Lec 4 18/01/17 (Wed)

Lemma 1.3: $\{\tilde{g}_n\}$, indep standard $\mathbb{R}$-valued Gaussian r.v.'s

Then, $$P\left[\left(\sum_{n=1}^{M} \tilde{g}_n^2\right)^{1/2} \geq R\right] \leq e^{-\frac{1}{4}R^2}, \quad R \geq 3M^{1/2}, \; M \geq 1$$

Pf: $0 < t < 1/2$

$$(\text{LHS}) \overset{\text{Chebyshev}}{\leq} e^{-tR^2} \underbrace{\mathbb{E}\left[e^{t\sum_{n=1}^{M}\tilde{g}_n^2}\right]}_{=\prod_{n=1}^{M}\mathbb{E}\left[e^{t\tilde{g}_n^2}\right]}$$

$$= (1-2t)^{-M/2} e^{-tR^2}$$

Choose $t = \frac{1}{2}\left(1 - \frac{M}{R^2}\right)$

$$(\text{LHS}) \leq \left(\frac{R^2}{M}\right)^{M/2} e^{-\frac{1}{2}R^2 + \frac{1}{2}M}$$

$$\leq e^{\frac{M}{2}\ln\frac{R^2}{M} + \left(\frac{1}{18} - \frac{1}{2}\right)R^2} \leq e^{-\frac{1}{4}R^2}$$

$\left(\ln x \leq \frac{x}{4} \text{ for } x \geq 9\right)$

$\boxtimes$

Pf of Prop 1.2 :

By Bernstein's ineq

$$\| P_{\le M_0} u \|_{L^{p+1}} \le c\, M_0^{\frac{1}{2}-\frac{1}{p+1}} \underbrace{\| P_{\le M_0} u \|_{L^2}}_{\le r}$$

$$\left(\begin{array}{l} \frac{s}{1} \ge \frac{1}{2} - \frac{1}{p+1} \\ (LHS) \underset{\text{Sobolev}}{\lesssim} \| \langle \partial x \rangle^{\frac{1}{2}-\frac{1}{p+1}} P_{\le M_0} u \|_{L^2} \\ \qquad \langle m \rangle^{\frac{1}{2}-\frac{1}{p}} \lesssim M_0^{\frac{1}{2}-\frac{1}{p}} \end{array} \right.$$

Given $\lambda > 1$, choose M_0 s.t.

$$\frac{1}{2} \lambda = c\, M_0^{\frac{1}{2}-\frac{1}{p+1}} r$$

i.e. $\quad M_0 \sim \left(\frac{\lambda}{r} \right)^{\frac{1}{\frac{1}{2}-\frac{1}{p+1}}}$

Let $q = p+1$.

Let $M_j = 2^j M_0$. Then, we have

$$\| P_{M_j} u \|_{L^q} \le c\, M_j^{\frac{1}{2}-\frac{1}{q}} \| P_{M_j} u \|_{L^2}$$

Let $\{\sigma_j\}$ s.t. $\sum_{j\ge 1} \sigma_j = \frac{1}{2}$

$$(\Leftarrow \text{ set } \sigma_j = C\, 2^{-\varepsilon j} = C M_0^{\varepsilon} M_j^{-\varepsilon}$$

$\Rightarrow$ By dyadic pigeon hole principle,

$$\rho_1 \left(\| u \|_{L^q} > \lambda ,\ \| u \|_{L^2} \le r \right)$$

Recall $\| P_{\le M_0} u \|_{L^q} \le \frac{1}{2} \lambda$

$$\le \sum_{j=1}^{\infty} \rho_1 \left(\| P_{M_j} u \|_{L^q} > \sigma_j \lambda \right)$$

$$\sum_{j=1}^{\infty} \sigma_j \lambda = \frac{1}{2}\lambda < \| P_{>M_0} u \|_{L^q} \le \sum_{j=1}^{\infty} \| P_{M_j} u \|_{L^q}$$

$$\le \sum_{j=1}^{\infty} \rho_1 \Big(\underbrace{\| P_{M_j} u \|_{L^2}} > c\, \sigma_j M_j^{\frac{1}{q}-\frac{1}{2}} \lambda \Big)$$

$$= \left(\sum_{|n| \sim M_j} \frac{|g_n|^2}{\langle n \rangle^2} \right)^{1/2} \sim M_j^{-1} \left(\sum_{|n|\sim M_j} |g_n|^2 \right)^{1/2}$$

$$P\Big(\Big(\sum_{|n|\sim M_j}|g_n(\omega)|^2\Big)^{1/2}\gtrsim \underbrace{\sigma_j M_j^{\frac{1}{q}+\frac{1}{2}}\lambda}\Big)$$

$$=: R_j \gtrsim M_0^{\varepsilon} M_j^{\frac{1}{2}+\frac{1}{q}-\varepsilon}$$

$$\gg M_j^{\frac{1}{2}+}$$

$\Rightarrow$ Apply Lemma 1.3.

$$\rho_1\left(\|u\|_{L^q}>\lambda,\ \|u\|_{L^2}\le r\right)$$

$$\lesssim \sum_{j=1}^{\infty} e^{-cR_j^2} \qquad R_j^2 \sim M_0^{\frac{2}{q}+1}\lambda^2\, \underline{\underline{2^{(\frac{2}{q}+1-2\varepsilon)j}}}$$

$$\sim e^{-c\lambda^2 M_0^{\frac{2}{q}+1}} \qquad M_0\sim\left(\frac{\lambda}{r}\right)^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}$$

$$\sim e^{-c\lambda^{\frac{4q}{q-2}} r^{-\frac{2q+4}{q-2}}} \qquad \frac{4q}{q-2}>q \text{ for } q<6.$$

$$\lesssim \begin{cases} e^{-c\lambda^{q+\delta}}, & q<6 \iff p<5. \\ e^{-C\lambda^6}, & q=6 \iff p=5 \end{cases}$$

with $C\gg 1$ by taking $r\ll 1$

$$\left(\begin{aligned} \|f\|_{L^p}^p &= \int |f(x)|^p \, dx \\ &= p \int_0^\infty \alpha^{p-1} \mu\left(|f(x)| > \alpha \right) d\alpha \end{aligned} \right.$$

(X, μ) meas space

$$\left\| \mathbb{1}_{\{\|u\|_{L^2} \le r\}} e^{\frac{1}{p+1} \int |u|^{p+1} dx} \right\|_{L^{\tilde{q}}(d\rho_1)}$$

$\lambda^{p+1} \sim \ln \alpha$

$$\sim 1 + \int_1^\infty e^{c \lambda^{p+1}} \begin{pmatrix} e^{-c \lambda^{p+1+\delta}} \\ \text{or} \\ e^{-c \lambda^{p+1}} \end{pmatrix} d\lambda$$

$p < 5$

$p = 5$

$\boxed{/}$

Rmk: We went over the harmonic analytic proof by Bourgain '94.

The proof by LRS '88 is more probabilistic.

$$R_r(u) = \mathbb{1}_{\{\|u\|_{L^2} \le r\}} e^{\frac{1}{p+1}\int |u|^{p+1}}$$

$$R_{N,r}(u) = \mathbb{1}_{\{\|P_{\le N} u\|_{L^2} \le r\}} e^{\frac{1}{p+1}\int |P_{\le N} u|^{p+1}}$$

Cor 1.4: $\forall q < \infty$

$$R_{N,r}(u) \longrightarrow R_r(u) \text{ in } L^q(d\rho_1)$$

as $N \to \infty$

In particular, with $d\mu_N = Z_N^{-1} R_{N,r}(u) d\rho_1$

μ_N converges "uniformly" to μ

i.e. $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t.

$$|\mu_N(A) - \mu(A)| < \varepsilon, \quad \forall \text{ measurable } A, \ \forall N \ge N_0$$

$\Leftarrow$ stronger than weak convergence:

$$\lim_{N\to\infty} \int f(u)\, d\mu_N(u) = \int f(u)\, d\mu(u)$$

for any $f \in C_b(H^\sigma(\mathbb{T}))$, $\sigma < \frac{1}{2}$

<u>Pf:</u> $\int |P_{\le N} u|^{p+1} \longrightarrow \int |u|^{p+1}$ a.s.

$\left(\Leftarrow \| P_{>N} u \|_{L^{p+1}} \lesssim \| P_{>N} u \|_{H^{\frac{1}{2}-}} \to 0 . \right.$

$\Rightarrow$ $R_{N,r}(u) \longrightarrow R_r(u)$ a.s.

$\Rightarrow$ By Egoroff's thm,

$R_{N,r}(u) \longrightarrow R_r(u)$ almost uniformly

$\Rightarrow$ in meas

· Given $\varepsilon > 0$,

let $A_{N,\varepsilon} = \{ u \in H^{\frac{1}{2}-}(\mathbb{T}) : |R_{N,r}(u) - R_r(u)| \le \frac{1}{2}\varepsilon \}$.

Then, $\rho_1(A^c_{N,\varepsilon}) \longrightarrow 0$ as $N \to \infty$

$$\| R_{N,r} - R_r \|_{L^q(d\rho_1)}$$

$$\leq \| (R_{N,r} - R_r) \mathbb{1}_{A_{N,\varepsilon}} \|_{L^q}$$

$$+ \| (R_{N,r} - R_r) \mathbb{1}_{A^c_{N,\varepsilon}} \|_{L^q}$$

$$\leq \frac{1}{2}\varepsilon + \underbrace{\left(\| R_{N,r} \|_{L^{2q}} + \| R_r \|_{L^{2q}} \right)}_{\leq C < \infty} \underbrace{\left\{ \rho_1 (A^c_{N,\varepsilon}) \right\}^{\frac{1}{2q}}}_{\to 0}$$

$$< \varepsilon$$

for $N \gg 1$. $\boxtimes$

· <u>Digression:</u> Proof of Fernique's integrability theorem

(H, B, ρ), abstract Wiener space

$$d\rho = Z^{-1} e^{-\frac{1}{2}\|x\|_H^2} dx$$

Define Wiener measure with variance t

$$d\rho_t = Z_t^{-1} e^{-\frac{\|x\|^2}{2t}} dx, \quad t > 0$$

· $\{\rho_t\}_{t>0}$ forms a <u>contraction semigroup</u> in the Banach space of bounded, unif conti functions on B. $\|\rho_t\| \le 1$

Riesz rep thm: X, LCH

$$(C_0(X))^* = M(X)$$

Radon meas

vanishing at ∞

$\forall \varepsilon > 0$. $\{x : |f(x)| \ge \varepsilon\}$ is cpt.

$$C_0(X) = \overline{C_c(X)}^{\|\cdot\|_{unif}}$$