Lec 15 27/02/17 (Mon)

Sec 3.2: Hermite polynomials, white noise functional, and Wick ordering

- Hermite polynomials $H_k(x;\sigma)$ defined through the generating function:

$$G(t,x;\sigma) = e^{tx - \frac{1}{2}\sigma t^2}$$
$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma)$$

$H_0(x;\sigma) = 1$, $H_1(x;\sigma) = x$,

$H_2(x;\sigma) = x^2 - \sigma$, $H_3(x;\sigma) = x^3 - 3\sigma x$,

$H_4(x;\sigma) = x^4 - 6\sigma x^2 + 3\sigma^2$.

- $\partial_x H_k(x;\sigma) = k H_{k-1}(x;\sigma)$

- <u>White noise functional</u>:

$$d\rho_0 = Z^{-1} e^{-\frac{1}{2}\|u\|_{L^2}^2} du \quad \text{on } H^{-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$$

$$\leftrightsquigarrow \quad u_w(x) = \sum_{n\in\mathbb{Z}^d} g_n(\omega) e^{inx} \Leftarrow \text{distribution.}$$

- $W_{(\cdot)} : L^2(\mathbb{T}^d) \longrightarrow L^2(\Omega)$

by $W_f(\omega) = \langle f, u_w(\omega)\rangle_{L^2} = \sum_{n\in\mathbb{Z}^d} \hat{f}(n)\, \overline{g_n}(\omega)$

$\Rightarrow$ W_f is a Gaussian r.v. with mean 0.

and

$$\mathrm{Var}(W_f) = \mathbb{E}\Big[\sum_n \hat{f}(n)\,\overline{g_n} \;\; \overline{\sum_m \hat{f}(m)\,\overline{g_m}}\Big] \qquad g_{-n} = \overline{g_n}.$$

$$= \|f\|_{L^2}^2$$

Moreover, $\mathbb{E}\big[W_f\,\overline{W_h}\big] = \langle f, h\rangle_{L^2}$.

$$\text{deterministic}$$

$W_{(\cdot)}$ is unitary. ($\sim$ Wiener integral $\int \overset{\downarrow}{f}\, dB$)

Lemma 3.1: $f, h \in L^2(\mathbb{T}^2)$ with $\|f\|_{L^2} = \|h\|_{L^2} = 1$ ③

Then, $\mathbb{E}\left[H_k(W_f)\, H_m(W_h) \right] = \delta_{km}\, k! \left[\langle f, h \rangle_{L^2_x} \right]^k$

Rmk: In the complex-valued setting, we use the (generalized) Laguerre polynomials (Oh - Thomann '15)

Pf: First, recall

$$\int_\Omega e^{W_f}\, dP = e^{\frac{1}{2}\|f\|^2_{L^2}}$$

$\mathcal{I}$ = index set
$= "\mathbb{Z}^2/2"$
$= \mathbb{Z}\times\mathbb{N} \cup \mathbb{N}\times\{0\} \cup \{(0,0)\}$
$\mathcal{I}^* = \mathcal{I} \setminus \{(0,0)\}$

$$\text{(LHS)} = \prod_{n\in\mathcal{I}^*} \frac{1}{\pi}\int_{\mathbb{C}} e^{2\operatorname{Re}(\widehat{f}(n)\,\overline{g_n}) - |g_n|^2}\, dg_n$$

$$\times \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\widehat{f}(0)\, g_0}\, e^{-\frac{1}{2}g_0^2}\, dg_0$$

$$= \prod_{n\in\mathcal{I}^*} \frac{1}{\pi}\int_{\mathbb{R}} e^{2\operatorname{Re}\widehat{f}(n)\operatorname{Re}g_n} - e^{-(\operatorname{Re}g_n)^2}\, d\operatorname{Re}g_n$$

$$\times \int_{\mathbb{R}} e^{2\operatorname{Im}\widehat{f}(n)\operatorname{Im}g_n - (\operatorname{Im}g_n)^2}\, d\operatorname{Im}g_n$$

$$\times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cdots \, dg_0$$

$$= e^{\sum_{n \in I^*} |\hat{f}(n)|^2} e^{\frac{1}{2}(\hat{f}(0))^2} = e^{\frac{1}{2}\|f\|_{L^2}^2}$$

$$\Longrightarrow \quad \int_\Omega G(t, W_f)\, G(s, W_h)\, dP$$

$$= e^{-\frac{1}{2}(t^2+s^2)} \underbrace{\int_\Omega e^{W_{tf+sh}}\, dP}_{= e^{\frac{1}{2}\|tf+sh\|_{L^2}^2}}$$

$\uparrow$ by linearity of $W_{(\cdot)}$

$$= e^{ts\langle f, h\rangle_{L^2}} = \sum_{l=0}^{\infty} \frac{(ts)^l}{l!} (\langle f, h\rangle)^l.$$

On the other hand,

$$(\mathrm{LHS}) = \sum_{k,m=0}^{\infty} \frac{t^k s^m}{k!\, m!} \int_\Omega H_k(W_f)\, H_m(W_h)\, dP$$

$\square$

Under ρ_1, we have

$$u = \sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle} e^{in\cdot x}.$$

Let $u_N = P_{\leq N} u$.

$\Rightarrow$ $u_N(x)$ is a Gaussian r.v. with mean 0 and $\text{var} \sim \log N$ $\forall x \in \mathbb{T}^2$.

$$\text{var} = \sigma_N$$

$\Rightarrow$ Define the Wick ordered monomial

$$:u_N^k(x): \;=\; H_k(u_N(x); \sigma_N)$$

Let $G_N(u) = \frac{1}{2m}\int_{\mathbb{T}^2} :(P_{\leq N}u)^{2m}: dx$

Prop 3.2: $\{G_N(u)\}$ is a Cauchy seq in $L^q(\rho_1)$ for any finite $q \geq 2$. Moreover,

$$\| G_M(u) - G_N(u)\|_{L^q(\rho_1)} \leq C_m (q-1)^m \frac{1}{N^{1/2}}$$

for any $q \geq 1$, $M \geq N \geq 1$.

Rmk: Prop 3.2 allows us to define

$$G(u) = G_\infty(u)$$
$$= \frac{1}{2m} \int_{\mathbb{T}^2} :u^{2m}: dx = \lim_{N\to\infty} G_N(u) \text{ in } L^q(\rho_1)$$

and the Wick ordered Hamiltonian

$$H_{Wick}(u) = \frac{1}{2} \int_{\mathbb{T}^2} |\nabla u|^2 + \frac{1}{2m} \int_{\mathbb{T}^2} :u^{2m}:$$

Pf ($q = 2$): $\quad e_n(y) = e^{in\cdot y}$

$$\eta_N(x)(\cdot) = \frac{1}{\sigma_N^{1/2}} \sum_{|n|\le N} \frac{\overline{e_n(x)}}{\langle n \rangle} e_n(\cdot)$$

$$\gamma_N(\cdot) = \sum_{|n|\le N} \frac{1}{\langle n \rangle^2} e_n(\cdot) \qquad \sigma_N = \sum_{|n|\le N} \frac{1}{\langle n \rangle^2} \sim \log N$$

Note that $\|\eta_N(x)\|_{L^2(\mathbb{T}^2)} = 1$ for each fixed $x \in \mathbb{T}^2$.

and

(*) $$\langle \eta_M(x), \eta_N(y) \rangle_{L^2(\mathbb{T}^2)} = \frac{1}{\sigma_M^{1/2} \sigma_N^{1/2}} \gamma_N(y - x) \qquad M \ge N \ge 1$$

$$\bullet\quad u_N(x) = \sigma_N^{1/2}\,\frac{u_N(x)}{\sigma_N^{1/2}} = \sigma_N^{1/2}\,\overline{W}_{\eta_N(x)}$$

$$= \sigma_N^{1/2}\, W_{\eta_N(x)}$$

$$\Rightarrow\quad :u_N^{2m}(x): \;=\; H_{2m}(u_N(x);\,\sigma_N) \qquad\qquad H_k(x;\sigma) = \sigma^{k/2} H_k(x;1)$$

$$= \sigma_N^{m}\, H_{2m}\left(\frac{u_N(x)}{\sigma_N^{1/2}};\,1\right)$$

$$= \sigma_N^{m}\, H_{2m}(W_{\underline{\eta_N(x)}})$$

$\uparrow$ L^2_x-norm is 1.

$$\Rightarrow\quad (2m)^2\,\|G_M(u) - G_N(u)\|^2_{L^2(\rho_1)}$$

$$= \int_{\mathbb{T}^2_x\times\mathbb{T}^2_y}\int_\Omega \sigma_M^{2m} H_{2m}(W_{\eta_M(x)})\, H_{2m}(W_{\eta_M(y)})$$

$$- \sigma_M^m\sigma_N^m\, H_{2m}(W_{\eta_M(x)})\, H_{2m}(W_{\eta_N(y)})$$

$$- \sigma_N^m\sigma_M^m\, H_{2m}(W_{\eta_N(x)})\, H_{2m}(W_{\eta_M(y)})$$

$$+ \sigma_N^{2m}\, H_{2m}(W_{\eta_N(x)})\, H_{2m}(W_{\eta_N(y)})\, dP\,dx\,dy$$

Lemma 3.1 & ⊛

$$= C_m \int_{\mathbb{T}^2_x \times \mathbb{T}^2_y} (\gamma_M(y-x))^{2m} - (\gamma_N(y-x))^{2m} \, dx\,dy$$

$$= C_m \int_{\mathbb{T}^2} (\gamma_M(x))^{2m} - (\gamma_N(x))^{2m} \, dx$$

$$\lesssim \int_{\mathbb{T}^2} |\gamma_M(x) - \gamma_N(x)| \left(|\gamma_M(x)|^{2m-1} + |\gamma_N(x)|^{2m-1} \right) dx$$

$$\underset{C-S}{\leq} \underbrace{\| \gamma_M - \gamma_N \|_{L^2}} \left(\| \gamma_M^{2m-1} \|_{L^2} + \| \gamma_N^{2m-1} \|_{L^2} \right)$$

$$= \left(\sum_{N \leq |n| \leq M} \frac{1}{\langle n \rangle^4} \right)^{1/2} \lesssim \frac{1}{N}$$

$$\cdot \| \gamma_M^{2m-1} \|_{L^2} = \| \gamma_M \|_{L^{4m-2}}^{2m-1} \underset{H-Y}{\leq} \left(\sum_{|n| \leq M} \frac{1}{\langle n \rangle^{\boxed{2 \cdot \frac{4m-2}{4m-3}}}} \right)^{\frac{4m-3}{2}} \leq C_m < \infty$$

>2

indep of M.

$\boxtimes$