Lec 10 : 08/02/17 (Wed)

· Probabilistic Strichartz estimate

$S(t) = e^{it\Delta}$ linear Schrödinger semigroup

· Strichartz estimates on $\mathbb{R}^d$

$$\| S(t) f \|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$$

$2 \le q, r \le \infty$, $(q,r,d) \ne (2,\infty,2)$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

Prop 2.4: $f \in L^2(\mathbb{T}^d)$

Given $2 \le q, r < \infty$, we have

$$P\left(\| S(t) f^\omega \|_{L^q_t L^r_x([-T,T]\times\mathbb{T}^d)} > \lambda \right) \le C\exp\left(- c\frac{\lambda^2}{T^{\frac{2}{q}} \|f\|^2_{L^2}} \right)$$

for all $\lambda > 0$ and all $T > 0$.

$\left(\to 0 \text{ as } \lambda \to \infty \text{ or } T \to 0 \right)$

<u>Rmk</u>: • Set $\lambda = T^{\theta}\|f\|_{L^2}$, $\theta < \frac{1}{q}$.

Prop 2.4 $\Rightarrow$ $\|S(t)f^{\omega}\|_{L^q_T L^r_x} \leq T^{\theta}\|f\|_{L^2}$

outside a set of prob $\leq c\, e^{-cT^{2(\theta-\frac{1}{q})}}$ $(\to 0$ as $T \to 0)$

• Fix $T > 0$. Given $\varepsilon > 0$,

$$\|S(t)f^{\omega}\|_{L^q_T L^r_x} \leq C_T \left(\log \frac{1}{\varepsilon}\right)^{1/2} \|f\|_{L^2}$$

outside a set of prob $< \varepsilon$.

$\Bigg($ or $\leq C_T K \|f\|_{L^2}$
prob $< e^{-cK^2}$

<u>Pf</u>: Let $p \geq \max(q, r)$.

$$\Big\| \|S(t)f^{\omega}\|_{L^q_T L^r_x} \Big\|_{L^p(\Omega)} \overset{\text{Mink}}{\leq} \Big\| \|\underbrace{S(t)f^{\omega}}_{\sum e^{-it|n|^2}\hat{f}(n)e^{in\cdot x} g_n}\|_{L^p(\Omega)} \Big\|_{L^q_T L^r_x}$$

$$\sum \underbrace{e^{-it|n|^2}\hat{f}(n)e^{in\cdot x}}_{=C_n(t,x)} g_n$$

$$\lesssim \sqrt{p}\, \Big\| \|\hat{f}(n)\|_{\ell^2_n} \Big\|_{L^q_T L^r_x}$$

$$= \sqrt{p}\, T^{\frac{1}{q}} \|f\|_{L^2_x} \Rightarrow \text{Apply Chebyshev} \quad \square$$

- The same estimate holds for the half wave operator $e^{\pm it|\nabla|}$

$\Rightarrow$ Also for $\cos t|\nabla| = \dfrac{e^{it|\nabla|} + e^{-it|\nabla|}}{2}$.

- Next, we consider $\dfrac{\sin t|\nabla|}{|\nabla|}$.

<u>Prop 2.5</u>: (probabilistic Strichartz estimate for wave eqn)

Let $(f_0, f_1) \in \mathcal{H}^0 = L^2 \times H^{-1}$.

$$P\Big(\| \underset{\left(\cos t|\nabla|,\ \frac{\sin t|\nabla|}{|\nabla|}\right)}{\underset{\shortparallel}{S(t)}} (f_0^\omega, f_1^\omega) \|_{L^q_t L^r_x (I \times \mathbb{T}^d)} > \lambda \Big)$$

$I = [a, b]$

$$\leq C \exp\left(- c \frac{\lambda^2}{\max(1, a^2, b^2) |I|^{2/q} \|(f_0, f_1)\|^2_{\mathcal{H}^0}} \right).$$

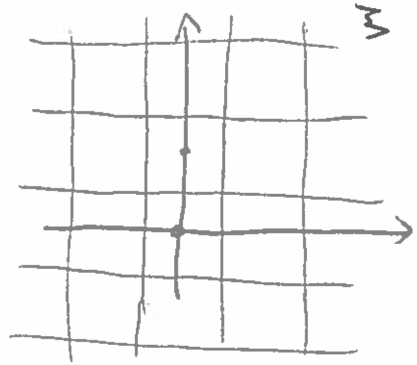
<u>Pf</u>: Repeat the proof of Prop 2.4, noting

$$\left| \frac{\sin t|n|}{|n|} \Big|_{n=0} \right| = |t| \leq \max(1, |a|, |b|)$$

See O. Pocovnicu arXiv '15.

· $\mathbb{R}^d$?

$$f(x) = \int \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi$$



$$= \sum_{n\in\mathbb{Z}^d} \int_{Q_n} \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi$$

$$Q_n = n + \left(-\tfrac{1}{2}, \tfrac{1}{2}\right]^d$$

$$\Rightarrow \quad \mathbb{R}^d_\xi = \bigcup_{n\in\mathbb{Z}^d} Q_n \quad \text{Wiener decomposition}$$

For practical purpose, let $\psi \in C_c^\infty(\mathbb{R}^d; [0,1])$ s.t

$$\operatorname{supp} \psi \subset [-1,1]^d$$

and $\displaystyle \sum_{n\in\mathbb{Z}^d} \underbrace{\psi(\xi - n)} \equiv 1$ smoothed version of 1_{Q_n}

$$\Rightarrow \quad f(x) = \sum_{n\in\mathbb{Z}^d} \underbrace{\psi(D-n) f}_{= \int \psi(\xi-n)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi}$$

Randomization of a function on $\mathbb{R}^d$

$$f^\omega = \sum_{n \in \mathbb{Z}^d} g_n \psi(D-n) f.$$

Rmk: · Why unit scale? ⇐ a matter of choice.

Can introduce a similar randomization based on dilated cubes. (e.g. Bényi-O-P '15)

· translation on $\mathbb{R}^d_\xi$ $\iff$ modulation $e^{ix\cdot\xi_0}$ on $\mathbb{R}^d_x$

$\rightsquigarrow$ modulation space $\|f\|_{M^{p,q}_s} = \big\| \langle n\rangle^s \|\psi(D-n)f\|_{L^p_x} \big\|_{\ell^q_n}$

Wiener amalgam space

$$\|f\|_{W^{p,q}} = \big\| \|\psi(D-n)f\|_{\ell^q_n} \big\|_{L^p_x}$$

(Rmk: On $\mathbb{T}^d$,

$$\|f\|_{M^{p,q}} = \|f\|_{W^{p,q}} = \|f\|_{\mathcal{F}L^q} = \|\hat{f}(n)\|_{\ell^q_n}$$

· What about randomization based on the LP-decomposition?

$$f = \sum_{\substack{N \ge 1 \\ \text{dyadic}}} P_N f, \quad P_N \overset{LP}{=} \text{proj onto } \{|\xi| \sim N\}.$$

Let's consider

$$f^{\omega} = \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \pm P_N f.$$

$1 < p < \infty$

$$\| f^{\omega} \|_{L^p} \overset{\text{LP theory}}{\sim} \Big\| \Big(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} | P_N f |^2 \Big)^{1/2} \Big\|_{L^p}$$

$$\sim \| f \|_{L^p} \Rightarrow \text{No gain of integrability!}$$

· Back to Wiener randomization on $\mathbb{R}^d$.

We can <u>not</u> use the finiteness of the domain.

$\Rightarrow$ Use Bernstein's inequality

$$\| P_N f \|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| f \|_{L^p}$$

In fact,

really smooth cutoff

$$\| \mathcal{F}^{-1} (\mathbb{1}_Q f) \|_{L^q} \lesssim |Q|^{\frac{1}{p} - \frac{1}{q}} \| f \|_{L^p}.$$

ξ_0 = center of Q

$$\| \underbrace{e^{ix \cdot \xi_0} \mathcal{F}^{-1} (\mathbb{1}_Q f)}_{} \|_{L^q} \quad \text{centered at } 0 \text{ on } \mathbb{R}^d_{\xi}$$

$$\Rightarrow \quad \| \psi(D-n) f \|_{L^r_x} \lesssim \| \psi(D-n) f \|_{L^2_x}$$

$\Rightarrow$ For $p \geq \max(q, r)$

$$\| \, \| S(t) f^{\omega} \|_{L^q_T L^r_x} \|_{L^p(\Omega)} \overset{\text{Mink}}{\leq} \Big\| \, \| \sum_n \underbrace{(\psi(D-n) S(t) f)}_{= c_n} g_n \|_{L^p(\Omega)} \Big\|_{L^q_T L^r_x}$$

$$\leq \sqrt{p} \, \| \, \| \psi(D-n) S(t) f \|_{\ell^2_n} \|_{L^q_T L^r_x}$$

$$\overset{\text{Mink}}{\leq} \sqrt{p} \, \| \, \underbrace{\| \psi(D-n) S(t) f \|_{L^r_x}}_{\underset{\text{Bernstein}}{\lesssim} \| \psi(D-n) \cancel{S(t)} f \|_{L^2_x}} \|_{L^q_T \ell^2_n}$$

unitarity

$$\lesssim \sqrt{p} \, \| \, \| f \|_{L^2(\mathbb{R}^d)} \|_{L^q_T}$$

$$\lesssim \sqrt{p} \, T^{\frac{1}{q}} \| f \|_{L^2(\mathbb{R}^d)} \Rightarrow \text{Prob. Strichartz estimate on } \mathbb{R}^d.$$

Prop 2.6: (Global-in-time prob. Strichartz estimate on $\mathbb{R}^d$)

Let (q, r) be Schrödinger admissible.

$q, r < \infty$.

Let $\tilde{r} \geq r$. Then,
finite.

$$P\left(\| S(t) f^\omega \|_{L_t^q L_x^{\tilde{r}} (\mathbb{R} \times \mathbb{R}^d)} > \lambda \right) \leq C \exp\left(-c \frac{\lambda^2}{\|f\|_{L^2}^2} \right)$$

$\Leftarrow$ Proof uses deteministic Strichartz estimate.