Lec 3 16/01/17 (Mon)

H, Hilbert space

"$d\rho = Z^{-1} e^{-\frac{1}{2}\|u\|_H^2} du$"

$\Rightarrow$ enlarge H to B = completion of H under a measurable norm $\|\cdot\|_B$

ex: $H = H^s(\mathbb{T}^d)$

$B = H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$, $u^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx}$

$W^{\sigma,p}(\mathbb{T}^d)$ $\quad \|u\|_{W^{\sigma,p}} = \|\mathcal{F}^{-1}(\langle n\rangle^\sigma \hat{u}(n))\|_{L^p_x(\mathbb{T}^d)}$, $p \le \infty$

Besov space

$B^\sigma_{p,q}(\mathbb{T}^d)$, $p \le \infty$, $q < \infty$

$\|u\|_{B^\sigma_{p,q}} = \Big\| N^\sigma \|P_N u\|_{L^p_x(\mathbb{T}^d)} \Big\|_{\ell^q_k(\mathbb{Z}_{\ge 0})}$ $\quad \Big(P_N u \text{"="} \sum_{|n|\sim N} \hat{u}_n e^{in\cdot x}\Big)$

$\Downarrow$

$\frac{1}{2}N \le |n| \le 2N$

"N, dyadic" $\quad N = 2^k$, $k \ge 0$, $k \in \mathbb{Z}$

- $B^\sigma_{2,2} = H^\sigma$

- Fourier - Lebesgue space: $\mathcal{F}L^{\sigma,p}(\mathbb{T}^d)$

$$\|u\|_{\mathcal{F}L^{\sigma,p}(\mathbb{T}^d)} = \|\langle n\rangle^{\sigma}\hat{u}_n\|_{\ell^p_n(\mathbb{Z}^d)}$$

- $\mathcal{F}L^{\sigma,2} = H^{\sigma}$.

$$\underline{(\sigma - s)p < -d}$$

- $$\mathbb{E}\left[\|u^{\omega}\|^p_{\mathcal{F}L^{\sigma,p}}\right] = \mathbb{E}\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{p(\sigma-s)}|g_n|^p$$

$$\sim \sum_{n\in\mathbb{Z}^d}\langle n\rangle^{p(\sigma-s)} < \infty \quad \text{iff } (\sigma-s)p < -d.$$

- <u>Lemma 1.1</u> (Tail estimate)

Let $\sigma < s - \frac{d}{2}$. Then,

$$\rho_s\left(\|u\|_{H^{\sigma}} > K\right) \le C e^{-cK^2}$$

for all $K > 0$.

<u>Rmk</u>: This follows from <u>Fernique's integrability theorem</u>:

$$\int_B e^{c\|u\|_B^2}\, d\rho(u) < \infty \quad \text{for some } c > 0.$$

$$\Leftrightarrow \quad \rho\left(\|u\|_B > K\right) \le c\, e^{-cK^2}$$

Pf of Lemma 1.1: By Chebyshev's ineq

$$e^{cK^2} \rho_s\left(B_K^c\right) \le \int_{H^\sigma} e^{c\|u\|_{H^\sigma}^2} d\rho_s(u)$$

$\sum_n \langle n \rangle^{2\sigma} |\hat{u}_n|^2$

$$\Leftarrow \rho_s\left(\|u\|_{H^\sigma} > K\right) = \rho_s\left(e^{c\|u\|_{H^\sigma}^2} > e^{cK^2}\right)$$

$$= \int_{H^\sigma} \mathbb{1}_{\frac{e^{c\|u\|_{H^\sigma}^2}}{e^{cK^2}} > 1}\, d\rho_s(u)$$

$$< \frac{1}{e^{cK^2}} (RHS)$$

(usual Chebyshev: $P(f(\omega) > K) \le \frac{\mathbb{E}\left[|f|^2\right]}{K^2}$)

$$= \prod_{n \in \mathbb{Z}^d} \int_{\mathbb{C}} e^{c\langle n \rangle^{2\sigma - 2s} |g_n|^2} e^{-\frac{1}{2}|g_n|^2} \frac{dg_n}{2\pi}$$

$$= \prod_{n \in \mathbb{Z}^d} \frac{1}{1 - 2c\langle n \rangle^{2\sigma - 2s}}$$

$$\mathbb{E}\left[e^{aX^2} \right] = \int e^{ax^2} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = \frac{1}{\sqrt{1-2a}}, \quad a < 1/2$$ ④

$X \sim \underbrace{N_{\mathbb{R}}(0,1)}$

real-valued Gaussian r.v.

mean 0, var 1.

$y = \sqrt{1-2a}\, x$

$$= \prod_{n \in \mathbb{Z}^d} \left(1 + \frac{2c \langle n \rangle^{2\sigma - 2s}}{1 - 2c \langle n \rangle^{2\sigma - 2s}} \right) < \infty$$

iff $\sigma < s - \frac{d}{2}$

($a_n > 0$. Then $\prod (1 + a_n) < \infty \iff \sum a_n < \infty$

$\boxtimes$

Construction of Gibbs measure on $\mathbb{T}$

<u>$d=1$</u>: Recall

$$d\mu = Z^{-1} e^{-H(u) - \frac{1}{2}M(u)} du$$

$$H(u) = \frac{1}{2}\int |\partial_x u|^2 \pm \frac{1}{p+1}\int |u|^{p+1}$$

$$M(u) = \int |u|^2$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}} \underbrace{e^{-\frac{1}{2}\|u\|_{H^1}^2} du}_{d\rho_1}$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}} d\rho_1 \quad \text{on } H^\sigma(\mathbb{T}),\ \sigma < 1 - \frac{1}{2} = \frac{1}{2}$$

- <u>Defocusing case</u>: By Sobolev ineq,

$$\int |u|^{p+1} = \|u\|_{L^{p+1}}^{p+1} \lesssim \|u\|_{H^{\frac{1}{2}-}}^{p+1} < \infty, \text{ a.s.}$$

$$\Rightarrow \quad 0 < e^{-\frac{1}{p+1}\int |u|^{p+1}} \leq 1, \text{ a.s.}$$

$\Rightarrow$ μ is a prob. meas on $H^\sigma(\mathbb{T})$, $\sigma < \frac{1}{2}$.

Sobolev ineq: On $\mathbb{R}^d$ or $\mathbb{T}^d$.

- $\mathbb{R}^d$: $\quad \|u\|_{L^q(\mathbb{R}^d)} \lesssim \|u\|_{\dot{W}^{s,p}(\mathbb{R}^d)}$

  homogeneous Sobolev space
  use $|\xi|^s$ instead of $\langle \xi \rangle^s$

  when $\quad \dfrac{s}{d} = \dfrac{1}{p} - \dfrac{1}{q}$

- $\mathbb{T}^d$; $\quad \|u\|_{L^q(\mathbb{T}^d)} \lesssim \|u\|_{W^{s,p}(\mathbb{T}^d)}$

  when $\quad \dfrac{s}{d} \geq \dfrac{1}{p} - \dfrac{1}{q}$.

ex: $d=1$ $\quad \|u\|_{L^q(\mathbb{T})} \lesssim \|u\|_{H^{\frac{1}{2}-\varepsilon}(\mathbb{T})}$

by choosing $\varepsilon = \varepsilon(q) > 0$ suff small.

- Focusing case: Let $p > 1$.

$$\int_{\mathbb{T}} |u|^{p+1} \geq \|u\|_{L^2}^{p+1} = \left(\sum |\hat{u}_n|^2\right)^{\frac{p+1}{2}}$$

$$\geq \sum \left|\frac{g_n}{\langle n\rangle}\right|^{p+1} \qquad \ell^2 \subset \ell^{p+1}$$

$$\Rightarrow \mathbb{E}_\rho\left[e^{\frac{1}{p+1}\int |u|^{p+1}}\right] \geq \prod_{n\in\mathbb{Z}} \mathbb{E}\left[e^{\left|\frac{g_n}{\langle n\rangle}\right|^{p+1}}\right] = \infty \quad \text{b/c } p+1 > 2.$$

$\Rightarrow$ can not construct μ as it is.

Idea: Mass is conserved

$\Rightarrow$ Introduce a mass cutoff.

$$\text{"} d\mu = Z^{-1} \mathbb{1}_{\{\|u\|_{L^2} \leq r\}} \underbrace{e^{\frac{1}{p+1}\int |u|^{p+1}} d\rho_1}_{= e^{-H(u) - \frac{1}{2}M(u)} du} \text{"}.$$

$p > 1$

Prop 1.2 (Lebowitz – Rose – Speer '88, Bourgain '94)

(i) $p < 5$

(*) $$R(u) = R_r(u) := \mathbb{1}_{\{\|u\|_{L^2} \le r\}} \, e^{\frac{1}{p+1}\int |u|^{p+1}} \in L^q(d\rho_1)$$

$\forall\, r, q < \infty$.

(ii) $p = 5$: (*) holds if r is sufficiently small.

In fact, $r < \|Q\|_{L^2(\mathbb{R})}$

ground state on $\mathbb{R}$

$p = 5$: mass-critical NLS $\quad i\partial_t u + \partial_x^2 u = -|u|^4 u$ on $\mathbb{T}$

$\Leftarrow$ lowest power for which we have a finite time blowup solution.

$M(Q) \leftarrow$ smallest mass for a finite time blowup solution