Lec 20 15 / 03 / 17 (Wed) 

## Sec 4.2: Local well-posedness of SNLS

$$(\text{SNLS}) \begin{cases} i\,du = (-\Delta u \pm |u|^2 u)\,dt + \phi\, dW & \text{on } \mathbb{T} \\ u|_{t=0} = u_0 \end{cases}$$

$\phi = \operatorname{diag} \phi_n$, HS on $L^2(\mathbb{T})$

$$\phi(e_n) = \phi_n e_n$$

Mild formulation: $S(t) = e^{it\Delta}$

$$u(t) = S(t)u_0 \mp i \int_0^t S(t-t')|u|^2 u(t')\,dt'$$

$$- \underbrace{i \int_0^t S(t-t')\,\phi\, dW(t')}_{\text{stochastic convolution } \Psi}$$

## Review of deterministic LWP of cubic NLS on $\mathbb{T}$

(Bourgain '93)

### Fourier restriction norm method

$$\|u\|_{X^{s,b}} = \| \langle n \rangle^s \langle \tau + n^2 \rangle^b \hat{u}(n,\tau) \|_{\ell^2_n L^2_\tau}$$

(lin Schrödinger: $i\partial_t u + \Delta u = 0$

$\underset{\text{F.T in spacetime}}{\Longrightarrow}$ $-(\tau + n^2)\,\hat{u}(n,\tau) = 0$

### local-in-time version: $T > 0$

$$\|u\|_{X^{s,b}_T} = \inf\{ \|v\|_{X^{s,b}} : v|_{[0,T]} = u \}.$$

### Duhamel formula:

$$u(t) = \Gamma_{u_0}(u)(t) = \eta(t) S(t) u_0 \mp i\, \eta\!\left(\tfrac{t}{T}\right) \int_0^t S(t-t') |u|^2 u(t')\,dt'.$$

$$\eta(t) \equiv \begin{cases} 1 & \text{on } [0,1] \\ 0 & \text{on } [-\frac{1}{2}, \frac{3}{2}]^c \end{cases}$$

$\Rightarrow$ If $u = \Gamma_{u_0}(u)$, then $u$ is a soln on $[0,T]$, $0 < T \le 1$.

## · Deterministic estimates

① homog lin $\| \eta(t) S(t) u_0 \|_{X^{s,b}} \leq C_b \| u_0 \|_{H^s}$.

$$\mathcal{F}_{x,t} ( \eta(t) S(t) u_0 ) = \hat{\eta} ( \tau + n^2 ) \hat{u}_0(n)$$

② nonhomog lin :

$$\left\| \eta\left(\tfrac{t}{T}\right) \int_0^t S(t-t') F(t') dt' \right\|_{X^{s,b}} \lesssim T^{\theta} \| F \|_{X^{s,b-1+\theta}}$$

· $b > 1/2$, $\theta > 0$ small.

③ $L^4$ - Strichatz estimate

$$\| u \|_{L^4_{x,t}(\mathbb{T} \times \mathbb{R})} \lesssim \| u \|_{X^{0,3/8}}$$

④ $\| u \|_{C_T H^s} \lesssim \| u \|_{X^{s,b}_T}$, $b > 1/2$.

($\Leftarrow$ 1-d Sobolev in time.)

$$\Rightarrow \|\Gamma_{u_0} u\|_{X_T^{0,\frac{1}{2}+}}$$

$$\le \|\eta(t)S(t)u_0\|_{X^{0,\frac{1}{2}+}} + \left\|\eta\left(\tfrac{t}{T}\right)\int_0^t S(t-t')|v|^2v(t')\,dt'\right\|_{X^{0,\frac{1}{2}+}}$$

( for any extension $v$ of $u$ )

$$\overset{①,②}{\lesssim} \|u_0\|_{L^2} + T^\theta \underbrace{\||v|^2v\|_{X^{0,-\frac{1}{2}+\theta+}}}$$

$$(X^{s,b})^* = X^{-s,-b}$$

$$= \sup_{\|w\|_{X^{0,\frac{1}{2}-\theta-}}=1} \underbrace{\left|\iint w\cdot|v|^2v\,dx\,dt\right|}$$

$$\le \|w\|_{L^4_{x,t}}\|v\|^3_{L^4_{x,t}}$$

$$\overset{③}{\lesssim} \|w\|_{X^{0,\frac{3}{8}}}\|v\|^3_{X^{0,3/8}}$$

$$\le \|v\|^3_{X^{0,3/8}}$$

Taking inf over $v$.

$$\Longrightarrow \|\Gamma_{u_0}u\|_{X_T^{0,1/2+}} \lesssim \|u_0\|_{L^2} + T^\theta\|u\|_{X_T^{0,1/2+}}.$$

- A difference estimate holds in a similar manner

$\Longrightarrow$ contraction in $B_R \subset X_T^{0,\frac{1}{2}+}$ $\qquad R = C\|u_0\|_{L^2}$

$0 < T = T(\|u_0\|_{L^2}) \ll 1$

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Back to SNLS.

- method 1: Show

$$\mathbb{E}\left[\|\Psi^\omega\|^2_{X_T^{0,\frac{1}{2}-}}\right] \lesssim 1 < \infty.$$

$\Longrightarrow$ fixed pt argument in $X_T^{0,\frac{1}{2}-}$

$$R^\omega \sim \|u_0\|_{L^2} + \|\Psi^\omega\|_{X_T^{0,\frac{1}{2}-}}$$

$$\Longrightarrow T = T^\omega > 0.$$

- Then, show the continuity in time.

$$u(t) = \underset{\text{conti}}{\text{lin}} + \underbrace{\text{nonlin}}_{\substack{\text{estimate} \\ \text{in } X_T^{0,\frac{1}{2}+}}} + \Psi$$

$\uparrow$ Kolmogorov conti criterion.

- method 2: write $u = \Psi + v$.

Need to estimate $\|\Psi\|_{L^q_T L^r_x}$

$p \geq q \vee r$

$$\Big[\mathbb{E}\big(\|\Psi\|^p_{L^q_T L^r_x}\big)\Big]^{1/p}$$

$\mathcal{H}^1$ = homog Wiener chaoses of order 1

$$\overset{\text{Minkowski}}{\leq} \Big\| \, \|\Psi(x,t)\|_{L^p(\Omega)} \Big\|_{L^q_T L^r_x}$$

$$\overset{\text{Wiener chaos est}}{\lesssim} p^{1/2} \Big\| \underbrace{\|\Psi(x,t)\|_{L^2(\Omega)}}_{\overset{\text{indep}}{=} \Big(\sum_n \underbrace{\mathbb{E}\Big[\Big|\int_0^t e^{-in^2(t-t')}\phi_n e^{inx} d\beta_n(t')\Big|^2\Big]}_{\overset{\text{Ito isometry}}{=} 2|\phi_n|^2 t}\Big)^{1/2}} \Big\|_{L^q_T L^r_x}$$

$$\lesssim p^{1/2} T^{\frac{1}{2}+\frac{1}{q}} \underbrace{\Big(\sum |\phi_n|^2\Big)^{1/2}}_{= \|\phi\|_{HS(L^2:L^2)}} \Rightarrow \text{chebyshev.}$$

$\Big( \|\phi\|_{HS(X:Y)} = \big(\sum \|\phi(e_n)\|^2_Y\big)^{1/2}$. $\{e_n\}$ ONB in $X$

· Fixed pt problem for $v$:

$$v(t) = \eta(t) S(t) u_0 \mp i\, \eta\!\left(\tfrac{t}{T}\right) \int_0^t S(t-t') \underbrace{|v+\Psi|^2(v+\Psi)(t')}_{\uparrow\ \mathbb{1}_{[0,T]}} dt'.$$

$\Rightarrow$ contraction in $X_T^{0,\frac{1}{2}+}$ (a bit smoother)

only change:

$$\|v+\Psi\|_{L^4_{x,T}} \le \|v\|_{L^4_{x,t}} + \underbrace{\|\Psi^\omega\|_{L^4_{x,T}}}_{<\infty,\ a.s.}$$

Thm 4.2: Stochastic cubic NLS on $\mathbb{T}$ (with $\phi \in HS(L^2; L^2)$) is pathwise locally well-posed in $L^2(\mathbb{T})$

# Sec 4.3 : Global well-posedness in $L^2(\mathbb{T})$



**Idea:** Control $M(u) = \sum_{n\in\mathbb{Z}} |\hat{u}_n|^2 = \sum p_n^2 + q_n^2$

$p_n = \mathrm{Re}\, \hat{u}_n$

$q_n = \mathrm{Im}\, \hat{u}_n$.

$$d\hat{u}_n = \left(-in^2 \hat{u}_n - i\,\mathcal{F}_x(|u|^2u)(n)\right)dt - i\,\phi_n\, d\beta_n$$

$$\Rightarrow \quad dp_n = \left(n^2 q_n + \mathrm{Im}\,\mathcal{F}_x(|u|^2u)(n)\right)dt + \mathrm{Im}(\phi_n\, d\beta_n)$$

$$dq_n = \left(-n^2 p_n - \mathrm{Re}\,\mathcal{F}_x(|u|^2u)(n)\right)dt \boxed{- \mathrm{Re}(\phi_n\, d\beta_n)}$$

$$\mathrm{Im}\,\phi_n\, d(\mathrm{Re}\,\beta_n) + \mathrm{Re}\,\phi_n\, d(\mathrm{Im}\,\beta_n)$$

$$-\mathrm{Re}\,\phi_n\, d(\mathrm{Re}\,\beta_n) + \mathrm{Im}\,\phi_n\, d(\mathrm{Im}\,\beta_n)$$

· <u>Ito's lemma</u>: $dX = f\, dt + g\, dB$

Consider $F(X)$

Then, $dF = \partial_x F\, dX + \frac{1}{2} \partial_x^2 F (dX)^2$

$= \partial_x F (f\, dt + g\, dB) + \frac{1}{2} \partial_x^2 F \cdot g^2 dt.$

$(dt^2) = 0$

$dt\, dB = dB\, dt = 0$

$(dB)^2 = dt$

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$$\Rightarrow dM \overset{\text{Ito}}{=} \underbrace{2 \sum_n (p_n\, dp_n + q_n\, dq_n)}_{= 2 \sum_n p_n \operatorname{Im}(\Phi_n d\beta_n) - q_n \operatorname{Re}(\Phi_n d\beta_n)} + \underbrace{\sum \left( (dp_n)^2 + (dq_n)^2 \right)}_{= 2 \sum |\Phi_n|^2 dt = 2 \|\Phi\|^2_{HS(L^2; L^2)} dt}$$

Integrate from 0 to $t$ (large), sup over $t \in [0, T]$, & take expecatation

$\Longrightarrow$ Apply Burkholder - Davis - Gundy ineq.

$$\mathbb{E}\left[ \sup_{t \in [0,T]} \|u(t)\|^2_{L^2} \right] \leq \|u_0\|^2_{L^2} + C(T, \|\Phi\|_{HS(L^2; L^2)}).$$

$\Longrightarrow$ a.s. GWP in $L^2(\mathbb{T})$ by iterating the local argument.

B-D-G ineq: $X$, local martingale

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|X(t)|^p\Big) \sim \mathbb{E}\Big[\langle X(t)\rangle_{[0,T]}^{p/2}\Big], \quad 1\le p<\infty$$

quadradic variation

Rmk: In order to justify the application of Ito's lemma, one should first consider the finite dim'l dynamics:

$$\begin{cases} i\,du^N = \left(-\Delta u^N \pm P_{\le N}(|u^N|^2 u^N)\right)dt + P_{\le N}(\phi\, dW) \\ u^N|_{t=0} = u_0^N := P_{\le N} u_0 \end{cases}$$

and obtain an a priori bound (indep of $N$)

Then, proceed with an approximation argument to obtain the same a priori bound for solutions to SNLS.

- Another alternative is to apply the infinite dim'l version of Ito's lemma (such as the one in Da Prato-Zabczyk.)