsilon}) \geq \overline{\lim}\, \mu_N(\Phi(t)K + \overline{B_\varepsilon})$$

- Fix $t \ll 1$. Then,

local theory

$$\Phi_N(t)(K + B_\delta) \subset \Phi_N(t)K + B_{\varepsilon/2}$$

$$\subset \Phi(t)K + B_\varepsilon$$

Approximation lemma.

$\Rightarrow$ By invariance of μ_N,

(2) $$\mu_N(K + B_\delta) \leq \mu_N(\Phi(t)K + B_\varepsilon).$$

Therefore,
$$\mu(K) \le \mu(K+B_\delta) \le \varliminf \mu_N(K+B_\delta) \le \varliminf \mu_N(\Phi(t)K+B_\varepsilon) \underset{(2)}{\le} \varlimsup \mu_N(\Phi(t)K+\overline{B_\varepsilon}) \underset{(1)}{\le} \mu(\Phi(t)K+\overline{B_\varepsilon}).$$

$\Rightarrow$ Let $\varepsilon \to 0$, $\mu(K) \le \mu(\Phi(t)K)$ for $t \ll 1$

$\Rightarrow$ for all $t \ge 0$.

$\Rightarrow$ for all $t \in \mathbb{R}$

$\square$