Lec 6 25/01/17 (Wed)

- LWP in a subcritical sense:

Linear estimate
nonlin estimate $\Rightarrow \|\Gamma_{u_0} u\|_{X^s([0,\delta))} \lesssim \|u_0\|_{H^s} + \delta^\theta \|u\|^p_{X^s([0,\delta))}$

$\Rightarrow$ By a fixed pt argument (Banach's contraction mapping thm)

$\Rightarrow$ $\exists!$ soln u with $u|_{t=0} = u_0$ on $[0,\delta)$

$$\delta \sim (1 + \|u_0\|_{H^s})^{-\gamma}, \quad \gamma > 0.$$

- Blowup alternative: let T^* be the forward maximal time of existence.

Then, either $T^* = \infty$ or $\lim_{t \nearrow T^*} \|u(t)\|_{H^s} = \infty$ $\quad [0, T^*)$

- The "only" way to construct global-in-time solns is to use conservation laws: (defocusing NLS)

$$H(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{p+1}\int |u|^{p+1}, \quad M(u) = \int |u|^2 dx$$

$$\Longrightarrow \quad \|u(t)\|_{H^1}^2 \lesssim H(u)(t) + M(u)(t) = H(u_0) + M(u_0) < \infty$$

Namely,

LWP in H^1 in a subcritical sense

$\Rightarrow$ GWP in H^1.

Issue: $\nexists$ conservation law at the level of the Gibbs meas μ

$s = \frac{1}{2}-$.

idea: use invariance of μ (in place of a conservation law) to construct global-in-time dynamics (on supp μ)

$$\begin{tikzcd} \text{invariance} \arrow[r, bend right] & \text{a.s. GWP} \arrow[l, bend right] \end{tikzcd}$$

Bourgain '94: use invariance of the "finite dim'l" Gibbs meas μ_N associated to the truncated dynamics

$\Rightarrow$ a.s. GWP $\Rightarrow$ invariance

CASE 3: No LWP (by a deterministic method)

(3.a): probabilistic local well-posedness

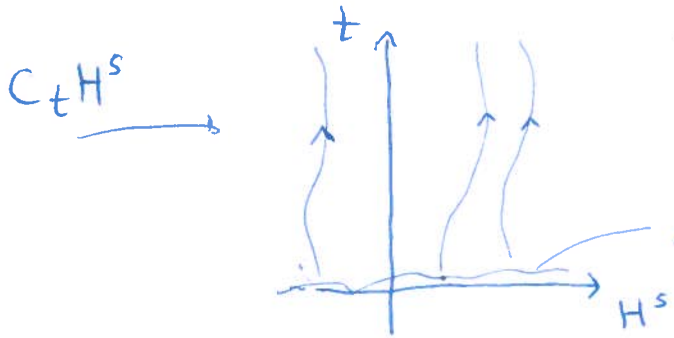
McKean '95, Bourgain '96, Burq–Tzvetkov '07.

Recall $\mu \ll \rho_1 \rightsquigarrow u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{inx}$

Aim: Exploit the randomness of initial data.

(3.b): compactness argument

(for measures on space–time functions)



$\Phi_N : u_{0,N} \longmapsto u_N$

induced prob meas

Define a prob meas ν_N by

$$\nu_N = \mu_N \circ \Phi_N^{-1}$$

prob meas on space–time functions

$\Rightarrow$ a.s. global existence (without uniqueness)

& "invariance" (in some mild sense)

- We first focus on CASE 2

$$\text{(NLS)} \quad i\partial_t u + \partial_x^2 u = \pm |u|^{p-1} u \quad \text{on } \mathbb{T}.$$

Bourgain '93: LWP in $H^s(\mathbb{T})$ for some $s = s(p) < \frac{1}{2}$.

in a subcritical sense

$$\delta \sim (1 + \|u_0\|_{H^s})^{-\gamma}$$

- Consider the following truncated dynamics:

$$\text{(FNLS)} \quad \begin{cases} i\partial_t u_N + \partial_x^2 u_N = \pm P_{\le N}\left(|P_{\le N} u_N|^{p-1} P_{\le N} u_N\right) \\ u_N|_{t=0} = u_0 \end{cases}$$

$$P_{\le N} f(x) = \sum_{|n| \le N} \hat{f}(n) e^{inx}$$

- (FNLS) is NOT finite dim'l.

high freq $\sim$ linear

low freq $\sim$ nonlinear

decoupled

- (FNLS) is locally well-posed by apply the LWP argument for (NLS): $\delta \sim (1 + \|u_0\|_{H^s})^{-\gamma}$, indep of N.

Let $H_N(u_N) = \frac{1}{2}\underbrace{\int |\partial_x u_N|^2}_{\text{lin. dynamics on high freq.}} \pm \frac{1}{p+1}\int |P_{\leq N} u|^{p+1}$.

$$\Rightarrow \quad \partial_t u_N = -i \frac{\partial H_N}{\partial \overline{u_N}}$$

① $u_{high} = P_{>N} u_N$ evolves linearly.

$$\partial_t \widehat{u}_{high}(n) = -i n^2 \widehat{u}_{high}(n), \quad |n| > N$$

$$\Rightarrow \widehat{u}_{high}(t,n) = e^{-itn^2}\widehat{u}_0(n), \quad |n| > N$$

In particular, u_{high} exists globally.

② $u_{low} = P_{\leq N} u_N$ satisfies

$$i\partial_t u_{low} + \partial_x^2 u_{low} = \pm P_{\leq N}(|u_{low}|^{p-1} u_{low})$$

$\Leftarrow$ finite dim'l system of (nonlin) ODEs (on the Fourier side) (with Lipschitz v.f.) $\Rightarrow$ local existence by Cauchy-Lipschitz thm.

- L^2-norm: $\|u_{low}\|_{L^2} = \left(\sum_{|n|\leq N} |\widehat{u}_{low}(n)|^2\right)^{1/2}$ is conserved.

$=$ Euclidean distance on $\mathbb{C}^{2N+1}$.

$\Rightarrow$ u_{low} exists globally.

$\Rightarrow$ $u_N = u_{low} + u_{high}$ exists globally in time.

Issue: (FNLS) is GWP for each $N \in \mathbb{N}$,

but there is NO uniform (in N) control on $\|u_N(t)\|_{H^s}$.

Write $\rho_1 = \rho_N \otimes \rho_N^{\perp}$

$$\cdot \; d\rho_N = Z_N^{-1} e^{-\frac{1}{2}\|P_{\leq N} u\|_{H^1}^2} d(P_{\leq N} u)$$

$$= Z_N^{-1} \prod_{|n| \leq N} e^{-\frac{1}{2}\langle n\rangle^2 |\hat{u}(n)|^2} d\hat{u}(n)$$

$$\cdot \; d\rho_N^{\perp} = \tilde{Z}_N^{-1} e^{-\frac{1}{2}\|P_{>N} u\|_{H^1}^2} d(P_{>N} u) \quad \text{on } H^s, \; s < \tfrac{1}{2}.$$

Then, ① $\rho_N^{\perp}$ is invariant under (FNLS$_{high}$).

$$P_{>N} u_0(x) = \sum_{|n|>N} \frac{g_n(\omega)}{\langle n\rangle} e^{inx} \Rightarrow u_{high}(t,x) = \sum_{|n|>N} \frac{\overbrace{e^{-itn^2} g_n(\omega)}^{\hat{g}_n(\omega)}}{\langle n\rangle} e^{inx}$$

$\Rightarrow$ Since g_N is invariant under a rotation, $\rho_N^\perp$ is invariant.

② $$d\mu_{N,low} = Z_N^{-1} \underbrace{R_{N,r}(u_{low})}_{\mathbb{1}_{\{\|u_{low}\|_{L^2} \le r\}} e^{\frac{1}{p+1}\int |u_{low}|^{p+1}}} d\rho_N$$

is invariant under $(FNLS_{low})$.

$$d\mu_{N,low} = Z_N^{-1} e^{\overbrace{-H_N(u_{low})}} \, \underbrace{d u_{low}}$$

conserved. inv by Liouville's thm

$\Rightarrow$ $\mu_N = \mu_{N,low} \otimes \rho_N^\perp$ is invariant under (FNLS).

Key Proposition (Bourgain '94)

$\forall T > 0, \ \varepsilon > 0, \ \exists \ \Omega_N = \Omega_N(T, \varepsilon)$ s.t.

(i) $\mu_N(\Omega_N^c) < \varepsilon$

(ii) For $u_0 \in \Omega_N$, the soln u_N to (FNLS) with $u_N|_{t=0} = u_0$ satisfies $\| u_N(t) \|_{H^s} \lesssim \left(\log \frac{T}{\varepsilon} \right)^{1/2}$, $|t| \le T$.

$\Leftarrow$ implicit const is independent of N

Pf: By local theory,

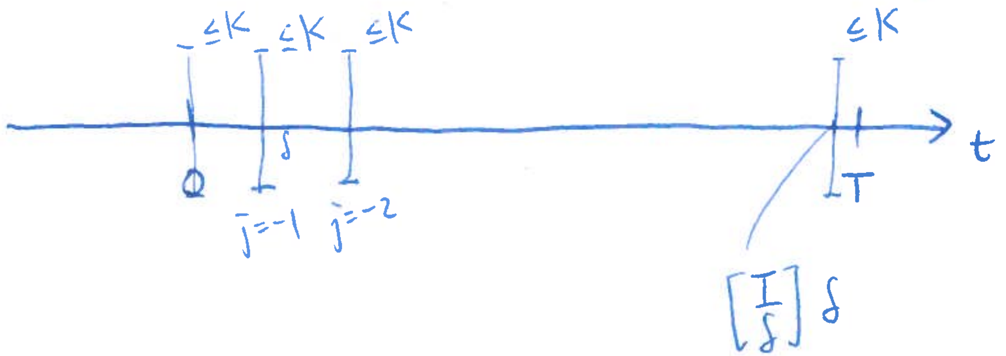
$$\|u_0\|_{H^s} \le K \implies \|u_N(t)\|_{H^s} \le CK$$

for $|t| \le \delta \sim K^{-\gamma}$, (indep of N)

Let

$$\Omega_N = \bigcap_{j=-[\frac{T}{\delta}]}^{[\frac{T}{\delta}]} \Phi_N(j\delta)\big(\underbrace{\{\|u_0\|_{H^s} \le K\}}_{= B_K}\big)$$

$\Phi_N(t): u_0 \longmapsto u_N(t)$, soln map for (FNLS)



(i)

using uniqueness

$$\mu_N(\Omega_N^c) \le \sum_{j=-[\frac{T}{\delta}]}^{[\frac{T}{\delta}]} \mu_N(\cancel{\Phi_N(j\delta)}(B_K^c))$$

By invariance of μ_N

$$\mu_N(\Phi(-t)A) = \mu_N(A)$$

$$\sim \frac{T}{\delta}\mu_N(B_K^c)$$

$$\lesssim T \cdot K^\gamma e^{-cK^2} < \varepsilon$$

Lemma 1.1 & Prop 1.2

$$\left(\mu_N(B_K^c) \underset{C\text{-}S}{\le} \underbrace{\|R_{N,r}\|_{L^2(d\rho_1)}}_{\le C \text{ indep of } N} \left(\rho_1(B_K^c)\right)^{1/2} \lesssim e^{-cK^2} \right)$$

by choosing $K \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$

(ii) By local theory,

$$\|u_N(t)\|_{H^s} \le CK \sim \left(\log\frac{T}{\varepsilon}\right)^{1/2}, \ \forall |t| \le T$$

$\boxed{}$