Lec 18   08/03/17 (Wed)

Idea of the proof of Thm 3.7 :

Let $X_j = i\partial_t \tilde{u}^{N_j} + \Delta \tilde{u}^{N_j} - F_{N_j}(\tilde{u}^{N_j})$

$\quad\;$ ↳ $\mathcal{D}'_{t,x}$-valued r.v.

$\Longrightarrow \quad \mathcal{L}(X_j) = \delta_0$

$\Longrightarrow \quad \mathcal{L}(X_\infty) = \delta_0$, i.e.

$u = \lim_{j\to\infty} \tilde{u}^{N_j}$ is a distributional soln to (WNLS)

↑

- $i\partial_t \tilde{u}^{N_j} + \Delta \tilde{u}^{N_j} \longrightarrow i\partial_t u + \Delta u$, easy
- $:|u|^{2(m-1)}u: \; = \lim_{j\to\infty} F_{N_j}(\tilde{u}^{N_j})$ exists in $L^q(\rho_1; H^{-\varepsilon}(\mathbb{T}^2))$, $\varepsilon > 0$.

$\Leftarrow$ analogous to Prop 3.2

First, study $\langle F_{N_j}(\tilde{u}^{N_j}), e_n \rangle_{L^2_x}$

- (defocusing) WNLW on $\mathbb{T}^2$

Oh–Thomann '15 <u>Thm 3.8</u>: WNLW is almost surely locally well-posed w.r.t. μ.

$\Longrightarrow$ a.s. GWP & invariance of μ.

- Key ingredient

① $z_N = P_{\le N} S(t)(u_0^\omega, u_1^\omega)$

$$\Longrightarrow P\left(\| \langle \nabla \rangle^{-\varepsilon} :z_N^k: \|_{L^q_T L^r_x} > \lambda \right) \le C \exp\left(-c \frac{\lambda^{2/k}}{T^{2/qk}} \right)$$

② $:u^{2m-1}: = \lim\limits_{N\to\infty} :u_N^{2m-1}:$ $\qquad u_N = z_N + v_N$

$$= \lim_{N\to\infty} \sum_{k=0}^{2m-1} \binom{2m-1}{k} \underline{:z_N^k:} \, \underset{\substack{\cap \\ H^s \text{ for some } s>0}}{v_N^{2m-1-k}}$$

Sec 3.6 : Further topics

① μ, invariant. $\mathcal{L}(u(t)) = \mu = \mathcal{L}(u(0))$

How are they related?

Difficult question: study space-time covariance

$$\mathbb{E}\left[u(t,x) \overline{u(s,y)} \right]$$

- Weakly nonlinear, and large box limit.
  $(\varepsilon \to 0)$ $(|\Lambda| \to \infty)$

- Dynamical property?
  - recurrence, ✓
  - ergodicity? (difficult)
    - Known for some SPDEs
      $\Leftarrow$ uniqueness of an invariant meas
      $\uparrow$
      NOT true for (deterministic) nonlin Hamil PDEs

e.g. δ_0.

$$d\rho_0 = Z^{-1} e^{-\frac{1}{2}\|u\|_{L^2}^2} du$$ $\leftarrow$ white noise inv for KdV.

Quastel-Valkó, Oh, OQV.

② $\mathbb{T}^3$.

- Gibbs meas on $\mathbb{T}^3$ is renormalizable only for cubic & defocusing

  i.e. Φ^4_3 — 4: deg of nonlin in Hamil, 3: dim

- Wick ordering is NOT enough.

  $\sigma_N = \mathbb{E}_{\rho_1}[u^2(x)] \sim N$ when $d=3$.

  but need to replace it by $C_1 N + C_2 \log N$.

- Stochastic quantization eqn

  $$\partial_t u = \Delta u - u - u^3 + \infty \cdot u + \xi$$

  space-time white noise

  "LWP": Hairer '14 (regularity structure)

  Catellier-Chouk '14 (paracontrolled distribution à la Gubinelli-Imkeller-Perkowski

  Kupiainen '14 (renormalization group method)

  Invariance: Hairer-Matetski '15

  GWP: Mourrat-Weber '16.

Q: NLS / NLW on $\mathbb{T}^3$?

Chap 4: Introduction to stochastic dispersive PDEs

Stochastic NLS (SNLS):

bdd op on L^2 (smoothing)

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u + \underline{\phi}\,\xi$$

(actually, Hilbert-Schmidt : $L^2 \to H^s$

Stochastic NLW (SNLW):

$$-\partial_t^2 u + \Delta u = u^p + \xi$$

- ξ = space time white noise = " $\dfrac{\partial^2 B}{\partial x \partial t}$ ".

- Ito's formulation:

$$i\,du = (-\Delta u \pm |u|^{p-1}u)\,dt + \phi\, \underline{\underline{dW}}$$

$W(t) = L^2$-cylindrical Wiener process

$$= \sum_{n \in \mathbb{Z}^d} \beta_n(t) e^{in\cdot x}, \quad \{\beta_n\} = \text{indep } \mathbb{C}\text{-valued Brownian motions.}$$

$$\beta_n = \beta_n^{(r)} + i\,\beta_n^{(i)}$$

indep $\mathbb{R}$-valued BM

Note: $X_t(\omega) = X(t, \omega)$

Sec 4.1 Basic stochastic calculus

$(\Omega, \mathcal{F}, P)$, prob space.

A BM. on $\mathbb{R}_+$ $(t \geq 0)$ is a stochastic process, satisfying

(i) $B(0) = 0$, a.s.

(ii) $B(t) - B(s) \sim N(0, \underset{\text{var}}{t-s})$, $t > s$

normal distri

(iii) $B(t_1) - B(s_1)$, $B(t_2) - B(s_2)$, $t_1 \geq s_1 \geq t_2 \geq s_2$

are indep (i.e. indep increment over disjoint intervals)

Properties: ① BM is a.s. conti. (i.e. $\exists$ "conti version".)

$$\mathbb{E}\left(|B(t) - B(s)|^{2k}\right) = \underbrace{\frac{(2k)!}{2^k k!}}_{=(2k-1)!!} |t-s|^k$$

i.e. $\mathbb{E}\left(|B(t) - B(s)|^p\right) \sim |t-s|^{p/2}$

Kolmogorov continuity criterion (Bass, EX 8.2)

$\{X_t\}$ with values in a metric space S. (separable?)

$p, \varepsilon > 0$ Suppose $\mathbb{E}\left(d(X_s, X_t)^p \right) \le c_0 |t-s|^{1+\varepsilon}$, $\forall t, s$

Then, $\forall 0 < \gamma < \frac{\varepsilon}{p}$

$$P\left(\sup_{s \ne t} \frac{d(X_s, X_t)}{|t-s|^{\varepsilon/p - \gamma}} \ge \lambda \right) \le \frac{C_1}{\lambda^p}.$$

$\Longrightarrow X_t$ is unif conti a.s.

Back to BM. $\frac{\varepsilon}{p} - = \frac{\frac{p}{2} - 1}{p} - = \frac{1}{2} - \frac{1}{p} - \to \frac{1}{2} -$ as $p \to \infty$.

- $B \in \dot{W}^{b,p}_{loc}$, $b < 1/2$, $p \le \infty$

$\underline{p < \infty}$:

$$\mathbb{E}\left[\| B(t) \|^p_{\dot{W}^{b,p}(I)} \right] = \int_I \int_I \frac{\mathbb{E}\left(|B(t) - B(t')|^p \right)}{|t-t'|^{1+bp}} dt' dt$$

$$\sim \iint_{I\,I} |t-t'|^{-1-bp+p/2} dt' dt < \infty$$

iff $b < 1/2$

$B \in B^{1/2}_{p,\infty}$, $p < \infty$ $\left(\text{but } B \in B^{1/2 -}_{\infty, \infty} \right)$

Roynette '93 SS rep

Bényi - Oh '11 Ad. Math

2 Covariance

$$\mathbb{E}[B(t)B(s)] = t \wedge s = \min(t, s)$$

$$\parallel$$

$$\mathbb{E}\big[B(s)\big(B(t) - B(s)\big)\big] + \mathbb{E}\big[B^2(s)\big] = s, \quad t > s$$

indep $= 0$

<u>White noise</u> : $\xi = dB$

$$\xi \rightsquigarrow \mathbb{E}[\xi(t)\xi(s)] = \delta(t-s), \quad \text{"}\xi \text{ scales like } \sqrt{\delta}\text{"}$$

$h > 0$

$$\phi_h(s) = \mathbb{E}\left[\frac{B(t+h) - B(t)}{h} \cdot \frac{B(s+h) - B(s)}{h}\right]$$

$$= \cdots = \frac{1}{h^2}\Big((t+h)\wedge(s+h) - (t+h)\wedge s - t \wedge (s+h) + t \wedge s\Big)$$

- $\int \phi_h(s)\,ds = 1$

$\phi_h(s) \longrightarrow \delta(t-s)$ as $h \to 0$