Lec 11 : 13/02/17 (Mon)

Prop 2.7: $f \in H^{\varepsilon}(\mathbb{T}^d)$, $\varepsilon > 0$.

Given $2 \le q < \infty$, we have

$$P\left(\|S(t) f^{\omega}\|_{L^q_t L^\infty_x([-T,T]\times\mathbb{T}^d)} > \lambda \right) \le C \exp\left(-c \frac{\lambda^2}{T^{2/q}\|f\|^2_{H^{\varepsilon}}} \right)$$

Pf: By Sobolev embedding,

$$\|S(t) f^{\omega}\|_{L^q_T L^\infty_x} \lesssim \|\langle\nabla\rangle^{\varepsilon} S(t) f^{\omega}\|_{L^q_T L^r_x} \qquad \varepsilon r > d$$

$$= \|S(t)(\langle\nabla\rangle^{\varepsilon} f^{\omega})\|_{L^q_T L^r_x}$$

$\Rightarrow$ Apply Prop 2.4.

$\square$

- The same estimate holds if $q = \infty$.

By Sobolev in time.

$$\| S(t) f^\omega \|_{L^\infty_T L^r_x} \lesssim \| \langle \partial_t \rangle^\varepsilon \| S(t) f^\omega \|_{L^r_x} \|_{L^q_T} \qquad \varepsilon q > 1$$

(*)

See also Oh-Pocovnicu '16 JMPA.

If we can place $\langle \partial_t \rangle^\varepsilon$ inside L^r_x-norm, we can replace $\langle \partial_t \rangle^\varepsilon$ by $\langle \nabla \rangle^{2\varepsilon}$ (Schrödinger case)

ex: $\varepsilon = 1$. consider

$$\| \partial_t \| F(t) \|_{L^r_x} \|_{L^q_T} = \| \lim_{h \to 0} \frac{\| F(t+h) \|_{L^r_x} - \| F(t) \|_{L^r_x}}{h} \|_{L^q_T}$$

$$\leq \| \lim_{h \to 0} \| \frac{F(t+h) - F(t)}{h} \|_{L^r_x} \|_{L^q_T}$$

$$= \| \| \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} \|_{L^r_x} \|_{L^q_T}$$

for nice F.

Justify (*) by interpolation.

2.3 Probabilistic well-posedness of NLW on $\mathbb{T}^3 / \mathbb{R}^3$

- Cubic NLW : $\begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \end{cases}$ defocusing (needed only for the global argument)

Recall $s_{crit} = 1/2$.

Assume $(u_0, u_1) \in \mathcal{H}^s \setminus \mathcal{H}^{1/2}$, $0 \le s < 1/2$

$\Rightarrow$ Randomize (u_0, u_1) and consider

$$(*) \quad \begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ (u, \partial_t u)\big|_{t=0} = (u_0^\omega, u_1^\omega) \end{cases}$$

Thm 2.8: The Cauchy problem $(*)$ is almost surely locally well-posed. More precisely, given small $T > 0$, $\exists\, \Omega_T$ (also depends on the fixed (u_0, u_1)) s.t. ① $P(\Omega_T^c) \le C e^{-c/T^\gamma}$ for some $\gamma > 0$.

② For each $\omega \in \Omega_T$, $\exists!$ soln $\underset{u^\omega}{\underset{\parallel}{u}}$ to $(*)$ with $(u, \partial_t u)\big|_{t=0} = (u_0^\omega, u_1^\omega)$

Rmk: • $\Sigma = \bigcup_{0<T\ll 1} \Omega_T$. Then, $P(\Sigma) = 1$.

• For each $\omega \in \Sigma$, u exists on $[-T_\omega, T_\omega]$.

• u lies in

$$S(t)(u_0^\omega, u_1^\omega) + C_T \underbrace{H^1}_{\text{smoother}}$$

Pf: Write $u = \underbrace{S(t)(u_0^\omega, u_1^\omega)}_{=: z^\omega} + v$ $\leftarrow \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} u^3(t')\,dt'$

$\Rightarrow$ v satisfies

$$\begin{cases} -\partial_t^2 v + \Delta v = (v+z)^3. \\ (v, \partial_t v)|_{t=0} = 0 \end{cases}$$

Let $\Gamma v = -\int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} (v+z)^3(t')\,dt'$.

$$\Rightarrow \|\Gamma v\|_{L_T^\infty \dot{H}^1} \le \Big\| \int_0^t \|(v+z)^3(t')\|_{L_x^2}\, dt' \Big\|_{L_T^\infty}$$

$$= \|(v+z)^3\|_{L_T^1 L_x^2} = \|v+z\|_{L_T^3 L_x^6}^3$$

$$\lesssim \|v\|^3_{L^3_T L^6_x} + \|z\|^3_{L^3_T L^6_x}$$

$$\overset{\text{Sob in } x}{\lesssim} T\,\|v\|^3_{L^\infty_T \dot H^1} + \underbrace{\|z\|^3_{L^3_T L^6_x}}_{\le 1 \text{ outside a set of prob. } \le C e^{-\frac{c}{T^\delta \|(u_0,u_1)\|^2_{L^2}}}}$$

- A difference estimate follows in a similar manner:

$$\|\Gamma v_1 - \Gamma v_2\|_{L^\infty_T \dot H^1} \lesssim T^{1/3}\Big(T^{2/3}\|v_1\|^2_{L^\infty_T \dot H^1} + T^{2/3}\|v_2\|^2_{L^\infty_T \dot H^1} + \|z\|^2_{L^3_T L^6_x}\Big)\|v_1 - v_2\|_{L^\infty_T \dot H^1}.$$

$\Rightarrow$ Γ^ω is a contraction on $B_{O(1)}$ in $L^\infty \dot H^1$ for every $\omega \in \Omega_T$

- Also, $$\|v(t)\|_{L^2_x} \overset{\text{Mink}}{\le} \int_0^t \underbrace{\|\partial_t v(t')\|_{L^2}}_{\text{controlled}}\, dt'$$

$$\overset{\text{FTC}}{=} \Big\|\int_0^t \partial_t v\, dt'\Big\|_{L^2_x}$$

$$\partial_t v = -\int_0^t \cos\big((t-t')|\nabla|\big)(v+z)^3(t')\,dt' \in C_T L^2_x$$

$\square$

The same proof works for

$$\begin{cases} -\partial_t^2 v + \Delta v = (v + z^\omega)^3 \\ (v, \partial_t v)\big|_{t=0} = (v_0, v_1) \in \mathcal{H}^1 \end{cases}$$

by performing a fixed pt argument in $B_{O(\|(v_0,v_1)\|_{\mathcal{H}^1})}$

$$\Gamma v = S(t)(v_0, v_1) - \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} (v+z)^3(t')\, dt'.$$

Rmk: In terms of the original NLW, we have a.s. LWP for

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ (u, \partial_t u) = \underset{\substack{\Uparrow \\ \mathcal{H}^1}}{(v_0, v_1)} + \underbrace{(u_0^\omega, u_1^\omega)}_{\text{rough \& random}} \end{cases}$$

$0 < s \leq 1$

Thm 2.9: The defocusing cubic NLW is a.s. GWP (with respect to (u_0^ω, u_1^ω).)

For $s=0$, see Burq–Tzvetkov JEMS '14

Pf: $E(u, \partial_t u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int (\partial_t u)^2 + \frac{1}{4}\int u^4$

- Estimate the growth of $E(v(t))$ ← in H^1.

$$\Rightarrow \partial_t E(v) = \int \partial_t v \big(\underbrace{\partial_t^2 v - \Delta v}_{= -(v+z)^3} + v^3 \big)\, dx$$

No v^3 term

$$\overset{C-S}{\leq} \left(\int (\partial_t v)^2 dx \right)^{1/2} \left(\int \left(v^2|z| + |v|z^2 + |z|^3 \right)^2 dx \right)^{1/2}$$

$z^{1/2} z^{3/2}$

$|v|z^2 \lesssim v^2|z| + |z|^3$

$v^4 z^2$

$$\lesssim E(v)\left(1 + \|z\|^2_{L^\infty_{T,x}}\right) + \|z\|^6_{L^\infty_T L^6_x}$$

Gronwall $\Rightarrow$

$$E(v)(t) \lesssim \|z\|^6_{L^6_{T,x}}\, e^{c\int_0^t 1 + \|z\|^2_{L^\infty_{T,x}} dt'}, \quad |t| \leq T.$$