01 / 03 / 17 (Wed)

Sec 3.3 : Hypercontractivity of the Ornstein-Uhlenbeck semigroup and Wiener chaos estimate

$$H = L^2(\mathbb{R}^d, \mu_d)$$

$\uparrow$ standard Gaussian meas on $\mathbb{R}^d$.

$$L = \Delta - x \cdot \nabla$$

= Hartree - Fock operator (OU operator)

Consider

$$\partial_t u = Lu \quad \text{on } \mathbb{R}^d.$$

Lemma 3.3 (hypercontractivity of OU semigroup) Nelson '65

$p > 1$, $q \geq 1$. Then, we have

$$\| e^{tL} f \|_{L^q(\mathbb{R}^d, \mu_d)} \leq \| f \|_{L^p(\mathbb{R}^d, \mu_d)}$$

for $t \geq \frac{1}{2} \log\left(\frac{q-1}{p-1}\right)$

<u>Rmk:</u> • The estimate holds indep of dim $d \geq 1$ (even $d = \infty$.)

• This is equivalent to Log Sobolev inequality.

Define a <u>homog Wiener chaos</u> of order k

$$\mathbb{H}_{\vec{k}}(x) = \prod_{j=1}^{d} H_{k_j}(x_j) \qquad x = (x_1, \cdots, x_d), \quad |\vec{k}| = k_1 + \cdots + k_d = k$$

• $\mathcal{H}_k$ = closure of homog Wiener chaoses of order k under $L^2(\mathbb{R}^d, \mu_d)$ (or any $L^p(\mathbb{R}^d, \mu_d)$.)

<u>Ito – Wiener decomposition</u>,

$$L^2(\mathbb{R}^d, \mu_d) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

- $F \in \mathcal{H}_k$ is an eigenfunction of L with eigenvalue $-k$.

$\Rightarrow$ <u>Cor 3.4</u>: $F \in \mathcal{H}_k$. Then for $q \geq 2$, we have

$$\|F\|_{L^q(\mathbb{R}^d, \mu_d)} \leq (q-1)^{k/2} \|F\|_{L^2(\mathbb{R}^d, \mu_d)}$$

($\Leftarrow$ $p = 2$, $t = \frac{1}{2} \log(q-1)$ in Lemma 3.3

<u>Lemma 3.5</u>: $S_k(\omega) = \sum_{\Gamma(k,d)} C(n_1, \cdots, n_k)\, g_{n_1}(\omega) \cdots g_{n_k}(\omega)$ $\in \bigoplus_{j=0}^{k} \mathcal{H}_j$

$\{g_n\}$ std indep Gaussian r.v's

where $\Gamma(k,d) = \{(n_1, \cdots, n_k) \in \mathbb{Z}^k : |n_j| \leq d\}$

$\Rightarrow$ Then, for $q \geq 2$

$$\|S_k\|_{L^q(\Omega)} \leq C\sqrt{k+1}\,(q-1)^{k/2} \|S_k\|_{L^2(\Omega)}$$

$\uparrow$ can be removed

Thomann–Tzvetkov

($\Leftarrow$ Cor 2.4 and $x^j = \sum_{m=0}^{[j/2]} c_{m,j}\, \sigma^m H_{j-2m}(x;\sigma)$

<u>ex</u>: $n \neq m$

$$g_n^2 \, g_m^3 = (g_n^2 - 1 + 1)(g_m^3 - 3g_m + 3g_m)$$

$$\Rightarrow g_n^2 g_m^3 = \underbrace{H_2(g_n)H_3(g_m)}_{\in \mathcal{H}_5} + \underbrace{H_0(g_n)H_3(g_m)}_{\in \mathcal{H}_3} + \underbrace{3H_2(g_n)H_1(g_m)}_{\in \mathcal{H}_3} + \underbrace{3H_0(g_n)H_1(g_m)}_{\in \mathcal{H}_1}$$

· <u>Back to Prop 3.2</u>: $u_N = \sum_{|n| \le N} \frac{g_n}{\langle n \rangle} e^{in\cdot x}$

$$u_N^{2m} = \sum_{|n|\le N} e^{in\cdot x} \left(\sum_{\substack{n = n_1 + \cdots + n_{2m} \\ |n_j| \le N}} \prod_{j=1}^{2m} (g_{n_j} / \langle n_j \rangle) \right) \leftarrow \text{2m fold product of Gaussian}$$

$$G_N(u) = \underbrace{\frac{1}{2m} \int \underbrace{: u_N^{2m} :}\, dx}_{\text{projection onto } \mathcal{H}_{2m}} \in \bigoplus_{k=0}^{2m} \mathcal{H}_k \quad (\text{actually in } \mathcal{H}_{2m})$$

$\Rightarrow$ Use Lemma 3.5 and Prop 3.2 with $q = 2$.

Namely,

$$\| G_M(u) - G_N(u) \|_{L^q(\rho_1)} \le C_m (q-1)^m \underbrace{\| G_M(u) - G_N(u) \|_{L^2(\rho_1)}}_{\lesssim \frac{1}{N^{1/2}}}$$

$\boxtimes$

Sec 3.4 : Construction of the Gibbs measure on $\mathbb{T}^2$

Goal : Construct

$$d\mu = Z^{-1} e^{-H_{\mathrm{Wick}}(u)} du$$
$$= Z^{-1} e^{-\frac{1}{2m} \int_{\mathbb{T}^2} :u^{2m}: dx} d\rho_1$$

(Φ_2^{2m} or φ_2^m Euclidean QFT)

(deg $\to$ $2m$, dim $\to$ 2)

Let $R_N(u) = e^{-G_N(u)} = e^{-\frac{1}{2m} \int :u_N^{2m}: dx}$.

Prop 3.6 : · $R_N(u) \in L^q(\rho_1)$, for any finite $q \ge 1$, with a unif bd in N (depends on q)

· $R_N(u)$ converges to some $R(u)$ in $L^q(\rho_1)$

- This allows us to define the Gibbs meas

$$d\mu = Z^{-1} R(u)\, d\rho_1 = Z^{-1} e^{-\frac{1}{2m}\int :u^{2m}: dx}\, d\rho_1$$

$$= \lim_{\substack{N\to\infty \\ \uparrow \\ \text{"unif" conv.}}} \underbrace{Z_N^{-1} R_N(u)\, d\rho_1}_{= d\mu_N}$$

Pf: Nelson's estimate. Next time.