Pf of Prop 3.6: $G_N(u)$ is not sign definite

Nonetheless, the defocusing nature plays an important role

Main observation: - $G_N(u)$ has a logarithmic upper bound.

$$-G_N(u) = -\frac{1}{2m}\int_{\mathbb{T}^2} \underbrace{H_{2m}(u_N;\sigma_N)}_{= \sigma_N^m H_{2m}\left(\frac{u_N}{\sigma_N^{1/2}}\right)} dx$$

$$\leq b_m (\log N)^m \qquad H_{2m}(x;1) \geq -\underbrace{a_m}_{>0}$$

- $$\underset{e^{-G_N(u)}}{\overset{\parallel}{\|R_N(u)\|}}{}_{L^q(\rho_1)}^{q} = \int_0^\infty \rho_1\left(e^{-qG_N(u)} > \alpha\right) d\alpha$$

$$\leq 1 + \int_1^\infty \rho_1\left(-qG_N(u) > \log\alpha\right) d\alpha$$

(*) $\Rightarrow$ Suffices to show $\rho_1(-qG_N(u) > \log\alpha) \leq C\alpha^{-(1+\delta)}$, $\forall \alpha \geq 1, N \in \mathbb{N}$

- Given $\lambda = \log \alpha$,

Choose $N_0 \in \mathbb{R}$ s.t. $\lambda = 2q\, b_m (\log N_0)^m$.

Then, by the log upper bound,

$$\rho_1 (\underbrace{-q\, G_N(u)}_{\leq q\, b_m (\log N)^m \leq \frac{1}{2}\lambda} > \lambda) = 0 \qquad \text{for all } N < N_0$$

- For $N \geq N_0$,

$$\rho_1 (-q\, G_N(u) > \lambda)$$

$$\leq \rho_1 (-q\, G_N(u) + q\, G_{N_0}(u) > \overbrace{\lambda - q\, b_m (\log N_0)^m}^{=\frac{1}{2}\lambda})$$

$$-q\, b_m (\log N_0)^m - q\, G_{N_0}(u) \leq 0$$

$$\leq \rho_1 (q\, | G_N(u) - G_{N_0}(u) | \geq \tfrac{1}{2}\lambda)$$

Prop 3.2 & Chebyshev

$$\leq C_{m,q}\, \underbrace{e^{-c N_0^{1/2m} \lambda^{1/m}}}_{\boxed{e^{-c\lambda^{1/m+1}} \times e^{c\lambda^{1/m}}}} \qquad N \geq N_0$$

(**)

$$\ll e^{-(1+\delta)\lambda} \sim \alpha^{-(1+\delta)} \Longrightarrow (*)$$

In $\circledast\!\circledast$, we used

$$\rho_1 (|G_M(u) - G_N(u)| > \tilde{\lambda}) \le C_m e^{-c_m N^{1/2m} \tilde{\lambda}^{\frac{1}{m}}}$$

$\Rightarrow$ $G_N(u)$ converges in measure

$\Rightarrow$ $R_N = e^{-G_N(u)}$ converges in measure

By repeating the argument in Cor 1.4,

$$R_N \to R = e^{-G(u)} \text{ in } L^q(\rho_1), \ \forall q \ge 1$$

$\boxtimes$

Sec 3.5: On the dynamical problem.

- Wick ordered NLS (WNLS):

$$i\partial_t u + \Delta u = \,:\!|u|^{2(m-1)}u\!: \quad \text{on } \mathbb{T}^2$$

- Wick ordered NLW (NLKG)

$$-\partial_t^2 u + \Delta u - u = \,:\!u^{2m-1}\!: \quad \text{on } \mathbb{T}^2.$$

- NLS: Scalining critical regularity $s_{crit} = 1 - \frac{1}{m-1}$ $(m \geq 2)$

cubic: $s_{crit} = 0$

quintic: $s_{crit} = 1/2$

but ρ_1 and μ are supported on $H^{-\varepsilon}(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$

$u = z + v$ $\Longrightarrow$

- Bourgain '96: The cubic WNLS is almost locally locally well-posed w.r.t. ρ_1 (and hence μ)
(Also, see Colliander-Oh, Duke '12)

"inv. meas argument" in Chap 1

$\Longrightarrow$ a.s. GWP & invariance of μ.

When $m \geq 3$, i.e. (super) quintic

$\Rightarrow$ regularity gap $> 1/2$ $\Leftarrow$ too large!!

· <u>Compactness argument</u> (on measures on spacetime functions)

(FWNLS) $\quad i\partial_t u^N + \Delta u^N = F_N(u^N)$,

where $F_N(u) = P_{\le N}\left(:|P_{\le N} u|^{2(m-1)} P_{\le N} u: \right)$

$$= (-1)^{m-1}(m-1)!\, \sigma_N^{m-1} \cdot P_{\le N}\left(L^{(1)}_{m-1}\left(\frac{|u_N|^2}{\sigma_N} \right) u_N \right)$$

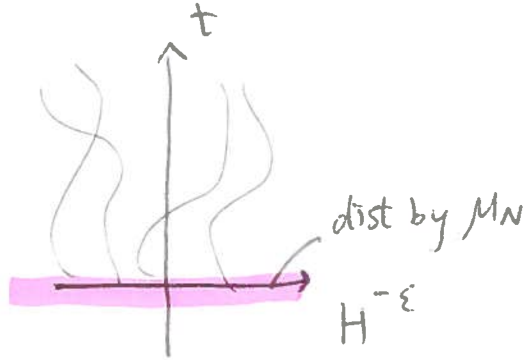
generalized Laguerre poly

· $d\mu_N = Z_N^{-1} R_N(u)\, d\rho$,

$=$ invariant Gibbs meas for (FWNLS).

① Extend μ_N = prob. meas on initial data

to ν_N = prob. meas on space-time functions



- (FWNLS) is $\underbrace{}_{\text{a.s.}}$ globally well-posed

$$\Rightarrow \exists\, \Phi_N : u_0 \in H^{-\varepsilon} \longmapsto u^N \in C(\mathbb{R}; H^{-\varepsilon})$$

- Define a prob. meas ν_N on $C(\mathbb{R}; H^{-\varepsilon})$

  by $\quad \nu_N = \mu_N \circ \Phi_N^{-1}$

  $\quad = $ induced prob meas under Φ_N.

- Note that by invariance of μ_N,

$$\mathcal{L}(u^N(t)) = \mu_N \quad \text{for any } t \in \mathbb{R}$$

law

2 (By a soft argument), show $\{\nu_N\}$ is <u>tight</u> (= precompact)

$$\overset{\text{Prokhorov}}{\Longrightarrow} \quad \nu_{N_j} \rightharpoonup \nu$$

3 <u>Skorokhod's Thm</u>: $\exists$ another prob space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$

$\widetilde{u}^{N_j}$, $\quad C(\mathbb{R}; H^{-\varepsilon})$-valued r.v's

u, $\quad$ "

s.t. $\mathcal{L}(\widetilde{u}^{N_j}) = \mathcal{L}(u^{N_j}) = \nu_{N_j}$.

$$\mathcal{L}(u) = \nu$$

and $\widetilde{u}^{N_j}$ converges to u in $C(\mathbb{R}; H^{-\varepsilon})$ almost surely.

We "upgrade" weak conv to a.s. conv.

Rmk: $\mathcal{L}(\widetilde{u}^{N_j}(t)) = \mathcal{L}(u^{N_j}(t)) = \mu_{N_j}$.

- By $\mu_N \xrightarrow[\text{"unif"}]{} \mu$, $\mathcal{L}(u(t)) = \mu$. = Gibbs meas

Oh-Thomann '15

Thm 3.7: $\exists$ a set Σ of full probability s.t.

$\forall \phi \in \Sigma$, $\exists$ global soln $u \in C(\mathbb{R}; H^{-\varepsilon})$ to WNLS with $u|_{t=0} = \phi$.

Moreover, $\mathcal{L}(u(t)) = \mu$, $\forall t \in \mathbb{R}$

Rmk: ① only a.s. global existence (NO uniqueness) ← uptness

"energy solns"

② only invariance of μ in mild sense