Lec 5 23/01/17 (Mon)

$$d\rho_t = Z_t^{-1} e^{-\|x\|^2/2t} dx, \quad t > 0 \qquad (H, B, \rho)$$

· Integrability of $e^{\alpha \|x\|^2}$

Ω = space of conti func ω on $[0, \infty)$
with values in B and $\omega(0) = 0$.

$\exists!$ prob meas P on the σ-field generated by the coordinate functions $\omega \mapsto \omega(t)$, $t > 0$, s.t.
if $0 = t_0 < t_1 < \cdots < t_n$, then

1. $\omega(t_j) - \omega(t_{j-1})$, $j = 1, \cdots, n$, are indep
2. $\omega(t_j) - \omega(t_{j-1})$ is distributed in B according to $\rho_{t_j - t_{j-1}}$.

· Define $W(t) : \Omega \to B$ by $W(t)(\omega) = \omega(t)$

Wiener process in B (starting at 0)

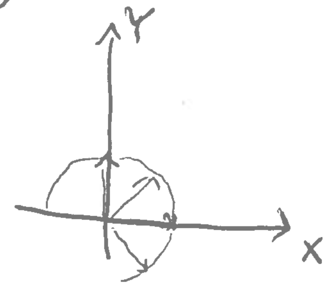
Note: $\displaystyle\int_B e^{\alpha\|x\|^2}\rho_1(dx) = \mathbb{E}\left[e^{\alpha\|W(1)\|^2}\right]$

- **Pf of Fernique's Thm.**

Let $X = W(1)$, $Y = W(2) - W(1)$ indep, distributed by ρ_1

Moreover, $\dfrac{X+Y}{\sqrt{2}}$ and $\dfrac{X-Y}{\sqrt{2}}$ are indep & distributed by ρ_1

(just rotate the coord axes in the XY plane.)



- For $t > s$, we have

$$P(\|W(1)\| \le s)\, P(\|W(1)\| > t)$$

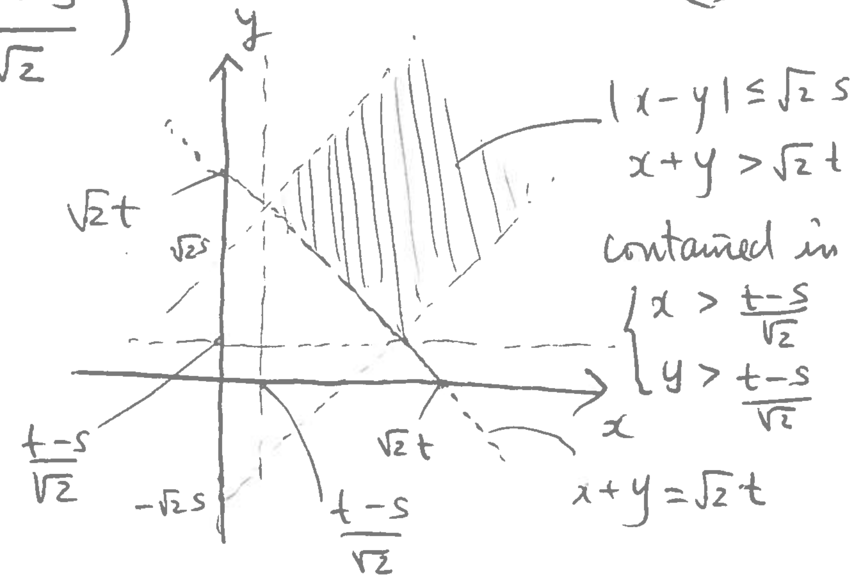
$$= P\left(\left\|\frac{X-Y}{\sqrt{2}}\right\| \le s\right) P\left(\left\|\frac{X+Y}{\sqrt{2}}\right\| > t\right)$$

$$\overset{\text{indep}}{=} P\left(\frac{\|X-Y\|}{\sqrt{2}} \le s \text{ and } \frac{\|X+Y\|}{\sqrt{2}} > t\right)$$

$$(*) \quad \overset{\text{triangle ineq}}{\le} P\left(\big|\|X\| - \|Y\|\big| \le \sqrt{2}s \text{ and } \|x\| + \|y\| > \sqrt{2}t\right)$$

$$\le P\left(\|X\| > \frac{t-s}{\sqrt{2}} \text{ and } \|Y\| > \frac{t-s}{\sqrt{2}} \right)$$

$$\overset{\text{indep}}{=} P\left(\|W(1)\| > \frac{t-s}{\sqrt{2}} \right)^2$$



Define t_n by $t_0 = s > 0$

$$t_{n+1} = s + \sqrt{2}\, t_n$$

$$\Rightarrow t_n = (1 + \sqrt{2} + \cdots + \sqrt{2}^n)\, s$$

$$= \frac{\sqrt{2}^{n+1} - 1}{\sqrt{2} - 1}\, s.$$

Let $\alpha_n = \dfrac{P(\|W(1)\| > t_n)}{P(\|W(1)\| \le s)}$, $n \ge 0$

$$\overset{\text{By }(*)}{\Longrightarrow} \quad \alpha_{n+1} \le \alpha_n^2 \Rightarrow \alpha_n \le \alpha_0^{2^n} = e^{2^n \log \alpha_0}$$

$$\Rightarrow P(\|W(1)\| > t_n) \le P(\|W(1)\| \le s)\, e^{2^n \log \alpha_0}$$

· Let $u_n = \sqrt{2}^{n+4} s$

$$\Rightarrow u_n > (\sqrt{2}^{n+1} - 1)(\sqrt{2} + 1)\, s = t_n$$

$$\Rightarrow \quad P(\|W(1)\| > u_n) \le P(\|W(1)\| \le s)\, e^{\frac{u_n}{16 s^2}\log \alpha_0}$$

- Choose $s \gg 1$ s.t. $0 < \alpha_0 < 1$.

  (By conti from above, $P(\|W(1)\| > s) \to 0$ as $s \to \infty$

  By conti from below, $P(\|W(1)\| \le s) \to 1$ as $s \to \infty$

Let

$$a = -\frac{1}{16 s^2}\log \alpha_0 > 0$$

$$b = P(\|W(1)\| \le s)$$

$$\Rightarrow \quad P(\|W(1)\| > u_n) \le b\, e^{-a u_n^2}$$

$$\Rightarrow \quad \int_{\|x\| > N} e^{\alpha \|x\|^2} \rho_1(dx) \le \sum_{k=0}^{\infty} \int_{2^k N \le \|x\| \le 2^{k+1} N} e^{\alpha \|x\|^2} \rho_1(dx)$$

$$\le b \sum_{k=0}^{\infty} e^{\alpha (2^{k+1} N)^2}\, e^{-a (2^k N)^2} < \infty$$

$$(4\alpha - a)(2^k N)^2$$

by choosing $\alpha \ll a$.

$\boxtimes$

• Invariance of the Gibbs measure ($d = 1$)

• 3 scenarios

CASE 1: We have an a priori (deterministic) global well-posedness in $\mathrm{supp}(\mu) \subset H^{\frac{1}{2}-}(\mathbb{T})$

ex: KdV, cubic, GWP in $L^2(\mathbb{T})$ (Bourgain '93)

$\Rightarrow$ Invariance of μ follows from the invariance of the "finite-dim'l" Gibbs meas μ_N associated to the truncated equation (easy)

CASE 2: We only know a priori local well-posedness in $\mathrm{supp}(\mu)$ (in subcritical sense).

$\Updownarrow$

local existence time $\delta > 0$ depends only on the size of initial data: $\delta \sim \|u_0\|_{H^\sigma}^{-\theta}$ for some $\theta > 0$.

NLS: $\quad i\partial_t u + \partial_x^2 u = \pm |u|^{p-1} u$

We say u is a solution to NLS if u satisfies the following Duhamel formulation:

$$u(t) = S(t)u_0 \mp i \int_0^t S(t-t')|u|^{p-1}u(t')\,dt'.$$

Note: $\quad i\partial_t u + \partial_x^2 u = F$

$$\underset{\text{F.T. in } x}{\Longrightarrow} \quad i\partial_t \hat{u}(n) - n^2 \hat{u}(n) = \hat{F}(n) \qquad \times\, e^{itn^2}$$

$$\Longrightarrow \quad i\partial_t\left(e^{itn^2}\hat{u}(n)\right) = e^{itn^2}\hat{F}(n).$$

$\Longrightarrow$ integrate from 0 to t. & invert F.T.

$$\Longrightarrow \quad u(t) = S(t)u_0 - i\int_0^t S(t-t')F(t')\,dt'$$

Let $\Gamma_{u_0}(u)(t) = S(t)u_0 \mp i\int_0^t S(t-t')|u|^{p-1}u(t')\,dt'.$

u, soln to NLS $\Longleftrightarrow$ $\Gamma_{u_0}(u) = u$. i.e. fixed pt for Γ.

Suppose that we have the following 2 estimates:

① <u>Linear estimate</u>:

$$\|\Gamma u\|_{X^s([0,\delta))} \lesssim \|u_0\|_{H^s} + \||u|^{p-1}u\|_{N^s}$$

δ^θ

② <u>Nonlinear estimate</u>:

$$\||u|^{p-1}u\|_{N^s} \lesssim \|u\|_{X^s}^p$$

$$\Longrightarrow \quad \|\Gamma u\|_{X^s([0,\delta))} \le C_1 \|u_0\|_{H^s} + C_2 \delta^\theta \|u\|_{X^s([0,\delta))}^p$$

$\Rightarrow$ Γ is a contraction on $B_R \subset X^s([0,\delta))$

Choose $R = 2C_1 \|u_0\|_{H^s}$

- $\|\Gamma u\|_{X^s([0,\delta))} \le \frac{1}{2} R + C_2 \delta^\theta R^p \le R$ for $\delta = \delta(R) \ll 1$

  $(\Leftarrow C_2 \delta^\theta R^{p-1} \le \frac{1}{2})$

- $\|\Gamma u - \Gamma v\|_{X^s([0,\delta))} \le \frac{1}{2} \|u - v\|_{X^s([0,\delta))}$ for $\delta = \delta(R) \ll 1$

$\Rightarrow \exists ! \ u \in B_R$ s.t. $\Gamma_{u_0}(u) = u$.

- examples of X^s:

Strichartz space: $\| \langle \nabla \rangle^s u \|_{L^q_t L^r_x([0,\delta) \times M)}$

Strichartz estimate for Schrödinger eqn on $\mathbb{R}^d$:

$$\| S(t) u_0 \|_{L^q_t L^r_x} \lesssim \| u_0 \|_{L^2_x(\mathbb{R}^d)}$$

where $q, r \geq 2$ satisfies

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \qquad (q, r, d) \neq (2, \infty, 2)$$

(q, r) is (Schrödinger) admissible

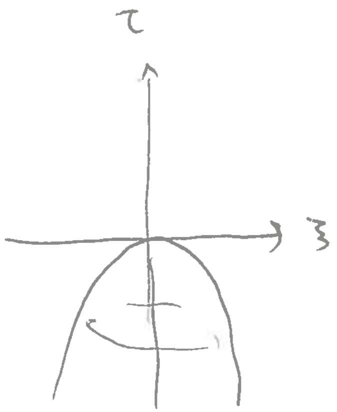
- $X^{s,b}$ - space (Fourier restriction norm method by Bourgain '93

$$\|u\|_{X^{s,b}} = \| \langle \xi \rangle^s \langle \tau + |\xi|^2 \rangle^b \widehat{u}(\tau, \xi) \|_{L^2_{\xi,\tau}}$$

linear Sch. eqn : $i \partial_t u + \Delta u = 0$

$$\underset{\text{space-time F.T.}}{\Longrightarrow} -(\tau + |\xi|^2) \widehat{u}(\tau, \xi) = 0.$$

$\Rightarrow$ $\widehat{u}(\tau, \xi) = \mathcal{F}_{t,x}(S(t)u_0)(\tau, \xi)$ is a measure supported on the paraboloid $\tau = -|\xi|^2$.



"measures" how far/close u is to being a linear soln.

When $b > 0$, $\langle \tau + |\xi|^2 \rangle^b$ penalizes functions away from linear solutions

$\Rightarrow$ suitable for study Γu in a perturbative manner.