Lec 19 13/03/17 (Mon)

Back to L^2-cylindrical Wiener process

$$W(t) = \sum_{n \in \mathbb{Z}^d} \beta_n(t) e^{inx} \in H^{-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$$

$$W^{-\frac{d}{2}-\varepsilon, \infty}(\mathbb{T}^d)$$

$$W \in W^{\frac{1}{2}-, \infty}_{t, loc} W^{-\frac{d}{2}-\varepsilon, \infty}_x (\mathbb{T}^d)$$

- (SNLS) $\Leftarrow$ study it in the mild formulation:

$$u(t) = S(t) u_0 \mp i \int_0^t S(t-t') |u|^{p-1} u \, dt'$$

$$- i \underbrace{\int_0^t S(t-t') \phi \, dW(t')}$$

$$= \text{stochastic convolution} = \Psi$$

$$\Psi = \int_0^t S(t-t') \phi \, dW(t') = \sum_{n \in \mathbb{Z}^d} \int_0^t e^{-in^2(t-t')} \phi(e^{in \cdot x}) \, d\beta_n(t')$$

For simplicity, we only consider $\phi = \text{diag}(\phi_n)$ i.e. $\phi(e_n) = \phi_n e_n$

translation invariant.

· Wiener integral (real-valued case)

$$\int_a^b f(t)\, dB(t) \sim \langle f, dB \rangle_{L^2_t(a,b)}$$

f ← deterministic

Step 1: Step func. $f(t) = \sum_{j=1}^{n} a_{j-1} \mathbb{1}_{[t_{j-1}, t_j)}(t)$

$\Rightarrow$ Define $I(f) = \sum_{j=1}^{n} a_{j-1} \left(B(t_j) - B(t_{j-1}) \right)$

(left endpt Riemann sum)

Then, ① $\mathbb{E}[I(f)] = 0$

② $\mathbb{E}\left[(I(f))^2 \right] \underset{\text{indep}}{=} \sum_{j=1}^{n} a_{j-1}^2 (t_j - t_{j-1})$

$$= \int_a^b f^2(t)\, dt$$

$$= \|f\|^2_{L^2(a,b)}$$

<u>step2</u>: Given $f \in L^2(a,b)$,

approximate f by step functions f_n in $L^2(a,b)$

and define $I(f) = \lim_{n\to\infty} I(f_n)$ ← Gaussian

$\Longrightarrow$ ① & ② hold.

$\mathcal{H}' \cap$ — Wiener homog. chaoses of order 1

Namely, $I : L^2(a,b) \longrightarrow L^2(\Omega)$ is an isometry.

(just like the white noise functional in Chap 3.)

<u>Rmk</u>: If $f \in C'$, we can define it as

Paley-Wiener-Zygmund integral:

$$I(f) = \int_a^b f\, dB = -\int_a^b f'(t) B(t)\, dt \quad (\text{if } f(b)=0)$$

<u>Rmk</u>: If B is $\mathbb{C}$-valued,

$$\mathbb{E}\left[(I(f))^2\right] = 2\, \|f\|^2_{L^2([a,b])}$$

- **Ito integral:**

  filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (σ-field) s.t. $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \subset \mathcal{F}$, $t_1 \leq t_2$

- We say $X(t)$ is adapted (non-anticipating) if $X(t)$ is $\mathcal{F}_t$-measurable, $\forall t \geq 0$.

  - progressively meas: $\forall T$,

    $$[0,T]\times\Omega \ni (t,\omega) \longmapsto X(t,\omega) \text{ is } \mathcal{B}_{[0,T]}\otimes\mathcal{F}_t\text{-meas}$$

  - p.m. $\rightarrow$ adapted
  - adapted & left (or right) conti $\rightarrow$ p.m.

    e.g. adapted & càdlàg func $\rightarrow$ p.m.

    càdlàg: continue à droite limite à gauche

"Assume" (i) $B(t)$ is $\mathcal{F}_t$ - meas

(ii) $B(t) - B(s)$ is indep of $\{\mathcal{F}_s\}_{s<t}$.

e.g. simply take $\mathcal{F}_t^B = \sigma(\{B(s) : s \le t\})$

- We set

$$L^2_{ad}([a,b] \times \Omega) = \{ f(t,\omega) :$$

① f is adapted to $\{\mathcal{F}_t\}$

② $\int_a^b \mathbb{E}(f^2(t))\, dt < \infty \}$

Define the Ito integral here.

<u>Rmk</u>: can define Ito integral on a larger class of stochastic processes.

e.g. $\mathcal{L}_{ad}(\Omega ; L^2([a,b]))$

① f, adapted

② $\int_a^b |f(t,\omega)|^2 dt < \infty$, a.s.

Step 1: Step stoch process

"does not peek in the future"

$$f(t,\omega) = \sum_{j=1}^{n} a_{j-1}(\omega)\, \mathbb{1}_{[t_{j-1}, t_j)}(t)$$

$$a_j, \ \mathcal{F}_{t_j}\text{-meas}$$

$$\sum \mathbb{E}(a_j^2) < \infty$$

· Define Ito integral:

$$I(f)(\omega) = \sum_{j=1}^{n} a_{j-1}(\omega)\,\big(B(t_j) - B(t_{j-1})\big)$$

(left endpt Riemann sum)

Lemma 4.1: ① $\mathbb{E}[I(f)] = 0$

② $\mathbb{E}[(I(f))^2] = \int_a^b \mathbb{E}[f^2]\, dt$. (Ito isometry)

Recall: Conditional expectation.

If $X \in L^1(\Omega; \mathcal{F})$ and $\mathcal{G} \subset \mathcal{F}$,

sub σ-field.

then define the conditional expectation of X given $\mathcal{G}$ by the unique r.v. Y s.t.

① Y is $\mathcal{G}$-meas

② $\int_A X\,dP = \int_A Y\,dP, \quad \forall A \in \mathcal{G}.$

Rmk: ① Y is given by Radon-Nikodym Thm:

$$\mu(A) = \int_A X\,dP, \quad A \in \mathcal{G}.$$

$\Rightarrow \mu \ll P|_{\mathcal{G}}$

$\overset{R-N}{\Rightarrow} \quad d\mu = \underset{\underset{L^1}{\Uparrow}}{Y}\,dP|_{\mathcal{G}}$. Denote it by $Y = \mathbb{E}[X|\mathcal{G}]$

② $X \in L^2(\Omega; \mathcal{F})$

$$\Rightarrow \mathbb{E}[X \mid \mathcal{G}] = P_{L^2(\Omega;\mathcal{G})}(X)$$

$L^2(\Omega;\mathcal{G}) \leftarrow$ closed

<u>Properties</u>: (i) $\mathbb{E}\big[\mathbb{E}[X \mid \mathcal{G}]\big] = \mathbb{E}(X)$

(In general, more useful to read it as

$$\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X \mid \mathcal{G}]\big]$$

i.e. compute expectation by conditioning)

(ii) If X is $\mathcal{G}$-meas,

$$\mathbb{E}[X \mid \mathcal{G}] = X$$

(iii) If X & $\mathcal{G}$ are indep, ($\{X \in U\}$ and $A \in \mathcal{G}$ are indep $\forall, U \in \mathcal{B}_{\mathbb{R}}, A \in \mathcal{G}$

$$\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$$

(iv) If Y is $\mathcal{G}$-meas and $\mathbb{E}[XY] < \infty$,

then $\mathbb{E}[XY \mid \mathcal{G}] = Y\,\mathbb{E}[X \mid \mathcal{G}]$

· Pf of Lemma 4.1 :

1. $$\mathbb{E}\left[a_{j-1} \left(B(t_j) - B(t_{j-1}) \right) \right]$$

$$\overset{(ii)}{=} \mathbb{E}\left[\mathbb{E}\left[a_{j-1} \left(B(t_j) - B(t_{j-1}) \right) \middle| \mathcal{F}_{t_{j-1}} \right] \right]$$

$$\overset{(iv)}{=} \mathbb{E}\left[a_{j-1} \underbrace{\mathbb{E}\left[B(t_j) - B(t_{j-1}) \middle| \mathcal{F}_{t_{j-1}} \right]}_{\overset{(iii)}{=} \mathbb{E}\left[B(t_j) - B(t_{j-1}) \right] = 0} \right] = 0$$

2. $i < j$.

$$\mathbb{E}\left[\underbrace{a_{i-1} a_{j-1} \left(B(t_i) - B(t_{i-1}) \right)}_{\mathcal{F}_{t_{j-1}}\text{ - meas}} \left(B(t_j) - B(t_{j-1}) \right) \right]$$

$$= \cdots \underset{(ii),(iv)}{=} \mathbb{E}\left[a_{i-1} a_{j-1} \left(B(t_i) - B(t_{i-1}) \right) \underbrace{\mathbb{E}\left[B(t_j) - B(t_{j-1}) \middle| \mathcal{F}_{t_{j-1}} \right]}_{\underset{(iii)}{=} 0} \right]$$

$$= 0$$

$\underline{i=j}$

$$\mathbb{E}\left[a_{j-1}^2 \left(B(t_j) - B(t_{j-1})\right)^2 \right]$$

$$\overset{(i),(iv),(iii)}{=} \mathbb{E}\left[a_{j-1}^2\right] (t_j - t_{j-1})$$

$\boxtimes$

<u>Step 2:</u> FACT: Given $f \in L^2_{ad}([a,b] \times \Omega)$, $\exists \{f_n\}$ of step stoch processes converging to f.

Define Ito integral:

$$I(f) = \int_a^b f(t,\omega)\, dB(t,\omega)$$

$$\overset{\text{def}}{=} \lim_{n\to\infty} I(f_n).$$

Properties: ① I, linear

② $\mathbb{E}(I(f)) = 0$

③ $\mathbb{E}[(I(f))^2] = \int_a^b \mathbb{E}[f^2]\, dt$ (Ito isometry)

④ $\mathbb{E}\left[\int_a^b f(t)\, dB \int_a^b h(t)\, dB \right]$

$$= \int_a^b \mathbb{E}[f(t)h(t)]\, dt.$$

$I : L^2_{ad}([a,b] \times \Omega) \longrightarrow L^2(\Omega)$ is an isometry