# Lec 13: 20/02/17 (Mon)



- <u>Digression</u>: Failure of the endpt Strichartz estimate (wave)

(*) False: $\| S(t)(u_0, u_1) \|_{L^2_t L^\infty_x} \lesssim \| (u_0, u_1) \|_{\dot{H}^1}$ $\quad d = 3$

$$(q, r, d) = (2, \infty, 3).$$

( Schrödinger: fails at $(2, \infty, 2)$ $\leftarrow$ ex: $\|f\|_{L^\infty_x(\mathbb{R}^2)} \not\lesssim \|f\|_{H^1(\mathbb{R}^2)}$

- <u>Nonhomog lin wave eqn</u>:

$$-\partial_t^2 u + \Delta u = F$$
$$(u, \partial_t u)|_{t=0} = (u_0, u_1) \qquad \text{on } \mathbb{R}^{3+1}$$

$$\Rightarrow u(t,x) = \frac{\partial}{\partial t} t \int_{S^2} u_0(x + t\omega)\, d\sigma(\omega)$$

Kirchoff

$$+ t \int_{S^2} u_1(x + t\omega)\, d\sigma(\omega)$$

$$+ \int_0^t t' \int_{S^2} F(t - t', x + t'\omega)\, d\sigma(\omega)\, dt'.$$

Idea: adaptation of Stein's counterexample for the spherical maximal function. (p469, Stein)

$$f \longmapsto \sup_{t>0} |A_t f| \quad \text{on } L^p$$

$$A_t f(x) = \int_{|y|=1} f(x-ty)\, d\sigma(y) = f * d\sigma_t$$

spherical average

$$\int g(x)\, d\sigma_t(x) = \int g(tx)\, d\sigma(x)$$

- Bounded: $d \geq 2$, $p > \frac{d}{d-1}$
- Fail: $d=1$, $p<\infty$
  $d \geq 2$; $p \leq \frac{d}{d-1}$

$d=1$: $A_t f(x) = \frac{1}{2}[f(x+t) + f(x-t)]$

$f$, positive, unbdd near $0$.

$$\Rightarrow \sup_t A_t f(x) = \infty$$

but $f \in L^p(\mathbb{R})$

$\underline{d \geq 2}$: $f(x) = \begin{cases} \dfrac{|x|^{1-d}}{\log 1/|x|} & \text{near } 0 \\ 0, & |x| \gg 1 \end{cases}$

$\Rightarrow \forall x$, $A_t f(x)$ is unbdd near $x$.

$\Rightarrow \sup_t A_t f(x) = \infty$ but $f \in L^p(\mathbb{R}^d)$, $p \leq \dfrac{d}{d-1}$.

---

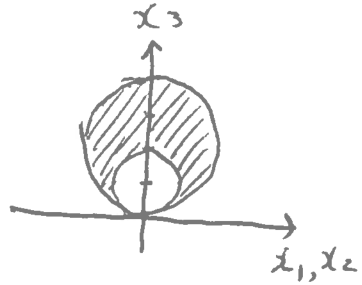
- Back to wave Strichartz.

$\underline{\text{Prop 2.10}}$: $u_0 = F = 0$.

$$u_1(x) = \frac{\mathbb{1}_A(x)}{|x|^2 (1 + \log|x|)^\alpha}, \quad \frac{1}{2} < \alpha \leq 1$$

$\underline{d=3}$

$$A = B(2\hat{e}_3, 2) \setminus B(\hat{e}_3, 1)$$




$\underline{\text{Claim:}}$ $u_1 \in L^2$ but

$u(t, t\hat{e}_3) = \infty$, $\forall\, 1 < t < 2$

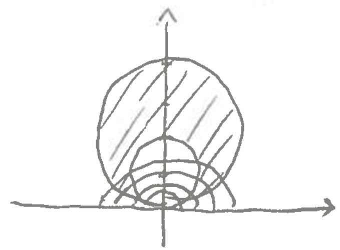
lin soln

$\Rightarrow$ (LHS) of ⊛ $= \infty$. $\Rightarrow$ ⊛ fails.

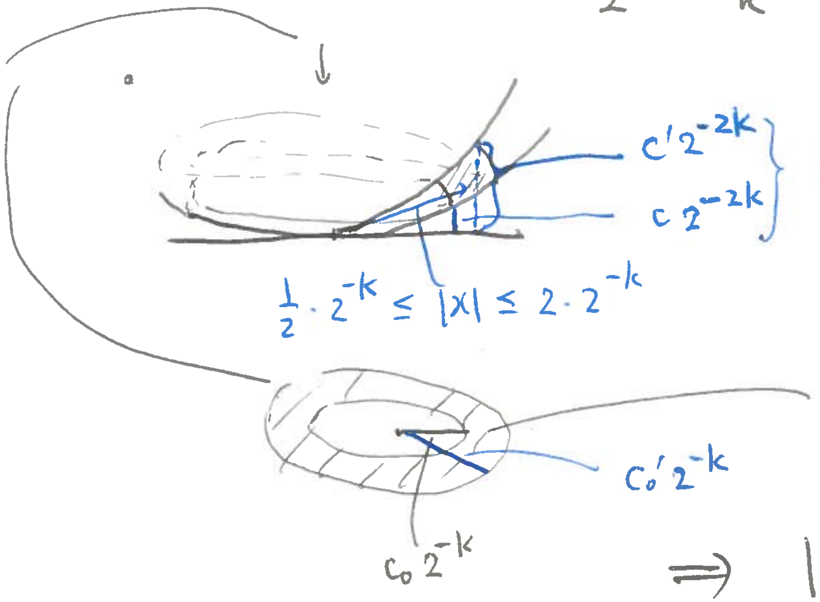
$\cdot$ Portion of $u_1$ in the dyadic shell $\{|x| \sim 2^{-k}\} \cap A = A_k$ lives in a set of volume $\sim 2^{-4k}$ and has height $\sim \dfrac{2^{2k}}{k^\alpha}$

$$\|u_1\|_{L^2} \sim \left( \sum_k \left( \frac{2^{2k}}{k^\alpha} \right)^2 \times 2^{-4k} \right)^{1/2}$$

$$\sim \left( \sum \frac{1}{k^{2\alpha}} \right)^{1/2} < \infty \,, \quad \alpha > 1/2$$



On $A_k$, $u_1(x) = \dfrac{1}{2^{-2k} k^\alpha}$.




height difference $\sim 2^{-2k}$

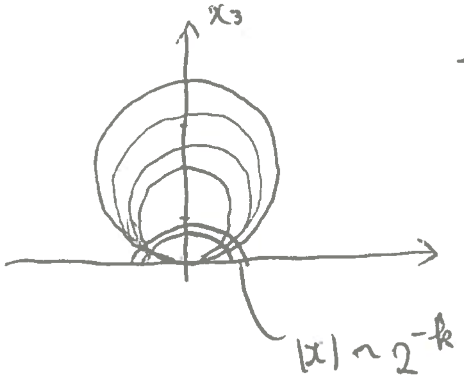
area $\sim 2^{-2k}$

$\Rightarrow |A_k| \sim 2^{-4k}$

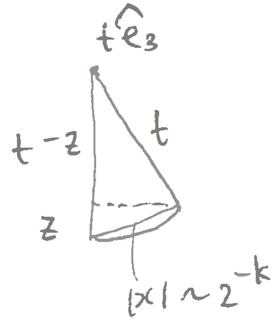

$$1^2 - (1-z)^2 = 2^{-2k} - z^2$$

$$\Rightarrow z \sim 2^{-2k}$$

- $u(t,x) = t\int_{S^2} u_1(x + t\omega)\, d\sigma(\omega)$

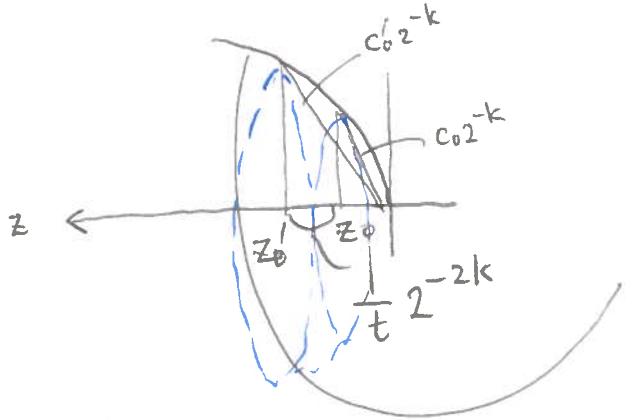



Take $x = t\hat{e}_3$, $\quad 1 < t < 2$




$$t^2 - (t-z)^2 = |x|^2 - z^2$$

$$\Rightarrow z \sim \frac{1}{t} 2^{-2k}$$




$f(z) = \sqrt{t^2 - (z-t)^2} \Leftarrow$ rotate around the $z$-axis

$$\text{surface area} = \int_{z_0}^{z_0'} 2\pi \cancel{f(z)} \sqrt{1 + \underbrace{(f'(z))^2}_{\frac{t-z}{\cancel{\sqrt{t^2-(z-t)^2}}}}}\, dz \sim 2^{-2k} \quad (\because 1 < t < 2)$$

$\Rightarrow$ contribution from $A_k$

$$2^{-2k} \times \frac{2^{2k}}{k^\alpha} = \frac{1}{k^\alpha} \notin l^1_k. \quad \alpha \le 1$$

$$\Rightarrow \quad u(t, t\hat{e}_3) = \infty, \qquad 1 < t < 2.$$

· Rmk: · localization in time does NOT help

· Assuming a higher regularity does NOT help

$$\|u\|_{L^2_t L^\infty_x} \not\lesssim \|u_1\|_{H^s}, \quad s \gg 1.$$

$\Leftarrow$ The estimate is $L^2$-scaling invariant.

· But if one assumes a higher regularity AND localization in time.

$$\Rightarrow \|u\|_{L^2_T L^\infty_x} \lesssim \underbrace{C(\varepsilon, T)}_{\begin{cases} C_\varepsilon (\log T)^{1/2}, & T \gg 1 \\ C_\varepsilon T^\varepsilon, & T \lesssim 1 \end{cases}} \|u_1\|_{H^\varepsilon}.$$

$\swarrow$ $TT^*$ argument.

$\swarrow$ scaling the $T \gg 1$ case

Q: What about replacing $L^\infty_x$ by the BMO norm?

or localize $\mathrm{supp}\, \hat{u}_1 \subset \{|\xi| \lesssim 1\}$?

ANS: Both fail!!

BMO estimate $\Longrightarrow$ freq localized estimate.
(if true)

Suppose $\| S(t)(u_0, u_1) \|_{L^2_t BMO_x} \lesssim \| (u_0, u_1) \|_{\dot{\mathcal{H}}^1}$

Suppose $u_0, u_1$ are frequency localized.

$\Rightarrow$ so is $u = S(t)(u_0, u_1)$

$\Rightarrow$ $\| u(t, \cdot) \|_{BMO_x} \sim \| u(t, \cdot) \|_{L^\infty_x}$

i.e. LP proj of a BMO function is in $L^\infty$.

<u>Goal</u>: Disprove freq localized estimate. (based Montgomery-Smith '98 Duke

Disprove

$$\left\| \int e^{it\sqrt{-\Delta}} F(t)\, dt \right\|_{L^2_x} \lesssim \| F \|_{L^2_t L^1_x}$$

loc in spatial freq

$|\xi| \sim 1$

$$\| e^{-it\sqrt{-\Delta}} f \|_{L^2_t L^\infty_x} \overset{?}{\lesssim} \| f \|_{L^2_x}$$

$$\| e^{-it\sqrt{-\Delta}} f \|_{L^2_t L^\infty_x} = \sup_{\substack{F \in L^2_t L^1_x \\ \|F\|_{L^2_t L^1_x} = 1}} \left| \langle e^{-it\sqrt{-\Delta}} f, F \rangle_{L^2_{t,x}} \right|$$

$$= \left| \int \langle f, e^{it\sqrt{-\Delta}} F \rangle \, dt \right|$$

$$= \left| \langle f, \int e^{it\sqrt{-\Delta}} F \, dt \rangle_{L^2_x} \right|$$

$$\leq \| f \|_{L^2_x} \underbrace{\left\| \int e^{it\sqrt{-\Delta}} F \, dt \right\|_{L^2_x}}_{\overset{?}{\lesssim} \| F \|_{L^2_t L^1_x} \leq 1}$$

$\operatorname{supp} \hat{f}(\xi) \subset \{ |\xi| \sim 1 \}$

$\Downarrow$

Can assume

$\operatorname{supp} \hat{F}(t, \xi) \subset \{ |\xi| \sim 1 \}$ $\forall t$.