Lec 2 11/01/17 (Wed)

- Finite dim'l Hamiltonian dynamics on $\mathbb{R}^{2n}$

$$\partial_t p_j = \frac{\partial H}{\partial q_j}, \quad \partial_t q_j = -\frac{\partial H}{\partial p_j}, \quad j = 1, \cdots, n$$

vec. field $= X$, $\quad X_j = \frac{\partial H}{\partial q_j}$

$$X_{n+j} = -\frac{\partial H}{\partial p_j}$$

① · Liouville's thm

$$\frac{d}{dt} \mathrm{vol} = \mathrm{div}\, X = \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} X_j + \frac{\partial}{\partial q_j} X_{n+j} \right]$$

$$= \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} \frac{\partial H}{\partial q_j} + \frac{\partial}{\partial q_j} \left(-\frac{\partial H}{\partial p_j} \right) \right] = 0.$$

$\Rightarrow$ $dp\,dq = \prod_{j=1}^{n} dp_j\, dq_j$ is <u>invariant</u>.

② Hamiltonian H is <u>conserved</u>

$$\frac{d}{dt} H(p(t), q(t)) = \frac{\partial H}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial H}{\partial q} \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial H}{\partial q} \left(-\frac{\partial H}{\partial p} \right) = 0$$

Gibbs meas: $d\mu = Z_\beta^{-1} e^{-\beta H(p,q)} dp\,dq$, $\beta > 0$ ← inverse temp

$$Z_\beta = \int_{\mathbb{R}^{2n}} e^{-\beta H(p,q)} dp\,dq < \infty$$

$=$ partition function

is invariant.

Invariance: $\Phi(t) : (p(0), q(0)) \longmapsto (p(t), q(t))$

$$\mu(\Phi(-t)A) = \mu(\{(p_0, q_0) \in \Phi(-t)A\})$$

$$= \mu(\{\Phi(t)(p_0, q_0) \in A\})$$

$$= Z_\beta^{-1} \int_A e^{-\beta \underbrace{H(p(t), q(t))}_{= H(p_0, q_0)}} \underbrace{dp(t)\,dq(t)}_{= dp_0\,dq_0}$$

$dp(t)\,dq(t) = \Phi(t)_*(dp\,dq)$

$$= \mu(A)$$

Rmk: Suppose that F is conserved under the dynamics

$$\Rightarrow \quad d\mu_F = Z^{-1} e^{-F(p,q)} \, dp \, dq$$

is invariant (for nice F).

Q: Why do we care about invariant meas?

① can view the system as a dynamical system with meas-preserving transf $T := \Phi(1)$

$$T : (p(0), q(0)) \mapsto (p(t), q(t)) \big|_{t=1}$$

- Poincaré recurrence thm:

(i) $\forall$ meas A with $\mu(A) > 0$.

$\exists N \in \mathbb{N}$ $\mu(A \cap T^{-N} A) > 0$.

(ii) a.e. $(p,q) \in \mathbb{R}^{2n}$ are stable according to Poisson. $\Leftarrow$ Zhidkov '01 Lec. Note in Math

i.e. $\exists \{t_j\}_{j \in \mathbb{N}} \to \infty$

s.t. $(p(t_j), q(t_j)) \longrightarrow (p(0), q(0))$

Also, <u>Furstenberg's multiple reccurence thm</u>

$\forall A$ with $\mu(A) > 0$ and $k \in \mathbb{N}$.

$\exists N \in \mathbb{N}$ s.t.

$$\mu(A \cap T^{-N} A \cap T^{-2N} A \cap \cdots \cap T^{-kN} A) > 0.$$

② μ is "supposed" to describe a "typical" long time behavior of solutions.

(NLS) $i\partial_t u + \Delta u = \pm |u|^{p-1} u$ on $\mathbb{T}^d$.

$$H(u) = \frac{1}{2}\int |\nabla u|^2 \pm \frac{1}{p+1}\int |u|^{p+1}$$

(NLS) $\iff \partial_t u = -i \dfrac{\partial H}{\partial \bar{u}}$.

Symplectic space $L^2(\mathbb{T}^d)$

Symplectic form $\omega(f,g) = \operatorname{Im} \int_{\mathbb{T}^d} f(x)\overline{g(x)}\,dx$

$$dH\big|_u(\phi) = \omega\left(\phi, -i\frac{\partial H}{\partial \bar{u}}\right)$$

Gâteaux derivative $= \dfrac{d}{dt} H(u + \varepsilon\phi)\Big|_{\varepsilon = 0}$

On Fourier side:

$$p_n = \operatorname{Re} \hat{u}_n \ , \qquad q_n = \operatorname{Im} \hat{u}_n$$

Linear case: $\displaystyle H(p,q) \underset{\substack{\uparrow \\ \text{Plancherel}}}{=} \frac{1}{2} \sum |n|^2 (p_n^2 + q_n^2)$

Plancherel: $\displaystyle \int_{\mathbb{T}^d} |\nabla u|^2 dx = \sum_{n \in \mathbb{Z}^d} |n|^2 |\hat{u}_n|^2$

Lin Schrödinger eqn: $\partial_t u = i \Delta u$

$$\Longleftrightarrow \quad \partial_t \hat{u}_n = -i |n|^2 \hat{u}_n , \qquad n \in \mathbb{Z}^d$$

$$\Longleftrightarrow \quad \partial_t \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} \dfrac{\partial H}{\partial q_n} \\ -\dfrac{\partial H}{\partial p_n} \end{pmatrix}$$

(6)

<u>Gibbs meas</u> $\quad d\mu \text{ "="} Z^{-1} e^{-H(u)} du$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}} \underbrace{e^{-\frac{1}{2}\int |\nabla u|^2} du}_{\text{Wiener meas on } \mathbb{T}^d}.$$

In fact, we consider

$$d\mu = Z^{-1} e^{-H(u) - \frac{1}{2}M(u)} du, \quad M(u) = \int |u|^2 dx$$

in order to avoid a problem at $n=0$.

$$e^{\mp \frac{1}{p+1}\int |u|^{p+1}} e^{-\frac{1}{2}\|u\|_{H^1}^2} du$$

· <u>Gaussian measures on periodic functions / distributions</u>

Consider $\quad d\rho_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du, \quad s \in \mathbb{R} \quad \mathcal{D}'(\mathbb{T}^d)$

$$\left(\text{on } \mathbb{R}^n, \quad d\rho = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx \right)$$

as a limit of

$$d\rho_{s,N} = Z_N^{-1} e^{-\frac{1}{2}\|P_{\leq N} u\|_{H^s}^2} d(P_{\leq N} u)$$

$$\left(P_{\leq N} u = \sum_{|n| \leq N} \hat{u}_n e^{in\cdot x} \right.$$

$$= Z_N^{-1} e^{-\frac{1}{2}\sum_{|n|\leq N} \langle n \rangle^{2s} |\hat{u}_n|^2} \prod_{|n|\leq N} \underbrace{d\hat{u}_n}$$

Lebesgue meas on $\mathbb{C} \cong \mathbb{R}^2$.

$$\langle n \rangle = (1+|n|^2)^{1/2}$$

$$= Z_N^{-1} \prod_{|n|\leq N} \underbrace{e^{-\frac{1}{2}\langle n \rangle^{2s} |\hat{u}_n|^2} d\hat{u}_n}$$

$$e^{-\frac{1}{2}(x^2+y^2)} dxdy$$

Gaussian meas on $\mathbb{C}$

mean 0, var $= 2\langle n \rangle^{-2s}$

$$c_n e^{-\frac{1}{2}|g_n|^2} dg_n$$

$$g_n = g_n(\omega) : \omega \in \Omega \longmapsto g_n(\omega) \in \mathbb{C}$$

$\mathbb{C}$-valued Gaussian random variable.

$$\Rightarrow \quad \hat{u}_n = \frac{g_n(\omega)}{\langle n \rangle^s}, \quad |n| \le N$$

$\{g_n\}_{|n|\le N}$ = seq of indep standard $\mathbb{C}$-valued Gaussian r.v.'s.

$$\text{mean} = \mathbb{E}[g_n] = \int_\Omega g_n(\omega)\, dP(\omega) = 0$$

$$\text{variance} = \mathbb{E}\big[\,|g_n - \underbrace{\mathbb{E}(g_n)}_{=0}|^2\big] = \int_\Omega |g_n|^2 dP = 2.$$

$$= \int_\Omega (\mathrm{Re}\, g_n)^2 dP + \int_\Omega (\mathrm{Im}\, g_n)^2 dP$$

$$= 1 + 1$$

$$e^{-\frac{1}{2}|g_n|^2} dg_n = e^{-\frac{1}{2}(\mathrm{Re}\, g_n)^2} d(\mathrm{Re}\, g_n) \times e^{-\frac{1}{2}(\mathrm{Im}\, g_n)^2} d(\mathrm{Im}\, g_n)$$

Take $N \to \infty$.

Problem: We can not take a limit as $N \to \infty$ in $H^s(\mathbb{T}^d)$.

$$\rho_{s,N} \sim u^N = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^s} e^{in\cdot x}$$

Let $\sigma \in \mathbb{R}$, $M > N$

$$\mathbb{E}\left[\| u^M - u^N \|^2_{H^\sigma(\mathbb{T}^d)} \right]$$

$$= \mathbb{E} \sum_{N < |n| \leq M} \frac{|g_n|^2}{\langle n \rangle^{2s - 2\sigma}} = 2 \sum_{N < |n| \leq M} \frac{1}{\langle n \rangle^{2s-2\sigma}} \to 0$$

iff $2s - 2\sigma > d$

$$\iff \boxed{\sigma < s - \frac{d}{2}}$$

Moral: topology matters!

Under $\sigma < s - \frac{d}{2}$,

u^N converges in $L^2(\Omega ; H^\sigma(\mathbb{T}^d))$.

to $u = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle^s} e^{in\cdot x}$.

- $d\rho_s = Z^{-1} e^{-\frac{1}{2}\|u\|^2_{H^s}} du$ is the induced probability meas under the map

$$\omega \in \Omega \xrightarrow{u} u = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle^s} e^{in\cdot x} \in H^\sigma(\mathbb{T}^d)$$

P

$$\rho_s = P \circ u^{-1}$$

Abstract Wiener space: Gross '65, Kuo '75 Lec. Notes in Math

$H = \infty$-dim'l separable Hilbert space

Consider "$d\rho = Z^{-1} e^{-\frac{1}{2}\|u\|^2_H} du$"

$\Leftarrow$ NOT countably additive if $\dim H = \infty$

- Must enlarge the space $H \subset B$ ← Banach space

dense & conti.

<u>Def</u>: We say (H, B, ρ) is an abstract Wiener space

if
$$\int_B e^{i \langle u, \varphi \rangle_{B, B^*}} \, d\rho(u) = e^{-\frac{1}{2} \|\varphi\|_{H^*}^2}$$

for all $\varphi \in B^* \subset H^* = H$

(i.e. $\langle u, \varphi \rangle$ is a stand. Gaussian r.v. $\forall \varphi$.

u ↑ random

<u>Another equiv def</u>: $B =$ completion of H under a "measurable" seminorm $\|\cdot\|_B$

$\forall \varepsilon > 0 \ \exists P_\varepsilon \in \{$ collection of finite dim'l proj$\}$

s.t. $\rho(\|Pu\|_B > \varepsilon) < \varepsilon \quad \forall \underset{\Psi}{P}$ s.t. $P \perp P_\varepsilon$.

e.g. H^s,

$$P_\varepsilon \sim P_{\leq N}$$

$$P^\perp P_\varepsilon \sim P_{N < \cdot \leq M}$$

Want $\rho(\|P_{>N} u\|_B > \varepsilon) < \varepsilon$.

ex: $B = W^{\sigma, p}$, $\sigma < s - \frac{d}{2}$