Lec 14 22/02/17

Goal: Disprove the freq loc. dual estimate:

$$(*) \quad \left\| \int e^{it\sqrt{-\Delta}} F(t)\, dt \right\|_{L^2_x} \lesssim \|F\|_{L^2_t L^1_x}$$

β_t = Brownian motion

$\beta_t - \beta_s$ = mean 0 Gaussian r.v. with Var $= t-s$

$$\boxed{\mathbb{E}\left[e^{\pm 2\pi i (\beta_t - \beta_s)\xi} \right] = e^{-2\pi |t-s| \xi^2}} \quad \forall \xi \in \mathbb{R}$$

Let $F(t,x) = \eta\left(\frac{t}{T}\right) \psi(x - t\hat{e}_3 - \beta_t \hat{e}_1)$

η: non-neg. smooth cutoff supp on $[-1,1]$

$\hat{\psi}(\xi)$ supported on $\{|\xi| \sim 1\}$

Suppose $(*)$ holds

With $\hat{F}(t,\xi) = \eta\left(\frac{t}{T}\right) e^{-2\pi i t \xi_3} e^{-2\pi i \beta_t \xi_1} \hat{\psi}(\xi)$

$$(*) \Rightarrow \left\| \int e^{2\pi i t|\xi|} e^{-2\pi i t \xi_3} e^{-2\pi i \beta_t \xi_1} \eta\left(\frac{t}{T}\right) dt \cdot \hat{\varphi}(\xi) \right\|_{L^2_\xi}$$ ②

$$\lesssim \|F\|_{L^2_t L^1_x} \sim \sqrt{T}.$$

$$(**) \quad \int_{|\xi|\sim 1} \left| \int_t \cdots dt \right|^2 d\xi \lesssim T$$

Let $A = \{\xi \in \mathbb{R}^3 : \xi_3 \sim 1, |\xi_2| \ll |\xi_1| \ll 1\}$.

$\Rightarrow$ We have the following Taylor approx:

$$|\xi| - \xi_3 \sim \xi_1^2, \quad \forall \xi \in A.$$

$$\left(|\xi| = \sqrt{\xi_3^2 + \xi_1^2 + \xi_2^2} = \xi_3 + \frac{1}{2|\xi_3|}(\xi_1^2 + \xi_2^2) + \text{error} \right.$$

- Restrict the domain to A and expand the square.

(LHS) of (**)

$$\sim \int_A \iint_{t\,t'} e^{2\pi i (t-t')(|\xi| - \xi_3)} \underline{e^{-2\pi i (\beta_t - \beta_{t'})\xi_1}} \eta\left(\frac{t}{T}\right)\eta\left(\frac{t'}{T}\right) dt' dt\, d\xi$$

Take $\mathbb{E}$

$$\int_A \iint e^{2\pi i (t-t')(|\xi| - \xi_3)} e^{-2\pi |t-t'| \xi_1^2} \eta\left(\frac{t}{T}\right)\eta\left(\frac{t'}{T}\right) dt' dt\, d\xi$$

$r+t'$

$$\lesssim \underline{\underline{T}}.$$

Let $r = t - t'$.

$$\int_A \int e^{2\pi i r (|\xi| - \xi_3)} e^{-2\pi |r| \xi_1^2} \eta * \eta\left(\frac{r}{T}\right) dr\, d\xi \ \underline{\underline{\lesssim 1}}.$$

- Let $T \to \infty$

$$\int_A \int e^{2\pi i r \underbrace{(|\xi| - \xi_3)}_{x}} e^{-2\pi |r| \underbrace{\xi_1^2}_{y}} dr\, d\xi \ \underline{\underline{\lesssim 1}}$$

$$\wr$$

$$\int_A \underline{\frac{1}{\xi_1^2} d\xi_1 d\xi_2 d\xi_3} \ \underline{\underline{\sim \infty}}$$

$\xi_3 \sim 1$

$|\xi_2| \ll |\xi_1| \ll 1 \quad \searrow \quad \int \frac{1}{\xi_1} d\xi_2 = \infty$

$$\left(\int e^{2\pi i r x} e^{-2\pi |r| y} dr = \frac{y}{\pi (x^2 + y^2)} \right) \Rightarrow \text{(X)}$$

contradiction.

Rmk: Without the BM factor, there would be NO $e^{-2\pi|t|\xi_1^2}$ in the last integral.

$$\Rightarrow \int_A \delta_0(\xi_1^2)\,d\xi \lesssim 1 \quad (\Leftarrow \text{No contradiction})$$

Recall $\|f\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |f - f_Q|$, $\quad f_Q = \operatorname{ave}_Q f$.

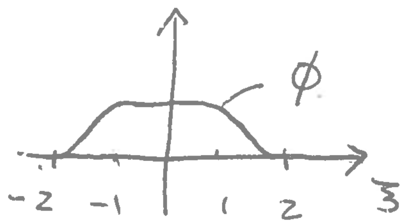
Lemma 2.11: $\operatorname{supp} \hat{f} \subset \{|\xi| \sim 2^k\}$, $k \in \mathbb{Z}$ (homog LP localization)

$$\Rightarrow \|f\|_{BMO} \sim \|f\|_{L^\infty}$$

Pf: By scaling, we may assume $k = 0$.

· Fix x.

$j \gg 1$. $\quad f(x) = P_{\le j} f(x) = \int f(x-y)\, 2^{jd}\, \check{\phi}(2^j y)\, dy$



$$\phi(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| \ge 2 \end{cases}.$$

We also have

$$\int f(x-y)\, 2^{-jd} \check{\Phi}(2^{-j}y)\, dy = \underset{\substack{\uparrow \\ \text{proj onto } \{|\xi| \lesssim 2^{-j}\}}}{P_{\lesssim -j}} f(x) = 0.$$

(5)

$$\Rightarrow f(x) = \int f(x-y)\left(2^{jd}\check{\Phi}(2^{j}y) - 2^{-jd}\check{\Phi}(2^{-j}y)\right) dy$$

Let $Q = \{y : |y| \le 2^{\alpha}\}$. and $Q_x = x + Q$.

Note that

$$\int f_{Q_x}\left(2^{jd}\check{\Phi}(2^{j}y) - 2^{-jd}\check{\Phi}(2^{-j}y)\right) dy$$

$$= f_{Q_x}\Big[\underbrace{\int 2^{jd}\check{\Phi}(2^{j}y)\, dy}_{=\int \check{\Phi}\, dy} - \underbrace{\int 2^{-jd}\check{\Phi}(2^{-j}y)\, dy}_{=\int \check{\Phi}\, dy}\Big] = 0$$

$$\Rightarrow f(x) = \int \left(f(x-y) - f_{Q_x}\right)\left(2^{jd}\check{\Phi}(2^{j}y) - 2^{-jd}\check{\Phi}(2^{-j}y)\right) dy$$

- If $y \in Q$, then $|f(x)| \lesssim 2^{d\alpha} \|f\|_{BMO}$.
- If $y \notin Q$, then $|f(x)| \le 2\|f\|_{L^\infty} \underbrace{\int_{|y| \ge 2^\alpha} |2^{jd}\check{\Phi}(2^{j}y)| + |2^{-jd}\check{\Phi}(2^{-j}y)|\, dy}_{< \frac{1}{4} \text{ by choosing } \alpha \gg 1 \text{ (indep of } x) \text{ (} j \text{ is fixed)}}$

$$\Rightarrow |f(x)| \le C_\alpha \|f\|_{BMO} + \frac{1}{2}\|f\|_{L^\infty}$$

$\Rightarrow$ Take sup in x. $\boxtimes$

Chap 3: Invariant Gibbs measure, Part 2

We'll focus on $\mathbb{T}^2$.

Sec 3.1: Gibbs meas on $\mathbb{T}^2$ (for NLS, NLW)

$$H(u) = \frac{1}{2}\int |\nabla u|^2 \pm \frac{1}{p+1}\int |u|^{p+1}.$$

$$\Rightarrow \text{"} d\mu = Z^{-1} e^{-H(u)} du$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}} d\rho_1 \text{"}$$

where $d\rho_1 = Z^{-1} e^{-\frac{1}{2}\|u\|^2_{H^1}} du \Leftarrow$ supported on $H^s(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$, $s<0$.

$$\Rightarrow \int |u|^{p+1} dx = \infty, \quad a.s.$$

- We need to renormalize $\int |u|^{p+1}$.

We only consider the defocusing case.

(Focusing case: Brydges–Slade '96.
We can not construct Gibbs measure for the focusing cubic NLS on $\mathbb{T}^2$
even if we consider the Wick ordered nonlinearity (with the Wick ordered

In the following, we focus on the real-valued setting (for simplicity) ⑦

- Under ρ_1, we have

indep, std Gaussian r.v

$$u = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x}, \qquad g_{-n} = \overline{g_n}, \quad \mathrm{Var}(g_n) = 1$$

Given $N \in \mathbb{N}$, let $u_N = P_{\le N} u$.

$\Rightarrow$ For each $x \in \mathbb{T}^2$, $u_N(x)$ is a mean 0 $\mathbb{R}$-valued Gaussian r.v. with variance:

$$\sigma_N := \mathbb{E}\big[u_N^2(x)\big] = \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} \sim \log N \nearrow \infty.$$

- Similarly,

$$\mathbb{E}\Big[\int_{\mathbb{T}^2} u_N^2(x)\,dx\Big] = \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} \sim \log N$$

but defining

$$\underbrace{:u_N^2(x):}_{\text{Wick ordered monomial}} \overset{\text{def}}{=} u_N^2(x) - \sigma_N$$

$$= H_2(u_N(x);\sigma_N)$$

Hermite poly $H_2(x;\sigma) = x^2 - \sigma$

We have

$$\int_{\mathbb{T}^2} :u_N^2(x): = \sum_{|n|\le N} \frac{|g_n|^2 - 1}{\langle n \rangle^2}$$

- $\mathbb{E}\left(\int_{\mathbb{T}^2} :u_N^2: dx\right] = 0$

- $\mathbb{E}\left[\left(\cdots \right)^2\right] = \mathbb{E}\left[(g_0^2 - 1)^2\right]$ "$I = \mathbb{Z}^2/2$"

$$+ 2\,\mathbb{E} \sum_{\substack{n \in I \\ |n|\le N}} \frac{(|g_n|^2 - 1)^2}{\langle n \rangle^4}$$

$$\lesssim 1 + \sum_{|n|\le N} \frac{1}{\langle n \rangle^4} \lesssim 1 \quad \text{indep of } N$$

$$\mathbb{E}\left[(|g_n|^2 - 1)(|g_m|^2 - 1)\right] = 0 \text{ unless } n = m.$$

$\Rightarrow$ $\int_{\mathbb{T}^2} :u_N^2: dx$ is a well-defined r.v.

$\int_{\mathbb{T}^2} :u^2: dx = \lim_{N\to\infty} \int :u_N^2: dx$ exists in $L^q(\Omega)$ for any $q < \infty$.