Lec 12: 15/02/17 (Wed)

Our goal is to prove a.s. GWP.

$\Leftarrow$ suffices to prove "almost a.s. GWP."

$\forall\, T, \varepsilon > 0 \;\exists\, \Omega_{T,\varepsilon}$ with $P(\Omega_{T,\varepsilon}^c) < \varepsilon$

s.t. $\forall\, \omega \in \Omega_{T,\varepsilon}$, $\exists!$ soln u^ω with $(u^\omega, \partial_t u^\omega)\big|_{t=0} = (u_0^\omega, u_1^\omega)$

on $[-T, T]$

Fix $T, \varepsilon > 0$.

$$A_\lambda = \{\omega \in \Omega : \|z^\omega\|_{L^6_{T,x} \cap L^\infty_{T,x}} \le \lambda\}$$

$\Rightarrow$ Can choose $\lambda = \lambda(\|(u_0, u_1)\|_{\mathcal{H}^\varepsilon}, T) \gg 1$ s.t.

$$P(A_\lambda^c) < \frac{\varepsilon}{2}$$

i.e. on A_λ, $\quad E(v)(t) \le C_T. \qquad \forall\, |t| \le T.$

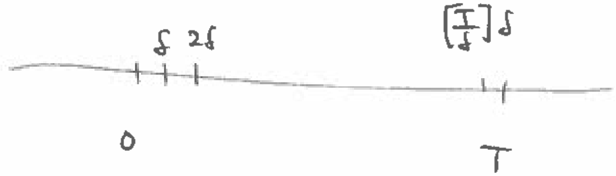
1. Choose local existence time

$$\delta = \delta\Big(\underbrace{\sup_{|t|\le T} \|(v, \partial_t v)(t)\|_{\dot{\mathcal{H}}^1}}_{\le C_T^{1/2}}\Big) > 0$$

2. By possibly making δ smaller, we have

$$P\Big(\underbrace{\|z\|_{L^3_{I_j} L^6_x} \ge 1}_{\text{set } =: B_j^c}\Big) \le e^{-c/\delta^{2/3}}, \qquad I_j = ((j-1)\delta, j\delta)$$

$$\Rightarrow \quad \forall \omega \in A_\lambda \cap \bigcap_{j=-[T/\delta]-1}^{[T/\delta]+1} B_j,$$



$$\exists!\ u = u^\omega \text{ on } [-T, T]$$

Lastly, $$P\Big(\big(\bigcap_j B_j\big)^c\Big) \lesssim \frac{T}{\delta} P(B_j^c) \le T\delta^{-1} e^{-c/\delta^{2/3}}$$

$< \varepsilon/2$ by choosing $\underset{\delta(T,\varepsilon)}{\underset{\shortparallel}{\delta}} \ll 1$.

$\boxtimes$

· Rmk: One actually need to work with v_N ← smooth

⇐ soln to $-\partial_t^2 v_N + \Delta v_N = (v_N + \underbrace{z_N^\omega}_{= P_{\leq N} z^\omega})^3$

to justify the switching of ∂_t and $\int dx$

in the probabilistic energy estimate (and let $N \to \infty$)

indep of N.

· $3 < p \leq 5$: $-\partial_t^2 u + \Delta u = \underbrace{|u|^{p-1}u}_{=: \mathcal{N}(u)}$ on $\mathbb{R}^3$ or $\mathbb{T}^3$

As before, set $u = z + v$

$$\Rightarrow \quad -\partial_t^2 v + \Delta v = \mathcal{N}(v+z).$$

only need some space-time control.

① a.s. LWP: Consider

$$\begin{cases} -\partial_t^2 v + \Delta v = \mathcal{N}(v+f) \\ (v, \partial_t v)\big|_{t=t_0} = (v_0, v_1) \end{cases} \quad \text{on } I \ni t_0$$

$$\|(\Gamma v, \partial_t \Gamma v)\|_{C_I \dot{\mathcal{H}}^1} + \|\Gamma v\|_{L_I^{2p/p-3} L_x^{2p}}$$

$$\lesssim \|(v_0, v_1)\|_{\dot{\mathcal{H}}^1} + \|N(v+f)\|_{L_I^1 L_x^2}$$

Strichartz pair $(s=1)$

$$\leq |I|^{\theta} \|v+f\|^{p}_{L_I^{\frac{2p}{p-3}} L_x^{2p}}$$

x: $\frac{1}{2} = \underbrace{\frac{1}{2p} + \cdots + \frac{1}{2p}}_{p \text{ factors}}$

t: $1 = p \cdot \frac{p-3}{2p} + \theta$

Need $\frac{p-3}{2} \leq 1 \iff p \leq 5$

$\theta = 0$ when $p = 5$

$$\lesssim \|(v_0, v_1)\|_{\dot{\mathcal{H}}^1} + |I|^{\theta} \|v\|^{p}_{L_I^{\frac{2p}{p-3}} L_x^{2p}} + |I|^{\theta} \|f\|^{p}_{L_I^{\frac{2p}{p-3}} L_x^{2p}}$$

- $p<5$: LWP on $|I| \leq \delta \sim \left(\|(v_0, v_1)\|_{\dot{\mathcal{H}}^1} + \|f\|^{p}_{L_I^{2p/p-3} L_x^{2p}}\right)^{-(p-1)}$
- $p=5$: δ depends on the profile of (v_0, v_1).

---

(2) a.s. GWP : $3 < p \leq 5$: ($p=5$: Oh - Pocovnicu '16)

side computation : $\partial_t (|v|^{p+1}) = \partial_t (v^2)^{\frac{p+1}{2}} = (p+1)|v|^{p-1} v \partial_t v$

Let $E(v) = \frac{1}{2}\int (\partial_t v)^2 + \frac{1}{2}\int |\nabla v|^2 + \frac{1}{p+1}\int |v|^{p+1}$

$$\Rightarrow \partial_t E(v) = \int \partial_t v \, (\underbrace{\partial_t^2 v - \Delta v}_{-|v+f|^{p-1}(v+f)} + |v|^{p-1} v)$$

$$F(x) = |x|^{p-1} x$$
$$F'(x) = p|x|^{p-1}$$
$$F''(x) = p(p-1)|x|^{p-3} x$$

$$\overset{\text{Taylor}}{\Longrightarrow} \; |v+f|^{p-1}(v+f) - |v|^{p-1} v = p|v|^{p-1} f + \frac{p(p-1)}{2!} |v+\theta f|^{p-3} (v+\theta f) \cdot f^2 .$$

$$\Rightarrow \partial_t E(v) = -p \int \partial_t v \cdot |v|^{p-1} f - \frac{p(p-1)}{2} \int \partial_t v \, \underbrace{|v+\theta f|^{p-2} (v+\theta f) \cdot f^2}_{\lesssim |v|^{p-2} f^2 + f^p .}$$

$$=: \mathrm{I} + \mathrm{II} .$$

$$\cdot \; |\mathrm{II}| \overset{C-S}{\lesssim} \left(\int (\partial_t v)^2 \right)^{1/2} \left(\|f(t)\|_{L^\infty_x}^4 \int |v|^{2(p-2)} + \|f(t)\|_{L^{2p}}^{2p} \right)^{1/2}$$

$$\lesssim \left(1 + \|f\|_{L^\infty_{T,x}}^{2\cdot} \right) E(v) + \|f(t)\|_{L^{2p}}^{2p} .$$

$2(p-2) \leq p+1$

$\Longleftrightarrow p \leq 5$

$$\cdot \int_0^t I = -p \int_0^t \int \underbrace{\partial_t v \, |v|^{p-1}}_{= \partial_t (|v|^{p-1} v)} f$$

$$\overset{\text{IBP}}{=} - \int_x |v|^{p-1} v \, f \Big|_0^t \qquad v(0) = 0$$

$$\left(\overset{\text{Young}}{\lesssim} \underbrace{\varepsilon \int_x |v(t)|^{p+1}}_{\varepsilon E(t)} + \underbrace{\frac{1}{\varepsilon} \int_x |f(t)|^{p+1}}_{= \frac{1}{\varepsilon} \|f\|^{p+1}_{L^\infty_T L^{p+1}_x}} \right.$$

$\varepsilon E(t) \Leftarrow$ hide it on LHS

$$+ p \int_0^t \int_x |v|^{p-1} v \underbrace{\partial_t f}_{=: \langle \nabla \rangle \tilde{f}.}$$

① place $\langle \nabla \rangle^{s-} \tilde{f}$ in $L^\infty_{T,x}$

$$\int_x \langle \nabla \rangle^{1-s+} (|v|^{p-1} v) \, \langle \nabla \rangle^{s-} \tilde{f}$$

② $\left(\int |v|^{(p-1)\frac{p+1}{p-1}} \right)^{\frac{p-1}{p+1}} \lesssim E^{\frac{p-1}{p+1}}$

If $f = z$, then

$$\partial_t z = \langle \nabla \rangle \tilde{z}$$

$$\cdot \ \tilde{z} = \tilde{S}(t)(u_0^\omega, u_1^\omega)$$

$$= -\frac{|\nabla|}{\langle \nabla \rangle} \sin(t|\nabla|) u_0^\omega + \frac{\cos t|\nabla|}{\langle \nabla \rangle} u_1^\omega.$$

Satisfies the same probabilistic Strichartz estimates

3 $\left(\int \left(\langle \nabla \rangle^{1-s+} v \right)^{\frac{p+1}{2}} \right)^{\frac{2}{p+1}}$

low freq easy

$= \| \langle \nabla \rangle^{1-s+} v \|_{L^{\frac{p+1}{2}}}$ $\Rightarrow$ Assume $\langle \nabla \rangle \sim |\nabla|$

$\lesssim \| \nabla v \|_{L^2}^{\theta} \, \| v \|_{L^{p+1}}^{1-\theta}$

$$\frac{2}{p+1} = \frac{\theta}{2} + \frac{1-\theta}{p+1} \Rightarrow \theta = \frac{2}{p-1}$$

$$1-s+ = \theta \cdot 1 + (1-\theta) \cdot 0$$

$$\boxed{s} > 1-\theta = 1 - \frac{2}{p-1} = \boxed{\frac{p-3}{p-1}}$$

$p=3$: $s>0$ Burq-Tzvet '14

$p=5$: $s>1/2$ O'-P '16

$3<p<5$: Xia. Lührman-Mendelson (weaker)

$\lesssim E^{\frac{\theta}{2} + \frac{1-\theta}{p+1}}$, $\quad \frac{\theta}{2} + \frac{1-\theta}{p+1} \le \frac{2}{p+1}$

$$\Rightarrow \int_0^t I_2 \lesssim \| \langle \nabla \rangle^{s-} f \|_{L^\infty_{T,x}} \int_0^t E(t')\, dt'.$$

$\Rightarrow$ Apply Gronwall.