Lec 2 11/01/17 (Wed)

- Finite dim'l Hamiltonian dynamics on $\mathbb{R}^{2n}$

$$\partial_t p_j = \frac{\partial H}{\partial q_j}, \qquad \partial_t q_j = -\frac{\partial H}{\partial p_j}, \qquad j = 1, \cdots, n$$

vec. field $= X$, $\quad X_j = \dfrac{\partial H}{\partial q_j}$

$$X_{n+j} = -\frac{\partial H}{\partial p_j}$$

① · Liouville's thm

$$\frac{d}{dt} \mathrm{vol} = \mathrm{div}\, X = \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} X_j + \frac{\partial}{\partial q_j} X_{n+j} \right]$$

$$= \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} \frac{\partial H}{\partial q_j} + \frac{\partial}{\partial q_j} \left(-\frac{\partial H}{\partial p_j} \right) \right] = 0.$$

$\Rightarrow$ $dp\,dq = \prod_{j=1}^{n} dp_j\, dq_j$ is <u>invariant</u>.

② Hamiltonian H is <u>conserved</u>

$$\frac{d}{dt} H(p(t), q(t)) = \frac{\partial H}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial H}{\partial q} \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial H}{\partial q} \left(-\frac{\partial H}{\partial p} \right) = 0$$

Gibbs meas: $d\mu = Z_\beta^{-1} e^{-\beta H(p,q)} dp\,dq$, $\beta > 0$ ← inverse temp

$$Z_\beta = \int_{\mathbb{R}^{2n}} e^{-\beta H(p,q)} dp\,dq < \infty$$

= partition function

is invariant.

Invariance: $\Phi(t) : (p(0), q(0)) \longmapsto (p(t), q(t))$

$$\mu(\Phi(-t)A) = \mu(\{(p_0, q_0) \in \Phi(-t)A\})$$

$$= \mu(\{\Phi(t)(p_0, q_0) \in A\})$$

$$= Z_\beta^{-1} \int_A e^{-\beta \underbrace{H(p(t), q(t))}_{= H(p_0, q_0)}} \underbrace{dp(t)\,dq(t)}_{= dp_0 dq_0}$$

$dp(t)\,dq(t) = \Phi(t)_* (dp\,dq)$

$$= \mu(A)$$

Rmk: Suppose that F is conserved under the dynamics

$$\Rightarrow \quad d\mu_F = Z^{-1} e^{-F(p,q)} \, dp \, dq$$

is invariant (for nice F).

Q: Why do we care about invariant meas?

1 can view the system as a dynamical system with meas-preserving transf $T := \Phi(1)$

$$T : (p(0), q(0)) \mapsto (p(t), q(t)) \big|_{t=1}$$

- Poincaré recurrence thm:

(i) $\forall$ meas A with $\mu(A) > 0$.

$\exists N \in \mathbb{N}$ $\mu(A \cap T^{-N}A) > 0$.

(ii) a.e. $(p,q) \in \mathbb{R}^{2n}$ are stable according to Poisson. $\Leftarrow$ Zhidkov '01 Lec. Note in Math

i.e. $\exists \{t_j\}_{j \in \mathbb{N}} \to \infty$

s.t. $(p(t_j), q(t_j)) \longrightarrow (p(0), q(0))$

Also, Furstenberg's multiple reccurence thm

$\forall A$ with $\mu(A) > 0$ and $k \in \mathbb{N}$.

$\exists N \in \mathbb{N}$ s.t.

$$\mu(A \cap T^{-N}A \cap T^{-2N}A \cap \cdots \cap T^{-kN}A) > 0.$$

② μ is "supposed" to describe a "typical" long time behavior of solutions.

(NLS) $i\partial_t u + \Delta u = \pm |u|^{p-1}u$ on $\mathbb{T}^d$.

$$H(u) = \frac{1}{2}\int |\nabla u|^2 \pm \frac{1}{p+1}\int |u|^{p+1}$$

(NLS) $\Longleftrightarrow$ $\partial_t u = -i \dfrac{\partial H}{\partial \bar{u}}$.

Symplectic space $L^2(\mathbb{T}^d)$

Symplectic form $\omega(f,g) = \operatorname{Im}\int_{\mathbb{T}^d} f(x)\overline{g(x)}\,dx$

$$dH\big|_u(\phi) = \omega\left(\phi, -i\frac{\partial H}{\partial \bar{u}}\right)$$

↑ Gâteaux derivative $= \dfrac{d}{dt} H(u + \varepsilon\phi)\Big|_{\varepsilon = 0}$

On Fourier side:

$$p_n = \operatorname{Re} \hat{u}_n \quad , \qquad q_n = \operatorname{Im} \hat{u}_n$$

Linear case: $$H(p,q) \underset{\substack{\uparrow \\ \text{Plancherel}}}{=} \frac{1}{2} \sum |n|^2 (p_n^2 + q_n^2)$$

Plancherel: $$\int_{\mathbb{T}^d} |\nabla u|^2 dx = \sum_{n \in \mathbb{Z}^d} |n|^2 |\hat{u}_n|^2$$

Lin Schrödinger eqn: $\partial_t u = i \Delta u$

$$\iff \partial_t \hat{u}_n = -i |n|^2 \hat{u}_n , \qquad n \in \mathbb{Z}^d$$

$$\iff \partial_t \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} \dfrac{\partial H}{\partial q_n} \\ -\dfrac{\partial H}{\partial p_n} \end{pmatrix}$$

Gibbs meas $\quad d\mu \text{ "=" } Z^{-1} e^{-H(u)} du$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}} \underbrace{e^{-\frac{1}{2}\int |\nabla u|^2} du}_{\text{Wiener meas on } \mathbb{T}^d}.$$

In fact, we consider

$$d\mu = Z^{-1} \underbrace{e^{-H(u) - \frac{1}{2}M(u)}}_{} du, \quad M(u) = \int |u|^2 dx$$

in order to avoid a problem at $n=0$.

$$e^{\mp \frac{1}{p+1}\int |u|^{p+1}} e^{-\frac{1}{2}\|u\|_{H^1}^2} du$$

- Gaussian measures on periodic functions / distributions

Consider $\quad d\rho_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du, \quad s \in \mathbb{R} \quad \mathcal{D}'(\mathbb{T}^d)$

$$\left(\text{on } \mathbb{R}^n, \quad d\rho = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx \right)$$

as a limit of

$$d\rho_{s,N} = Z_N^{-1} e^{-\frac{1}{2}\|P_{\le N} u\|_{H^s}^2} d(P_{\le N} u)$$

$$\left(P_{\le N} u = \sum_{|n|\le N} \hat{u}_n e^{in\cdot x} \right.$$

$$= Z_N^{-1} e^{-\frac{1}{2}\sum_{|n|\le N} \langle n\rangle^{2s} |\hat{u}_n|^2} \prod_{|n|\le N} \underbrace{d\hat{u}_n}$$

Lebesgue meas on $\mathbb{C} \cong \mathbb{R}^2$.

$$\langle n\rangle = (1+|n|^2)^{1/2}$$

$$= Z_N^{-1} \prod_{|n|\le N} \underbrace{e^{-\frac{1}{2}\langle n\rangle^{2s}|\hat{u}_n|^2} d\hat{u}_n}$$

$$e^{-\frac{1}{2}(x^2+y^2)} dx dy$$

Gaussian meas on $\mathbb{C}$

mean 0, var $= 2\langle n\rangle^{-2s}$

$$c_n e^{-\frac{1}{2}|g_n|^2} dg_n$$

$g_n = g_n(\omega) : \omega \in \Omega \longmapsto g_n(\omega) \in \mathbb{C}$

$\mathbb{C}$-valued Gaussian random variable.

$$\Rightarrow \quad \hat{u}_n = \frac{g_n(\omega)}{\langle n \rangle^s}, \quad |n| \le N$$

$\{g_n\}_{|n|\le N}$ = seq of indep standard $\mathbb{C}$-valued Gaussian r.v.'s.

$$\text{mean} = \mathbb{E}[g_n] = \int_\Omega g_n(\omega)\, dP(\omega) = 0$$

$$\text{variance} = \mathbb{E}\big[|g_n - \underbrace{\mathbb{E}(g_n)}_{=0}|^2\big] = \int_\Omega |g_n|^2 dP = 2.$$

$$= \int_\Omega (\mathrm{Re}\, g_n)^2 dP + \int_\Omega (\mathrm{Im}\, g_n)^2 dP$$

$$= 1 + 1$$

$$e^{-\frac{1}{2}|g_n|^2} dg_n = e^{-\frac{1}{2}(\mathrm{Re}\, g_n)^2} d(\mathrm{Re}\, g_n) \times e^{-\frac{1}{2}(\mathrm{Im}\, g_n)^2} d(\mathrm{Im}\, g_n)$$

Take $N \to \infty$.

Problem: We can not take a limit as $N \to \infty$ in $H^s(\mathbb{T}^d)$.

$$\rho_{s,N} \sim u^N = \sum_{|n| \le N} \frac{g_n(\omega)}{\langle n \rangle^s} e^{in\cdot x}$$

Let $\sigma \in \mathbb{R}$, $M > N$

$$\mathbb{E}\left[\| u^M - u^N \|^2_{H^\sigma(\mathbb{T}^d)} \right]$$

$$= \mathbb{E} \sum_{N < |n| \le M} \frac{|g_n|^2}{\langle n \rangle^{2s-2\sigma}} = 2 \sum_{N < |n| \le M} \frac{1}{\langle n \rangle^{2s-2\sigma}} \to 0$$

iff $2s - 2\sigma > d$

$$\Longleftrightarrow \quad \boxed{\sigma < s - \frac{d}{2}}$$

Moral: topology matters!

Under $\sigma < s - \frac{d}{2}$,

u^N converges in $L^2(\Omega; H^\sigma(\mathbb{T}^d))$.

to $$u = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle^s} e^{in\cdot x}$$

- $d\rho_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du$ is the induced probability meas under the map

$$\omega \in \Omega \xrightarrow{u} u = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle^s} e^{in\cdot x} \in H^\sigma(\mathbb{T}^d)$$

P $\qquad\qquad \rho_s = P \circ u^{-1}$

Abstract Wiener space: Gross '65, Kuo '75 Lec. Notes in Math

$H = \infty$-dim'l separable Hilbert space

Consider " $d\rho = Z^{-1} e^{-\frac{1}{2}\|u\|_H^2} du$ "

$\Leftarrow$ NOT countably additive if $\dim H = \infty$

- Must enlarge the space $H \subset B$ ← Banach space

dense & conti.

<u>Def</u>: We say (H, B, ρ) is an abstract Wiener space

if
$$\int_B e^{i\langle u, \varphi\rangle_{B, B^*}} \, d\rho(u) = e^{-\frac{1}{2}\|\varphi\|_{H^*}^2}$$

for all $\varphi \in B^* \subset H^* = H$

(i.e. $\langle u, \varphi\rangle$ is a stand. Gaussian r.v. $\forall \varphi$.

u ↑ random

<u>Another equiv def</u>: $B =$ completion of H under a "measurable" seminorm $\|\cdot\|_B$

$\forall \varepsilon > 0 \ \exists P_\varepsilon \in \{$ collection of finite dim'l proj$\}$

s.t. $\rho(\|Pu\|_B > \varepsilon) < \varepsilon \quad \forall \underset{\cup}{P}$ s.t. $P \perp P_\varepsilon$.

e.g. H^s,

$$P_\varepsilon \sim P_{\leq N}$$

$$P^\perp P_\varepsilon \sim P_{N < \cdot \leq M}$$

Want $\rho\left(\| P_{>N} u \|_B > \varepsilon \right) < \varepsilon$.

ex: $B = W^{\sigma, p}$, $\sigma < s - \frac{d}{2}$

H, Hilbert space

"$d\rho = Z^{-1} e^{-\frac{1}{2}\|u\|_H^2} du$"

$\Rightarrow$ enlarge H to B = completion of H under a measurable norm $\|\cdot\|_B$

ex: $H = H^s(\mathbb{T}^d)$

$B = H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$, $u^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx}$

$W^{\sigma,p}(\mathbb{T}^d)$ $\quad \|u\|_{W^{\sigma,p}} = \|\mathcal{F}^{-1}(\langle n\rangle^\sigma \hat{u}(n))\|_{L^p_x(\mathbb{T}^d)}$, $p \le \infty$

Besov space

$B^\sigma_{p,q}(\mathbb{T}^d)$, $p \le \infty$, $q < \infty$

$\|u\|_{B^\sigma_{p,q}} = \big\| N^\sigma \|P_N u\|_{L^p_x(\mathbb{T}^d)} \big\|_{\ell^q_k(\mathbb{Z}_{\ge 0})}$ $\left(P_N u \text{"="} \sum_{|n|\sim N} \hat{u}_n e^{in\cdot x} \right)$

$\Downarrow$

$\frac{1}{2}N \le |n| \le 2N$

"N, dyadic" $\quad N = 2^k$, $k \ge 0$

$k \in \mathbb{Z}$

- $B^\sigma_{2,2} = H^\sigma$

- Fourier - Lebesgue space : $\mathcal{F}L^{\sigma,p}(\mathbb{T}^d)$

$$\|u\|_{\mathcal{F}L^{\sigma,p}(\mathbb{T}^d)} = \|\langle n\rangle^{\sigma}\hat{u}_n\|_{\ell^p_n(\mathbb{Z}^d)}$$

- $\mathcal{F}L^{\sigma,2} = H^{\sigma}$.

$$\underline{(\sigma-s)p < -d}$$

- $$\mathbb{E}\left[\|u^\omega\|^p_{\mathcal{F}L^{\sigma,p}}\right] = \mathbb{E}\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{p(\sigma-s)}|g_n|^p$$

$$\sim \sum_{n\in\mathbb{Z}^d}\langle n\rangle^{p(\sigma-s)} < \infty \quad \text{iff } (\sigma-s)p < -d.$$

- <u>Lemma 1.1</u> (Tail estimate)

Let $\sigma < s - \frac{d}{2}$. Then,

$$\rho_s\left(\|u\|_{H^\sigma} > K\right) \le C e^{-cK^2}$$

for all $K > 0$.

<u>Rmk</u>: This follows from <u>Fernique's integrability theorem</u>:

$$\int_B e^{c\|u\|_B^2}\, d\rho(u) < \infty \quad \text{for some } c > 0.$$

$$\iff \quad \rho\left(\|u\|_B > K\right) \le C e^{-cK^2}$$

<u>Pf of Lemma 1.1</u>: By Chebyshev's ineq

$$\sum_n \langle n\rangle^{2\sigma} |\hat{u}_n|^2$$

$$e^{cK^2} \rho_s\left(B_K^c\right) \le \int_{H^\sigma} e^{c\|u\|_{H^\sigma}^2} d\rho_s(u)$$

$$\Leftarrow \quad \rho_s\left(\|u\|_{H^\sigma} > K\right) = \rho_s\left(e^{c\|u\|_{H^\sigma}^2} > e^{cK^2}\right)$$

$$= \int_{H^\sigma} \mathbb{1}_{\frac{e^{c\|u\|_{H^\sigma}^2}}{e^{cK^2}} > 1} \, d\rho_s(u)$$

$$< \frac{1}{e^{cK^2}} (\text{RHS})$$

usual Chebyshev: $P(f(\omega) > K) \le \dfrac{\mathbb{E}[|f|^2]}{K^2}$

$$= \prod_{n\in\mathbb{Z}^d} \int_{\mathbb{C}} e^{c\langle n\rangle^{2\sigma-2s}|g_n|^2} e^{-\frac{1}{2}|g_n|^2} \frac{dg_n}{2\pi}$$

$$= \prod_{n\in\mathbb{Z}^d} \frac{1}{1-2c\langle n\rangle^{2\sigma-2s}}$$

$$\mathbb{E}\left[e^{aX^2}\right] = \int e^{ax^2} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = \frac{1}{\sqrt{1-2a}}, \quad a < 1/2$$ ④

$y = \sqrt{1-2a}\, x$

$$X \sim \underbrace{N_{\mathbb{R}}(0,1)}_{\text{real-valued Gaussian r.v.}}$$

mean 0, var 1.

$$= \prod_{n \in \mathbb{Z}^d} \left(1 + \frac{2c\langle n\rangle^{2\sigma-2s}}{1-2c\langle n\rangle^{2\sigma-2s}}\right) < \infty$$

iff $\sigma < s - \frac{d}{2}$

($a_n > 0$. Then $\prod (1+a_n) < \infty \iff \sum a_n < \infty$

$\boxtimes$

Construction of Gibbs measure on $\mathbb{T}$

<u>$d=1$</u>: Recall

$$d\mu = Z^{-1} e^{-H(u) - \frac{1}{2}M(u)} du$$

$$H(u) = \frac{1}{2}\int |\partial_x u|^2 \pm \frac{1}{p+1}\int |u|^{p+1}$$

$$M(u) = \int |u|^2$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}} \underbrace{e^{-\frac{1}{2}\|u\|^2_{H^1}} du}_{d\rho_1}$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}} d\rho_1 \quad \text{on } H^\sigma(\mathbb{T}), \ \sigma < 1 - \frac{1}{2} = \frac{1}{2}$$

- <u>Defocusing case</u>: By Sobolev ineq,

$$\int |u|^{p+1} = \|u\|^{p+1}_{L^{p+1}} \lesssim \|u\|^{p+1}_{H^{\frac{1}{2}-}} < \infty, \text{ a.s.}$$

$$\Rightarrow \quad 0 < e^{-\frac{1}{p+1}\int |u|^{p+1}} \leq 1, \text{ a.s.}$$

$\Rightarrow$ μ is a prob. meas on $H^\sigma(\mathbb{T})$, $\sigma < \frac{1}{2}$.

Sobolev ineq: On $\mathbb{R}^d$ or $\mathbb{T}^d$.

- $\mathbb{R}^d$: $\|u\|_{L^q(\mathbb{R}^d)} \lesssim \|u\|_{\dot{W}^{s,p}(\mathbb{R}^d)}$

homogeneous Sobolev space
use $|\xi|^s$
instead of $\langle \xi \rangle^s$

when $\frac{s}{d} = \frac{1}{p} - \frac{1}{q}$

- $\mathbb{T}^d$; $\|u\|_{L^q(\mathbb{T}^d)} \lesssim \|u\|_{W^{s,p}(\mathbb{T}^d)}$

when $\frac{s}{d} \geq \frac{1}{p} - \frac{1}{q}$.

ex: $d=1$: $\|u\|_{L^q(\mathbb{T})} \lesssim \|u\|_{H^{\frac{1}{2}-\varepsilon}(\mathbb{T})}$
by choosing $\varepsilon = \varepsilon(q) > 0$ suff small.

- Focusing case: Let $p > 1$.

$$\int_{\mathbb{T}} |u|^{p+1} \geq \|u\|_{L^2}^{p+1} = \left(\sum |\hat{u}_n|^2\right)^{\frac{p+1}{2}}$$

$$\geq \sum \left|\frac{g_n}{\langle n\rangle}\right|^{p+1} \qquad \ell^2 \subset \ell^{p+1}$$

$$\Rightarrow \mathbb{E}_\rho\left[e^{\frac{1}{p+1}\int |u|^{p+1}}\right] \geq \prod_{n\in\mathbb{Z}} \mathbb{E}\left[e^{\left|\frac{g_n}{\langle n\rangle}\right|^{p+1}}\right] = \infty \quad b/c \ p+1 > 2.$$

$\Rightarrow$ can not construct μ as it is.

Idea: Mass is conserved

$\Rightarrow$ Introduce a mass cutoff.

$$\text{"} d\mu = Z^{-1} \mathbb{1}_{\{\|u\|_{L^2} \leq r\}} \underbrace{e^{\frac{1}{p+1}\int |u|^{p+1}} d\rho_1}_{= e^{-H(u) - \frac{1}{2}M(u)} du} \text{"}.$$

$p > 1$

Prop 1.2 (Lebowitz – Rose – Speer '88, Bourgain '94)

(i) $p < 5$

(*) $$R(u) = R_r(u) := \mathbb{1}_{\{\|u\|_{L^2} \leq r\}} e^{\frac{1}{p+1}\int |u|^{p+1}} \in L^q(d\rho_1)$$

$\forall\, r, q < \infty$.

(ii) $p = 5$: (*) holds if r is sufficiently small.

In fact, $r < \|Q\|_{L^2(\mathbb{R})}$

ground state on $\mathbb{R}$

$p = 5$: mass-critical NLS $i\partial_t u + \partial_x^2 u = -|u|^4 u$ on $\mathbb{T}$

$\Leftarrow$ lowest power for which we have a finite time blowup solution.

$M(Q) \leftarrow$ smallest mass for a finite time blowup solution

Lec 4 18/01/17 (Wed)

Lemma 1.3: $\{\tilde{g}_n\}$, indep standard $\mathbb{R}$-valued Gaussian r.v.'s

Then, $$P\left[\left(\sum_{n=1}^{M} \tilde{g}_n^2\right)^{1/2} \geq R\right] \leq e^{-\frac{1}{4}R^2}, \quad R \geq 3M^{1/2}, \quad M \geq 1$$

Pf: $0 < t < 1/2$

$$(\text{LHS}) \overset{\text{Chebyshev}}{\leq} e^{-tR^2} \underbrace{\mathbb{E}\left[e^{t\sum_{n=1}^{M} \tilde{g}_n^2}\right]}_{= \prod_{n=1}^{M} \mathbb{E}[e^{t\tilde{g}_n^2}]}$$

$$= (1-2t)^{-M/2} e^{-tR^2}$$

Choose $t = \frac{1}{2}\left(1 - \frac{M}{R^2}\right)$

$$(\text{LHS}) \leq \left(\frac{R^2}{M}\right)^{M/2} e^{-\frac{1}{2}R^2 + \frac{1}{2}M}$$

$$\leq e^{\frac{M}{2}\ln\frac{R^2}{M} + \left(\frac{1}{18} - \frac{1}{2}\right)R^2} \leq e^{-\frac{1}{4}R^2}$$

$\left(\ln x \leq \frac{x}{4} \text{ for } x \geq 9\right)$

☑

Pf of Prop 1.2:

By Bernstein's ineq

$$\| P_{\leq M_0} u \|_{L^{p+1}} \leq c\, M_0^{\frac{1}{2}-\frac{1}{p+1}} \underbrace{\| P_{\leq M_0} u \|_{L^2}}_{\leq r}$$

$$\left(\begin{array}{l} \frac{s}{1} \geq \frac{1}{2} - \frac{1}{p+1} \\ (LHS) \underset{\text{Sobolev}}{\lesssim} \| \langle \partial x \rangle^{\frac{1}{2}-\frac{1}{p+1}} P_{\leq M_0} u \|_{L^2} \\ \qquad \langle m \rangle^{\frac{1}{2}-\frac{1}{p}} \lesssim M_0^{\frac{1}{2}-\frac{1}{p}} \end{array} \right.$$

Given $\lambda > 1$, choose M_0 s.t.

$$\frac{1}{2}\lambda = c\, M_0^{\frac{1}{2}-\frac{1}{p+1}} r$$

i.e. $$M_0 \sim \left(\frac{\lambda}{r}\right)^{\frac{1}{\frac{1}{2}-\frac{1}{p+1}}}$$

Let $q = p+1$.

Let $M_j = 2^j M_0$. Then, we have

$$\| P_{M_j} u \|_{L^q} \leq c\, M_j^{\frac{1}{2}-\frac{1}{q}} \| P_{M_j} u \|_{L^2}$$

Let $\{\sigma_j\}$ s.t. $\displaystyle\sum_{j\geq 1} \sigma_j = \frac{1}{2}$

$\left(\Leftarrow \text{ set } \sigma_j = C\, 2^{-\varepsilon j} = C M_0^{\varepsilon} M_j^{-\varepsilon} \right.$

$\Rightarrow$ By dyadic pigeon hole principle,

$$\rho_1 \left(\|u\|_{L^q} > \lambda,\ \|u\|_{L^2} \leq r \right)$$

Recall $\| P_{\leq M_0} u \|_{L^q} \leq \frac{1}{2} \lambda$

$$\leq \sum_{j=1}^{\infty} \rho_1 \left(\| P_{M_j} u \|_{L^q} > \sigma_j \lambda \right)$$

$$\sum_{j=1}^{\infty} \sigma_j \lambda = \frac{1}{2} \lambda < \| P_{>M_0} u \|_{L^q} \leq \sum_{j=1}^{\infty} \| P_{M_j} u \|_{L^q}$$

$$\leq \sum_{j=1}^{\infty} \rho_1 \Big(\underbrace{\| P_{M_j} u \|_{L^2}} > c\, \sigma_j M_j^{\frac{1}{q}-\frac{1}{2}} \lambda \Big)$$

$$= \left(\sum_{|n| \sim M_j} \frac{|g_n|^2}{\langle n \rangle^2} \right)^{1/2} \sim M_j^{-1} \left(\sum_{|n| \sim M_j} |g_n|^2 \right)^{1/2}$$

$$P\left(\left(\sum_{|n|\sim M_j}|g_n(\omega)|^2\right)^{1/2}\gtrsim \underbrace{\sigma_j M_j^{\frac{1}{q}+\frac{1}{2}}\lambda}\right)$$

$$=: R_j \gtrsim M_0^{\varepsilon} M_j^{\frac{1}{2}+\frac{1}{q}-\varepsilon}$$

$$\gg M_j^{\frac{1}{2}+}$$

$\Rightarrow$ Apply Lemma 1.3.

$$\rho_1\left(\|u\|_{L^q}>\lambda,\ \|u\|_{L^2}\le r\right)$$

$$\le \sum_{j=1}^{\infty} e^{-cR_j^2} \qquad R_j^2 \sim M_0^{\frac{2}{q}+1}\lambda^2\, \underline{\underline{2^{(\frac{2}{q}+1-2\varepsilon)j}}}$$

$$\sim e^{-c\lambda^2 M_0^{\frac{2}{q}+1}} \qquad M_0\sim\left(\frac{\lambda}{r}\right)^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}$$

$$\sim e^{-c\lambda^{\frac{4q}{q-2}} r^{-\frac{2q+4}{q-2}}} \qquad \frac{4q}{q-2}>q \text{ for } q<6.$$

$$\lesssim \begin{cases} e^{-c\lambda^{q+\delta}}, & q<6 \iff p<5. \\ e^{-C\lambda^6}, & q=6 \iff p=5 \end{cases}$$

with $C\gg 1$ by taking $r\ll 1$

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$$\left(\begin{aligned} \|f\|_{L^p}^p &= \int |f(x)|^p \, dx \\ &= p \int_0^\infty \alpha^{p-1} \mu(|f(x)| > \alpha) \, d\alpha \end{aligned} \right.$$

(X, μ) meas space

$$\left\| \mathbb{1}_{\{\|u\|_{L^2} \le r\}} e^{\frac{1}{p+1} \int |u|^{p+1} dx} \right\|_{L^{\tilde{q}}(d\rho_1)}$$

$\lambda^{p+1} \sim \ln \alpha$

$$\sim 1 + \int_1^\infty e^{c \lambda^{p+1}} \left(\begin{array}{l} e^{-c\lambda^{p+1+\delta}} \\ \text{or} \\ e^{-c\lambda^{p+1}} \end{array} \right) d\lambda$$

$p < 5$

$p = 5$

$\boxtimes$

Rmk: We went over the harmonic analytic proof by Bourgain '94.

The proof by LRS '88 is more probabilistic.

$$R_r(u) = \mathbb{1}_{\{\|u\|_{L^2} \le r\}} \, e^{\frac{1}{p+1}\int |u|^{p+1}}$$

$$R_{N,r}(u) = \mathbb{1}_{\{\|P_{\le N} u\|_{L^2} \le r\}} \, e^{\frac{1}{p+1}\int |P_{\le N} u|^{p+1}}$$

Cor 1.4: $\forall q < \infty$

$$R_{N,r}(u) \longrightarrow R_r(u) \text{ in } L^q(d\rho_1)$$

as $N \to \infty$

In particular, with $d\mu_N = Z_N^{-1} R_{N,r}(u) d\rho_1$

μ_N converges "uniformly" to μ

i.e. $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t.

$$|\mu_N(A) - \mu(A)| < \varepsilon, \quad \forall \text{ measurable } A, \ \forall N \ge N_0$$

⑦

$\Leftarrow$ stronger than weak convergence:

$$\lim_{N\to\infty} \int f(u)\, d\mu_N(u) = \int f(u)\, d\mu(u)$$

for any $f \in C_b(H^\sigma(\mathbb{T}))$, $\sigma < \frac{1}{2}$

Pf: $\int |P_{\le N} u|^{p+1} \longrightarrow \int |u|^{p+1}$ a.s.

$\left(\Leftarrow \| P_{>N} u \|_{L^{p+1}} \lesssim \| P_{>N} u \|_{H^{\frac{1}{2}-}} \to 0. \right.$

$\Rightarrow$ $R_{N,r}(u) \longrightarrow R_r(u)$ a.s.

$\Rightarrow$ By Egoroff's thm,

$R_{N,r}(u) \longrightarrow R_r(u)$ almost uniformly

$\Rightarrow$ in meas

· Given $\varepsilon > 0$,

let $A_{N,\varepsilon} = \{ u \in H^{\frac{1}{2}-}(\mathbb{T}) : |R_{N,r}(u) - R_r(u)| \le \frac{1}{2}\varepsilon \}$.

Then, $\rho_1 (A^c_{N,\varepsilon}) \longrightarrow 0$ as $N \to \infty$

⑧

$$\| R_{N,r} - R_r \|_{L^q(d\rho_1)}$$

$$\leq \| (R_{N,r} - R_r) \mathbb{1}_{A_{N,\varepsilon}} \|_{L^q}$$

$$+ \| (R_{N,r} - R_r) \mathbb{1}_{A^c_{N,\varepsilon}} \|_{L^q}$$

$$\leq \frac{1}{2}\varepsilon + \underbrace{\left(\| R_{N,r} \|_{L^{2q}} + \| R_r \|_{L^{2q}} \right)}_{\leq C < \infty} \underbrace{\left\{ \rho_1 (A^c_{N,\varepsilon}) \right\}^{\frac{1}{2q}}}_{\to 0}$$

$$< \varepsilon$$

for $N \gg 1$. $\boxtimes$

- <u>Digression:</u> Proof of Fernique's integrability theorem

(H, B, ρ), abstract Wiener space

$$d\rho = Z^{-1} e^{-\frac{1}{2}\|x\|_H^2} dx$$

Define Wiener measure with variance t

$$d\rho_t = Z_t^{-1} e^{-\frac{\|x\|^2}{2t}} dx, \quad t > 0$$

- $\{\rho_t\}_{t>0}$ forms a <u>contraction semigroup</u> in the Banach space of bounded, unif conti functions on B. $\to \|\rho_t\| \leq 1$

Riesz rep thm: X, LCH

$$(C_0(X))^* = M(X)$$

$C_0(X)$: vanishing at ∞

$M(X)$: Radon meas

$\forall \varepsilon > 0$. $\{x : |f(x)| \geq \varepsilon\}$ is cpt.

$$C_0(X) = \overline{C_c(X)}^{\|\cdot\|_{unif}}$$

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$$d\rho_t = Z_t^{-1} e^{-\|x\|^2/2t} dx, \quad t > 0 \qquad (H, B, \rho)$$

· **Integrability of $e^{\alpha\|x\|^2}$**

Ω = space of conti func ω on $[0, \infty)$ with values in B and $\omega(0) = 0$.

$\exists!$ prob meas P on the σ-field generated by the coordinate functions $\omega \mapsto \omega(t)$, $t > 0$, s.t. if $0 = t_0 < t_1 < \cdots < t_n$, then

1. $\omega(t_j) - \omega(t_{j-1})$, $j = 1, \cdots, n$, are indep
2. $\omega(t_j) - \omega(t_{j-1})$ is distributed in B according to $\rho_{t_j - t_{j-1}}$.

· Define $W(t) : \Omega \to B$ by $W(t)(\omega) = \omega(t)$

Wiener process in B (starting at 0)

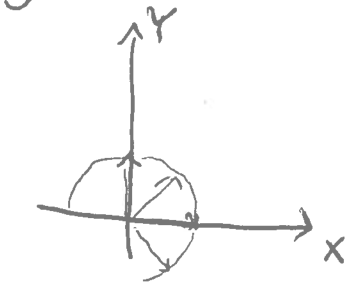
Note: $\displaystyle\int_B e^{\alpha \|x\|^2} \rho_1(dx) = \mathbb{E}\left[e^{\alpha \|W(1)\|^2} \right]$

- Pf of Fernique's Thm.

Let $X = W(1)$, $Y = W(2) - W(1)$ indep, distributed by ρ_1

Moreover, $\dfrac{X+Y}{\sqrt{2}}$ and $\dfrac{X-Y}{\sqrt{2}}$ are indep & distributed by ρ_1

(just rotate the coord axes in the XY plane.)



- For $t > s$, we have

$$P(\|W(1)\| \le s) P(\|W(1)\| > t)$$

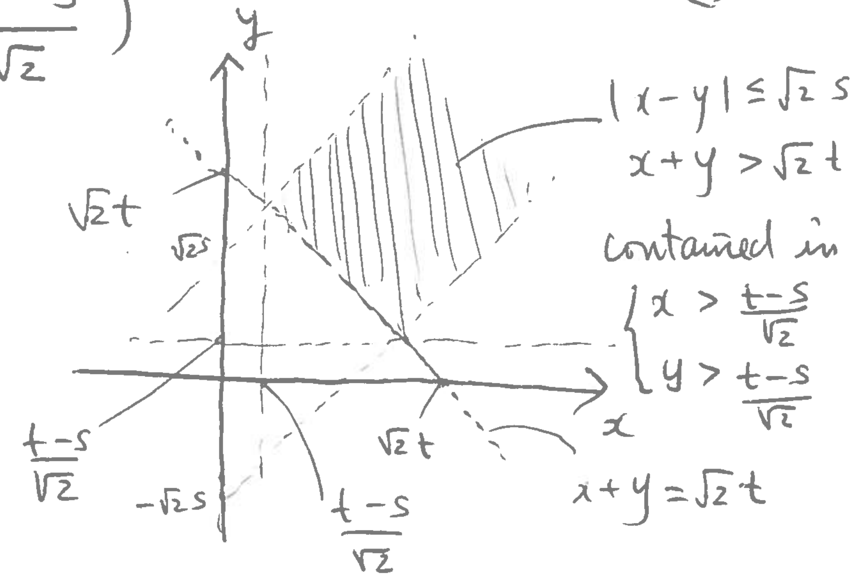
$$= P\left(\left\| \frac{X-Y}{\sqrt{2}} \right\| \le s \right) P\left(\left\| \frac{X+Y}{\sqrt{2}} \right\| > t \right)$$

$$\overset{\text{indep}}{=} P\left(\frac{\|X-Y\|}{\sqrt{2}} \le s \text{ and } \frac{\|X+Y\|}{\sqrt{2}} > t \right)$$

$(*)$ $$\overset{\text{triangle ineq}}{\le} P\left(\big| \|X\| - \|Y\| \big| \le \sqrt{2}s \text{ and } \|x\| + \|y\| > \sqrt{2}t \right)$$

$$\leq P\left(\|X\| > \frac{t-s}{\sqrt{2}} \text{ and } \|Y\| > \frac{t-s}{\sqrt{2}} \right)$$

$$\overset{\text{indep}}{=} P\left(\|W(1)\| > \frac{t-s}{\sqrt{2}} \right)^2$$



Define t_n by $t_0 = s > 0$

$$t_{n+1} = s + \sqrt{2}\, t_n$$

$$\Rightarrow t_n = (1 + \sqrt{2} + \cdots + \sqrt{2}^n)\, s$$

$$= \frac{\sqrt{2}^{n+1} - 1}{\sqrt{2} - 1}\, s.$$

Let $\alpha_n = \dfrac{P(\|W(1)\| > t_n)}{P(\|W(1)\| \leq s)}$, $n \geq 0$

$$\overset{\text{By } (*)}{\Longrightarrow} \quad \alpha_{n+1} \leq \alpha_n^2 \Rightarrow \alpha_n \leq \alpha_0^{2^n} = e^{2^n \log \alpha_0}$$

$$\Rightarrow P(\|W(1)\| > t_n) \leq P(\|W(1)\| \leq s)\, e^{2^n \log \alpha_0}$$

- Let $u_n = \sqrt{2}^{n+4} s$

$$\Rightarrow u_n > (\sqrt{2}^{n+1} - 1)(\sqrt{2} + 1)\, s = t_n$$

$$\Rightarrow P(\|W(1)\| > u_n) \le P(\|W(1)\| \le s)\, e^{\frac{u_n}{16s^2}\log \alpha_0}$$

- Choose $s \gg 1$ s.t. $0 < \alpha_0 < 1$.

(By conti from above, $P(\|W(1)\| > s) \to 0$ as $s \to \infty$
By conti from below, $P(\|W(1)\| \le s) \to 1$ as $s \to \infty$

Let

$$a = -\frac{1}{16s^2}\log \alpha_0 > 0$$

$$b = P(\|W(1)\| \le s)$$

$$\Rightarrow P(\|W(1)\| > u_n) \le b\, e^{-a u_n^2}$$

$$\Rightarrow \int_{\|x\|>N} e^{\alpha\|x\|^2} \rho_1(dx) \le \sum_{k=0}^{\infty} \int_{2^k N \le \|x\| \le 2^{k+1}N} e^{\alpha\|x\|^2} \rho_1(dx)$$

$$\le b \sum_{k=0}^{\infty} e^{\alpha(2^{k+1}N)^2} e^{-a(2^k N)^2} < \infty$$

$(4\alpha - a)(2^k N)^2$

by choosing $\alpha \ll a$.

$\square$

• Invariance of the Gibbs measure ($d=1$)

• 3 scenarios

CASE 1: We have an a priori (deterministic) global well-posedness in $\mathrm{supp}(\mu) \subset H^{\frac{1}{2}-}(\mathbb{T})$

ex: KdV, cubic, GWP in $L^2(\mathbb{T})$ (Bourgain '93)

$\Rightarrow$ Invariance of μ follows from the invariance of the "finite-dim'l" Gibbs meas μ_N associated to the truncated equation (easy)

CASE 2: We only know a priori local well-posedness in $\mathrm{supp}(\mu)$ (in subcritical sense).

$\updownarrow$

local existence time $\delta > 0$ depends only on the size of initial data: $\delta \sim \|u_0\|_{H^\sigma}^{-\theta}$ for some $\theta > 0$.

NLS: $\quad i\partial_t u + \partial_x^2 u = \pm |u|^{p-1} u$

We say u is a solution to NLS if u satisfies the following Duhamel formulation:

$$u(t) = S(t)u_0 \mp i \int_0^t S(t-t')|u|^{p-1}u(t')\,dt'.$$

Note: $i\partial_t u + \partial_x^2 u = F$

$$\underset{\text{F.T. in } x}{\Longrightarrow} \quad i\partial_t \hat{u}(n) - n^2 \hat{u}(n) = \hat{F}(n) \qquad \times e^{itn^2}$$

$$\Longrightarrow \quad i\partial_t \left(e^{itn^2}\hat{u}(n)\right) = e^{itn^2}\hat{F}(n).$$

$\Longrightarrow$ integrate from 0 to t. & invert F.T.

$$\Longrightarrow \quad u(t) = S(t)u_0 - i\int_0^t S(t-t')F(t')\,dt'$$

Let $\Gamma_{u_0}(u)(t) = S(t)u_0 \mp i\int_0^t S(t-t')|u|^{p-1}u(t')\,dt'$.

u, soln to NLS $\Longleftrightarrow$ $\Gamma_{u_0}(u) = u$. i.e. fixed pt for Γ.

Suppose that we have the following 2 estimates:

1. Linear estimate :

$$\|\Gamma u\|_{X^s([0,\delta))} \lesssim \|u_0\|_{H^s} + \|\,|u|^{p-1}u\|_{N^s}$$

δ^θ

2. Nonlinear estimate:

$$\|\,|u|^{p-1}u\|_{N^s} \lesssim \|u\|^p_{X^s}$$

$$\Longrightarrow \quad \|\Gamma u\|_{X^s([0,\delta))} \le C_1\|u_0\|_{H^s} + C_2\delta^\theta \|u\|^p_{X^s([0,\delta))}$$

$\Rightarrow$ Γ is a contraction on $B_R \subset X^s([0,\delta))$

Choose $R = 2C_1\|u_0\|_{H^s}$

- $\|\Gamma u\|_{X^s([0,\delta])} \le \frac{1}{2}R + C_2\delta^\theta R^p \le R$ for $\delta = \delta(R) \ll 1$
  $(\Leftarrow C_2\delta^\theta R^{p-1} \le \frac{1}{2})$
- $\|\Gamma u - \Gamma v\|_{X^s([0,\delta))} \le \frac{1}{2}\|u-v\|_{X^s([0,\delta))}$ for $\delta = \delta(R) \ll 1$

$$\Rightarrow \exists ! \, u \in B_R \quad \text{s.t.} \quad \Gamma_{u_0}(u) = u.$$

- examples of X^s:

Strichartz space: $\| \langle \nabla \rangle^s u \|_{L^q_t L^r_x([0,\delta) \times M)}$

Strichartz estimate for Schrödinger eqn on $\mathbb{R}^d$:

$$\| S(t) u_0 \|_{L^q_t L^r_x} \lesssim \| u_0 \|_{L^2_x(\mathbb{R}^d)}$$

where $q, r \geq 2$ satisfies

$$\underline{\frac{2}{q} + \frac{d}{r} = \frac{d}{2}}, \qquad (q, r, d) \neq (2, \infty, 2)$$

(q, r) is (Schrödinger) admissible

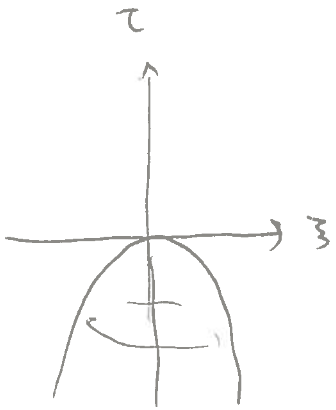
- $X^{s,b}$ - space (Fourier restriction norm method by Bourgain '93

$$\|u\|_{X^{s,b}} = \| \langle \xi \rangle^s \langle \tau + |\xi|^2 \rangle^b \widehat{u}(\tau,\xi) \|_{L^2_{\xi,\tau}}$$

linear Sch. eqn : $i\partial_t u + \Delta u = 0$

$$\underset{\text{space-time F.T.}}{\Longrightarrow} -(\tau + |\xi|^2)\widehat{u}(\tau,\xi) = 0.$$

$\Rightarrow$ $\widehat{u}(\tau,\xi) = \mathcal{F}_{t,x}(S(t)u_0)(\tau,\xi)$ is a measure supported on the paraboloid $\tau = -|\xi|^2$.



"measures" how far/close u is to being a linear soln.

When $b > 0$, $\langle \tau + |\xi|^2 \rangle^b$ penalizes functions away from linear solutions

$\Rightarrow$ suitable for study Γu in a perturbative manner.

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· LWP in a subcritical sense:

Linear estimate
nonlin estimate $\Rightarrow \|\Gamma_{u_0} u\|_{X^s([0,\delta))} \lesssim \|u_0\|_{H^s} + \delta^\theta \|u\|^p_{X^s([0,\delta))}$

$\Rightarrow$ By a fixed pt argument (Banach's contraction mapping thm)

$\Rightarrow$ $\exists!$ soln u with $u|_{t=0} = u_0$ on $[0,\delta)$

$$\delta \sim (1 + \|u_0\|_{H^s})^{-\gamma}, \quad \gamma > 0.$$

· Blowup alternative: let T^* be the forward maximal time of existence.

Then, either $T^* = \infty$ or $\lim_{t \nearrow T^*} \|u(t)\|_{H^s} = \infty$ $[0, T^*)$

· The "only" way to construct global-in-time solns is to use conservation laws: (defocusing NLS)

$$H(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{p+1}\int |u|^{p+1}, \quad M(u) = \int |u|^2 dx$$

$$\Longrightarrow \quad \|u(t)\|_{H^1}^2 \lesssim H(u)(t) + M(u)(t) = H(u_0) + M(u_0) < \infty$$

Namely,

LWP in H^1 in a subcritical sense

$\Rightarrow$ GWP in H^1.

Issue: $\nexists$ conservation law at the level of the Gibbs meas μ

$s = \frac{1}{2}-$.

idea: use invariance of μ (in place of a conservation law)

to construct global-in-time dynamics (on supp μ)

$$\begin{tikzcd} \text{invariance} \arrow[r, bend right] & \text{a.s. GWP} \arrow[l, bend right] \end{tikzcd}$$

Bourgain '94: use invariance of the "finite dim'l" Gibbs meas μ_N

associated to the truncated dynamics

$\Longrightarrow$ a.s. GWP $\Rightarrow$ invariance

CASE 3: No LWP (by a deterministic method)

(3.a): probabilistic local well-posedness

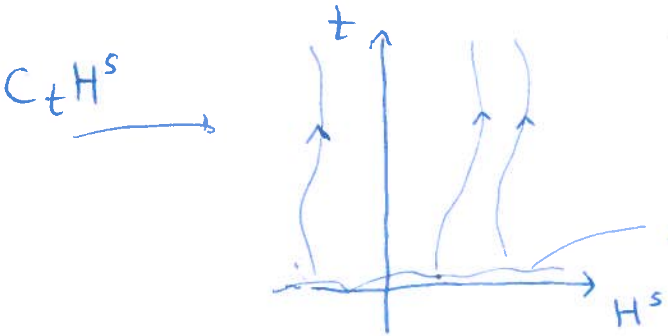
McKean '95, Bourgain '96, Burq–Tzvetkov '07.

Recall $\mu \ll \rho_1 \rightsquigarrow u_0^\omega(x) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{inx}$

Aim: Exploit the randomness of initial data.

(3.b): compactness argument

(for measures on space-time functions)



$\Phi_N : u_{0,N} \longmapsto u_N$

induced prob meas

Define a prob meas ν_N by

$$\nu_N = \mu_N \circ \Phi_N^{-1}$$

prob meas on space-time functions

$\Rightarrow$ a.s. global existence (without uniqueness)

& "invariance" (in some mild sense)

- We first focus on <u>CASE 2</u>.

$$\text{(NLS)} \quad i\partial_t u + \partial_x^2 u = \pm |u|^{p-1} u \quad \text{on } \mathbb{T}.$$

Bourgain '93: LWP in $H^s(\mathbb{T})$ for some $s = s(p) < \frac{1}{2}$.

in a subcritical sense

$$\delta \sim (1 + \|u_0\|_{H^s})^{-\gamma}$$

- Consider the following truncated dynamics:

$$\text{(FNLS)} \quad \begin{cases} i\partial_t u_N + \partial_x^2 u_N = \pm P_{\le N}\left(|P_{\le N} u_N|^{p-1} P_{\le N} u_N\right) \\ u_N|_{t=0} = u_0 \end{cases}$$

$$P_{\le N} f(x) = \sum_{|n| \le N} \hat{f}(n) e^{inx}$$

- (FNLS) is NOT finite dim'l.

high freq $\sim$ linear

low freq $\sim$ nonlinear

decoupled

- (FNLS) is locally well-posed by apply the LWP argument for (NLS): $\delta \sim (1 + \|u_0\|_{H^s})^{-\gamma}$, <u>indep of N</u>.

Let $H_N(u_N) = \frac{1}{2} \underbrace{\int |\partial_x u_N|^2}_{\text{lin. dynamics on high freq.}} \pm \frac{1}{p+1} \int |P_{\leq N} u|^{p+1}$.

$$\Rightarrow \quad \partial_t u_N = -i \frac{\partial H_N}{\partial \overline{u_N}}$$

① $u_{high} = P_{>N} u_N$ evolves linearly.

$$\partial_t \widehat{u}_{high}(n) = -i n^2 \widehat{u}_{high}(n), \quad |n| > N$$

$$\Rightarrow \widehat{u}_{high}(t, n) = e^{-itn^2} \widehat{u}_0(n), \quad |n| > N$$

In particular, u_{high} exists globally.

② $u_{low} = P_{\leq N} u_N$ satisfies

$$i \partial_t u_{low} + \partial_x^2 u_{low} = \pm P_{\leq N} (|u_{low}|^{p-1} u_{low})$$

$\Leftarrow$ finite dim'l system of (nonlin) ODEs (on the Fourier side) (with Lipschitz v.f.) $\Rightarrow$ local existence by Cauchy-Lipschitz thm.

- L^2-norm : $\| u_{low} \|_{L^2} = \left(\sum_{|n| \leq N} | \widehat{u}_{low}(n)|^2 \right)^{1/2}$ is conserved.

$=$ Euclidean distance on $\mathbb{C}^{2N+1}$.

$\Rightarrow$ u_{low} exists globally.

$\Rightarrow$ $u_N = u_{low} + u_{high}$ exists globally in time.

Issue: (FNLS) is GWP for each $N \in \mathbb{N}$,

but there is NO uniform (in N) control on $\|u_N(t)\|_{H^s}$.

Write $\rho_1 = \rho_N \otimes \rho_N^{\perp}$

- $$d\rho_N = Z_N^{-1} e^{-\frac{1}{2}\|P_{\leq N} u\|_{H^1}^2} d(P_{\leq N} u)$$

$$= Z_N^{-1} \prod_{|n| \leq N} e^{-\frac{1}{2}\langle n\rangle^2 |\hat{u}(n)|^2} d\hat{u}(n)$$

- $$d\rho_N^{\perp} = \tilde{Z}_N^{-1} e^{-\frac{1}{2}\|P_{>N} u\|_{H^1}^2} d(P_{>N} u) \quad \text{on } H^s,\ s < \tfrac{1}{2}.$$

Then, ① $\rho_N^{\perp}$ is invariant under (FNLS_{high}).

$$P_{>N} u_0(x) = \sum_{|n|>N} \frac{g_n(\omega)}{\langle n\rangle} e^{inx} \Rightarrow u_{high}(t,x) = \sum_{|n|>N} \frac{\overbrace{e^{-itn^2} g_n(\omega)}^{\hat{g}_n(\omega)}}{\langle n\rangle} e^{inx}$$

$\Rightarrow$ Since g_n is invariant under a rotation,
$\rho_N^\perp$ is invariant.

② $d\mu_{N,\text{low}} = Z_N^{-1} \underbrace{R_{N,r}(u_{\text{low}})}_{\mathbb{1}_{\{\|u_{\text{low}}\|_{L^2} \le r\}} e^{\frac{1}{p+1}\int |u_{\text{low}}|^{p+1}}} d\rho_N$

is invariant under $(\text{FNLS}_{\text{low}})$.

$$d\mu_{N,\text{low}} = Z_N^{-1} e^{-\underbrace{H_N(u_{\text{low}})}_{\text{conserved}}} \underbrace{du_{\text{low}}}_{\text{inv by Liouville's thm}}$$

$\Rightarrow$ $\mu_N = \mu_{N,\text{low}} \otimes \rho_N^\perp$ is invariant under (FNLS).

Key Proposition (Bourgain '94)

$\forall T > 0,\ \varepsilon > 0,\ \exists\, \Omega_N = \Omega_N(T, \varepsilon)$ s.t.

(i) $\mu_N(\Omega_N^c) < \varepsilon$

(ii) For $u_0 \in \Omega_N$, the soln u_N to (FNLS) with $u_N|_{t=0} = u_0$ satisfies $\|u_N(t)\|_{H^s} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$, $|t| \le T$.

$\Leftarrow$ implicit const is independent of N

Pf: By local theory,

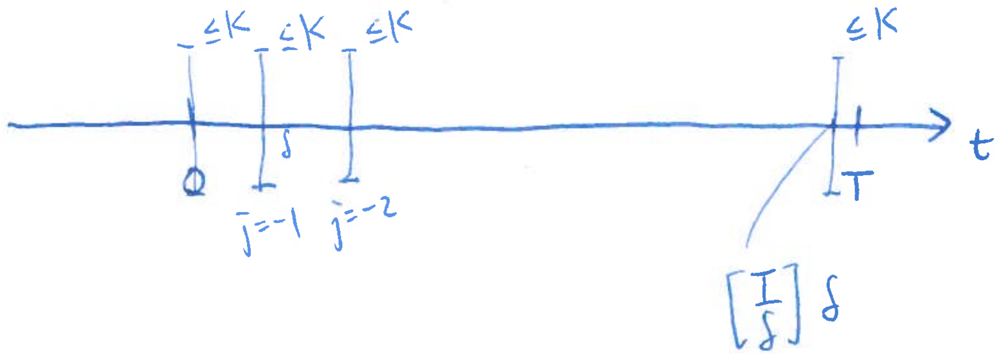
$$\|u_0\|_{H^s} \le K \implies \|u_N(t)\|_{H^s} \le CK$$

for $|t| \le \delta \sim K^{-\theta}$, (indep of N)

Let

$$\Omega_N = \bigcap_{j=-[\frac{T}{\delta}]}^{[\frac{T}{\delta}]} \Phi_N(j\delta)\Big(\underbrace{\{\|u_0\|_{H^s} \le K\}}_{= B_K}\Big)$$

$\Phi_N(t) : u_0 \longmapsto u_N(t)$, soln map for (FNLS)



(i)

using uniqueness

$$\mu_N(\Omega_N^c) \le \sum_{j=-[\frac{T}{\delta}]}^{[\frac{T}{\delta}]} \mu_N(\cancel{\Phi_N(j\delta)}(B_K^c))$$

By invariance of μ_N

$\mu_N(\Phi(-t)A) = \mu_N(A)$

$$\sim \frac{T}{\delta}\, \mu_N(B_K^c)$$

$$\lesssim T \cdot K^{\gamma} e^{-cK^2} < \varepsilon$$

Lemma 1.1 & Prop 1.2

$$\left(\mu_N(B_K^c) \underset{C\text{-}S}{\le} \underbrace{\|R_{N,r}\|_{L^2(d\rho_1)}}_{\le C \text{ indep. of } N} (\rho_1(B_K^c))^{1/2} \lesssim e^{-cK^2}. \right)$$

by choosing $K \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$

(ii) By local theory,

$$\|u_N(t)\|_{H^s} \le CK \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}, \quad \forall |t| \le T$$

$\square$

Lec 7: 30/01/17 (Mon)

Lemma 1.6: (Approximation lemma) $s < 1/2$

$u_0 \in H^s$ with $\|u_0\|_{H^s} \le K$

Suppose soln u_N to (FNLS) with $u_N|_{t=0} = u_0$ satisfies

$$\|u_N(t)\|_{H^s} \le K, \quad |t| \le T$$

Then, $\exists!$ soln u to (NLS) on $[-T, T]$ with $u|_{t=0} = u_0$.

Moreover, we have

$$\|u(t) - P_{\le N} u_N(t)\|_{H^{s_1}} \le C_0 e^{C_1(1+K)^{C_2} T} K \underbrace{N^{s_1 - s}}_{\to 0}$$

for $s_1 < s$. (for suff. large $N \in \mathbb{N}$.)

Pf: sketch (FNLS) & (NLS) with $u_N|_{t=0} = u|_{t=0} = u_0$ are locally well-posed on $[-\delta, \delta]$, $\delta \sim (1+K)^{-\gamma}$, indep of N.

Infact, choose this as $\delta \sim (1+2K)^{-\gamma}$

Let $v_N = P_{\le N} u_N$.

$$\|u - v_N\|_{X^{s_1}([0,\delta])} \lesssim \|\underbrace{u_0 - P_{\le N} u_0}_{= P_{>N} u_0}\|_{H^{s_1}} + \text{nonlin term} \qquad ②$$

$$\underbrace{\qquad\qquad\qquad\qquad}$$

$$\lesssim N^{s_1 - s} \|u_0\|_{H^s}$$

$$\le N^{s_1 - s} K$$

Write $|u|^{p-1}u = (Id - P_{\le N})(|u|^{p-1}u) + P_{\le N}(|u|^{p-1}u)$

① $\underbrace{(Id - P_{\le N})(|u|^{p-1}u)}$

$$\lesssim \delta^\theta N^{s_1 - s} K^p.$$

$\uparrow$ LWP of u in $X^s([0,\delta])$

$$\|u\|_{X^s([0,\delta])} \le CK.$$

$+$ ② $P_{\le N}(|u|^{p-1}u - |v_N|^{p-1}v_N)$

$$\lesssim \delta^\theta K^{p-1} \|u - v_N\|_{X^{s_1}([0,\delta])}$$

$$\Rightarrow \quad \|u - v_N\|_{X^{s_1}([0,\delta])} \le C K N^{s_1 - s} + \frac{1}{2} \|u - v_N\|_{X^{s_1}([0,\delta])}$$

$$\left(\Leftarrow \ \delta^{\theta} K^{p-1} \ll 1 \right.$$

$$\Rightarrow \quad \|u - v_N\|_{X^{s_1}([0,\delta])} \le 2CK N^{s_1 - s}$$

- Now iterate the argument $\sim T/\delta$ many times.

$$\|u - v_N\|_{X^{s_1}([\delta, 2\delta])} \lesssim \underbrace{\|u(\delta) - v_N(\delta)\|_{H^{s_1}}}_{\lesssim K N^{s_1 - s}} + \underbrace{\text{nonlin term}}_{\text{can be handled exactly as before.}}$$

$$\left(X^{s_1} \subset C_t H^{s_1} \right.$$

etc.

$$\Rightarrow \|u(t) - v_N(t)\|_{H^{s_1}} \lesssim \underbrace{e^{c\frac{T}{\delta}}}_{\hookrightarrow\ e^{c(1+K)^{\gamma} T}} K N^{s_1 - s}$$

$\boxtimes$

Prop 1.7 (Almost a.s. GWP) $s < 1/2$

Given $T, \varepsilon > 0$, $\exists\, \Omega_{T,\varepsilon} \subset H^s(\mathbb{T})$ s.t

(i) $\mu(\Omega^c_{T,\varepsilon}) < \varepsilon$

(ii) For $u_0 \in \Omega_{T,\varepsilon}$, $\exists!$ soln u to (NLS) on $[-T, T]$

s.t.
$$\| u(t) \|_{H^s} \lesssim \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \quad |t| \le T.$$

Pf: Let $\Omega_N(T, \varepsilon)$ be as in Key Prop.

$$\Rightarrow \| \Phi_N(t)(u_0) \|_{H^s} \le CK \quad \text{for } |t| \le T \text{ and } u_0 \in \Omega_N$$

· By Lemma 1.6, $\exists\, N_1 \gg 1$ s.t.

$$\| u(t) - u_N(t) \|_{H^{s_1}} \ll 1, \qquad |t| \le T$$

for $N \ge N_1$

$$\Rightarrow \| u(t) \|_{H^{s_1}} \lesssim K \sim \left(\log \frac{T}{\varepsilon} \right)^{1/2}, \qquad |t| \le T.$$

· Also
$$\mu(\Omega_N^c) \overset{C\text{-}S}{\le} \underbrace{\| R_N \|_{L^2(d\rho_1)}}_{\left(e^{\frac{1}{p+1}\int |P_{\le N} u|^{p+1}}\right.} \left(\int_{\Omega_N^c} \underbrace{\mathbb{1}_{\{\|u\|_{L^2} \le r\}}}_{\le \mathbb{1}_{\{\|P_{\le N} u\|_{L^2} \le r\}}} d\rho_1 \right)^{1/2}$$

$$\Big(\mathbb{1}_{\{\|P_{\le N} u\|_{L^2} \le r\}} \, d\rho_1 \ll \mu_N$$

$\gg$

$$\overset{C-S}{\lesssim} \big(\mu_N(\Omega_N^c)\big)^{\frac{1}{4}} < \varepsilon.$$

$\boxtimes$

Thm 1.8: NLS is globally well-posed almost surely with respect to the Gibbs measure μ.

Pf: Fix $\varepsilon > 0$. Let $T_j = 2^j$, $\varepsilon_j = \varepsilon / 2^j$.

$\Rightarrow$ Construct $\Omega_j = \Omega_{T_j, \varepsilon_j}$

Let $\Omega_\varepsilon = \bigcap_{j=1}^{\infty} \Omega_j$

(i) $\mu(\Omega_\varepsilon^c) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$

(ii) If $u_0 \in \Omega_\varepsilon$, then the soln u to (NLS) with $u|_{t=0} = u_0$ exists globally in time.

- Let $\Sigma = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$.

(i) $\mu(\Sigma^c) \leq \inf_{\varepsilon>0} \varepsilon = 0.$ ⑥

(ii) If $u_0 \in \Sigma$, then the soln u to (NLS) with $u|_{t=0} = u_0$ exists globally in time.

$\boxed{/}$

<u>Rmk:</u> We have the following probabilistic bound: $s < 1/2$

$$\|u(t)\|_{H^s} \lesssim C(u_0^{\omega}) \left(\log(1+|t|) \right)^{1/2}, \quad \forall t \in \mathbb{R}.$$

<u>Thm 1.9:</u> The Gibbs measure μ is invariant under the flow of NLS.

<u>Pf:</u> By time reversibility of $\Phi(t)$, it suffices to show

$$(*) \qquad \mu(A) \leq \mu(\Phi(t) A)$$

for all measurable set $A \subset H^s$ and $t \in \mathbb{R}$

(If u solves NLS, so does $\bar{u}(-t)$.)

By inner regularity,

$$\mu(A) = \sup_{\substack{F \subset A \\ \text{closed in } H^s}} \mu(F).$$

i.e. $\exists \{F_n\}$ closed sets in H^s s.t.

$$F_n \subset A \quad \text{and} \quad \mu(A) = \lim_{n\to\infty} \mu(F_n)$$

- <u>Claim</u>: Suffices to prove (*) for closed sets.

$$\Leftarrow \quad \mu(A) = \lim_{n\to\infty} \mu(F_n)$$
$$\leq \overline{\lim_{n\to\infty}} \mu(\Phi(t) F_n)$$
$$\leq \mu(\Phi(t) A)$$

$F_n \subset A$ and uniqueness.

- Given a closed set $F \subset H^s$,

let $K_n = \{ u \in F : \|u\|_{H^\sigma} \leq n \}$, $\quad s < \sigma < 1/2$.

Then, K_n is compact in H^s. (Rellich's lemma)

$\Rightarrow$ Suffices to prove ⊛ for compact sets.

$$\mu(F) = \lim_{n\to\infty} \mu(K_n)$$

Lem 1.1. & Prop 1.2

- Let K be a cpt set in H^s.

Since $\mu_N \to \mu$ & Portmanteau theorem,

we have

(1) $$\mu(\Phi(t)K + \overline{B_\varepsilon}) \geq \overline{\lim} \, \mu_N(\Phi(t)K + \overline{B_\varepsilon})$$

- Fix $t \ll 1$. Then,

local theory

$$\Phi_N(t)(K + B_\delta) \subset \Phi_N(t)K + B_{\varepsilon/2}$$

$$\subset \Phi(t)K + B_\varepsilon$$

Approximation lemma.

$\Rightarrow$ By invariance of μ_N,

(2) $$\mu_N(K + B_\delta) \leq \mu_N(\Phi(t)K + B_\varepsilon).$$

Therefore, $\mu(K) \le \mu(K + B_\delta)$

$$\le \varliminf \mu_N(K + B_\delta)$$

$$\underset{(2)}{\le} \varliminf \mu_N(\Phi(t)K + B_\varepsilon)$$

$$\le \varlimsup \mu_N(\Phi(t)K + \overline{B_\varepsilon})$$

$$\underset{(1)}{\le} \mu(\Phi(t)K + \overline{B_\varepsilon}).$$

$\Rightarrow$ Let $\varepsilon \to 0$. $\mu(K) \le \mu(\Phi(t)K)$ for $t \ll 1$

$\Rightarrow$ for all $t \ge 0$.

$\Rightarrow$ for all $t \in \mathbb{R}$

$\square$

Lec 8: 01 / 02 / 17 (Wed)

Chap 2: Probabilistic well-posedness of dispersive PDEs

· Nonlinear wave equations on $\mathbb{T}^d$ (or $\mathbb{R}^d$)

(NLW) $$\underbrace{(-\partial_t^2 + \Delta)}_{= \Box} u = \pm |u|^{p-1} u$$

2.1 Review on deterministic theory of NLW.

· Consider the non-homog linear wave equation

$$\begin{cases} \Box u = \pm F \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \end{cases}$$

Duhamel
$$\Longleftrightarrow \quad u(t) = \underbrace{\cos(t|\nabla|) u_0 + \frac{\sin t|\nabla|}{|\nabla|} u_1}_{S(t)(u_0, u_1)} \mp \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} F(t')\, dt'$$

With $v = \partial_t u$,

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \mp \begin{pmatrix} 0 \\ F \end{pmatrix}$$

②

Basic tool: Strichartz estimates: $d \geq 2$

$s \geq 0$, $2 \leq q \leq \infty$, $2 \leq r < \infty$, $(q, r, d) \neq (2, \infty, 3)$

Scaling condition: $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s$

admissibility condition: $\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}$. ($\Leftarrow$ Knapp's counterexample)

- Suppose (q, r), s-admissible

  $(\tilde{q}, \tilde{r})$, $(1-s)$-admissible $\quad 0 \leq s \leq 1$

Then, we have

On $\mathbb{R}^d$

$$\boxed{\begin{aligned} &\|(u, \partial_t u)\|_{L^\infty_t \dot{\mathcal{H}}^s} + \|u\|_{L^q_t L^r_x} \\ &\lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^s} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \end{aligned}}$$

$\dot{\mathcal{H}}^s = \dot{H}^s \times \dot{H}^{s-1}$
$\Downarrow$
(u_0, u_1)

Rmk: Thanks to $|\nabla|^{-1}$ in the Duhamel term, Sobolev inquality is also very effective.

---

ex: Cubic NLW on $\mathbb{R}^3$:

$$-\partial_t^2 u + \Delta u = \pm u^3.$$

- LWP in $\dot{\mathcal{H}}^1(\mathbb{R}^3)$

$$\Gamma_{(u_0,u_1)}(u) = S(t)(u_0, u_1) \mp \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} u^3(t')\,dt'.$$

Sobolev: $\|f\|_{L^6_x(\mathbb{R}^3)} \lesssim \|f\|_{\dot{H}^1(\mathbb{R}^3)}$ $\quad \frac{1}{3} = \frac{1}{2} - \frac{1}{6}$ ($\frac{1}{3}$ ← s/d)

$$\left(2^* = \frac{2d}{d-2}\right)$$

$$\|\Gamma u\|_{L^\infty_T \dot{H}^1} \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^1} \leftarrow \text{By unitarity of } e^{\pm it|\nabla|},$$

$$+ \int_0^T \underbrace{\|u^3\|_{L^\infty_T L^2_x}}_{= \|u\|^3_{L^\infty_T L^6_x} \lesssim \|u\|^3_{L^\infty_T \dot{H}^1}} dt',$$

$$\Rightarrow \quad \|\Gamma u\|_{L^\infty_T \dot H^1} \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^1} + T \|u\|^3_{L^\infty_T \dot H^1}$$

$$\|\Gamma u - \Gamma v\|_{L^\infty_T \dot H^1} \lesssim T \left(\|u\|^2_{L^\infty_T \dot H^1} + \|v\|^2_{L^\infty_T \dot H^1} \right) \|u - v\|_{L^\infty_T \dot H^1}.$$

$\Rightarrow$ By the fixed pt argument, cubic NLW on $\mathbb{R}^3$ is LWP in $\dot{\mathcal{H}}^1(\mathbb{R}^3)$

- The same proof works on $\mathbb{T}^3$.

Q: How low can we go?

scaling / dilation symmetry: $u(x,t)$ soln to NLW (with u^p)

$$\Rightarrow \quad u_\lambda(x,t) = \frac{1}{\lambda^{2/p-1}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right) \text{ is also a soln.}$$

We say that s is a critical Sobolev index if

$$\|f_\lambda\|_{\dot H^s(\mathbb{R}^d)} = \|f\|_{\dot H^s(\mathbb{R}^d)}, \quad f_\lambda(x) = \frac{1}{\lambda^{2/p-1}} f\left(\frac{x}{\lambda}\right)$$

$$\boxed{s_{crit} = \frac{d}{2} - \frac{2}{p-1}}$$

$d = 3$, $p = 3$; $s_{crit} = \frac{1}{2}$

Moral: Given $(u_0, u_1) \in \dot{H}^s(\mathbb{R}^d)$,

(i) $s > s_{crit}$: subcritical $\Rightarrow$ (expect) well-posedness

(ii) $s = s_{crit}$: critical $\Rightarrow$ (expect) well-posedness but more delicated

(iii) $s < s_{crit}$: supercritical $\Rightarrow$ bad behavior (ill-posedness)

Norm inflation: $\forall \varepsilon > 0$, $\exists$ smooth soln u_ε and $t_\varepsilon \in (0, \varepsilon)$

s.t. ① $\| (u_\varepsilon(0), \partial_t u_\varepsilon(0)) \|_{\dot{H}^s} < \varepsilon$

$s < s_{crit}$ but ② $\| (u_\varepsilon(t_\varepsilon), \partial_t u_\varepsilon(t_\varepsilon)) \|_{\dot{H}^s} > \varepsilon^{-1}$.

$\Rightarrow$ failure of continuity of the soln map $\Phi : (u_0, u_1) \mapsto (u, \partial_t u)$ (at $(0,0)$)

$$\begin{array}{ccc} (u_0, u_1) & \longmapsto & (u, \partial_t u) \\ \cap & & \cap \\ \dot{H}^s & \longrightarrow & C_t \dot{H}^s \end{array}$$

Christ - Colliander - Tao '03 arXiv

• LWP of cubic NLW in $\dot{\mathcal{H}}^{\frac{1}{2}}(\mathbb{R}^3)$

↑ lowest possible regularity.

Use Strichartz with

$s = \frac{1}{2}, \quad (q, r) = (4, 4)$
$= (\hat{q}, \tilde{r})$

$\frac{1}{4} + \frac{3}{4} = \frac{3}{2} - \frac{1}{2}$

$\frac{1}{4} + \frac{3-1}{2 \cdot 4} = \frac{1}{2} \leq \frac{3-1}{4}$

$\Rightarrow \quad \|\Gamma u\|_{\boxed{L^\infty_T \dot{H}^{1/2} \cap L^4_T L^4_x}} = X^{1/2}$

$(*) \quad \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^{1/2}} + \underbrace{\|u^3\|_{L^{4/3}_T L^{4/3}_x}}_{= \|u\|^3_{L^4_{T,x}}} \leq \|u\|^3_{L^\infty_T \dot{H}^{1/2} \cap L^4_{T,x}} \leq \eta^2 \|u\|_{X^{1/2}}$

No T^θ !!

⇐ manifestation of criticality.

Can carry out a contraction mapping principle in B_η

where $\eta = 2C \|(u_0, u_1)\|_{\dot{\mathcal{H}}^1} \ll 1$

↑ const on lin soln

Rmk: can take $T=\infty$ if initial data is suff. small. ⑦

(small data global well-posedness)

For large data, note that

$$\underbrace{\| S(t)(u_0, u_1) \|_{L^4_{T,x}}}_{\left(\lesssim \|(u_0,u_1)\|_{\dot{H}^{1/2}} < \infty . \right)} \to 0 \text{ as } T \to 0 \quad (\text{by DCT})$$

By Strichartz estimate

$$\| \Gamma u \|_{L^4_{T,x}} \leq \underbrace{\| S(t)(u_0, u_1) \|_{L^4_{T,x}}}_{\to 0} + C \| u \|^3_{L^4_{T,x}} .$$

$\Rightarrow$ contraction on B_η in $L^4_{T,x}$

$$\eta = 2 \| S(t)(u_0, u_1) \|_{L^4_{T,x}} \ll 1 .$$

$\Rightarrow$ By ⊛, $u \in C_T \dot{H}^{1/2}$.

Rmk: ① The same LWP holds on $\mathbb{T}^3$

② Local existence time $T = T((u_0, u_1))$.

(i.e. more info than $\|(u_0, u_1)\|_{\mathcal{H}^{1/2}}$ is needed.)

Let $s < 1/2$. Consider $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3) \setminus \mathcal{H}^{1/2}(\mathbb{T}^3)$

Q: Can we construct a sol'n u with $(u, \partial_t u)\big|_{t=0} = (u_0, u_1)$?

Idea: "Randomize" (u_0, u_1)

f on $\mathbb{T}^d$. $\quad f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in\cdot x}$

Randomization of f:

$$f^\omega(x) = \sum_{n \in \mathbb{Z}^d} \underline{\underline{g_n(\omega)}} \hat{f}(n) e^{in\cdot x}$$

Lec 9: 06/02/17 (Mon)

2.2 Randomization of a function on $\mathbb{T}^d$ (and $\mathbb{R}^d$)

$$f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in\cdot x}, \quad x \in \mathbb{T}^d.$$

· Let $\{g_n\}_{n \in \mathbb{Z}^d}$ be a collection of <u>independent</u> mean-zero random variables with <u>distributions</u> of real and imaginary parts

$\mu_n^{(r)}, \mu_n^{(i)}$

assume indep.

$$\int e^{\gamma\cdot x} d\mu_n^{(r)}(x) \le e^{c\gamma^2}$$
$$\int e^{\gamma\cdot x} d\mu_n^{(i)}(x) \le e^{c\gamma^2}, \qquad \forall\, n \in \mathbb{Z}^d,\ \gamma \in \mathbb{R}$$

$\Leftarrow$ satisfied by standard $\mathbb{C}$-valued Gaussian r.v.'s, Bernoulli (or uniform distri on $S^1 \subset \mathbb{C}$)

Rmk: In many situations, it suffices to assume (uniform) bddness of the k^{th} moment $\mathbb{E}[|g_n|^k] \leq c < \infty$

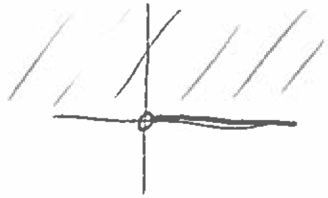
$\Rightarrow$ Randomization f^ω of f:

$$f^\omega(x) = \sum_{n \in \mathbb{Z}^d} \underline{g_n(\omega)}\, \hat{f}(n)\, e^{in\cdot x}$$

Rmk: If f is real, i.e. $\hat{f}(-n) = \overline{\hat{f}(n)}$, $\forall n \in \mathbb{Z}^d$.

Let $I = \bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{Z}_+ \times \{0\}^{d-k-1}$ "$=$" $\mathbb{Z}^d / 2$

$\Rightarrow$ $\mathbb{Z}^d = I \cup \{-I\} \cup \{0\}$.



Then, introduce $\{g_n\}_{n \in I \cup \{0\}}$

and set $g_{-n} = \overline{g_n}$, $\forall n \in \mathbb{Z}^d$. ($g_0$ is real-valued.)

$\Rightarrow$ $f^\omega(x) = \sum_{n \in \mathbb{Z}^d} g_n(\omega)\, \hat{f}(n)\, e^{in\cdot x}$ is <u>real-valued</u>.

<u>Moral</u>: If $f \in H^s \setminus H^{s+\varepsilon}$, then $f^\omega \in H^s \setminus H^{s+\varepsilon}$ a.s.

i.e. there is no gain in differentiability.

- If $f \in L^2$, then $f^\omega \in L^p$, $\forall\ 2 \le p < \infty$, a.s.

$\Rightarrow$ gain of integrability.

<u>Lemma 2.1</u>: (i) Given $f \in H^s(\mathbb{T}^d)$, let f^ω be the randomization defined above. Then, the following tail estimate holds:

$$P\left(\|f^\omega\|_{H^s} > \lambda\right) \le C \exp\left(-c \frac{\lambda^2}{\|f\|_{H^s}^2}\right) \longrightarrow 0 \text{ as } \lambda \to \infty$$

for all $\lambda > 0$. In particular, $f^\omega \in H^s$, a.s.

(ii) Let $f \in L^2(\mathbb{T}^d)$. Then, for any finite $p \ge 2$, we have

$$P\left(\|f^\omega\|_{L^p} > \lambda\right) \le C \exp\left(-c \frac{\lambda^2}{\|f\|_{L^2}^2}\right).$$

for all $\lambda > 0$. In particular, $f^\omega \in L^p$, a.s.

Paley-Zygmund '30

Kahane's book

Lemma 2.2: Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be as above. Then, we have

$$\Big\| \sum_{n\in\mathbb{Z}^d} c_n g_n \Big\|_{L^p(\Omega)} \lesssim \sqrt{p}\, \| c_n \|_{\ell^2_n}.$$

Pf: We first prove

$$P\big(|\Sigma c_n g_n| > \lambda \big) \le C e^{-c\lambda^2 / \|c_n\|^2_{\ell^2_n}}.$$

$$\cdot \int e^{t\Sigma c_n g_n}\, dP \overset{\text{indep}}{=} \prod_n \int e^{t c_n g_n}\, dP$$

$$\le \prod_n e^{\alpha (t c_n)^2} = e^{\alpha^2 t^2 \Sigma c_n^2}.$$

$$\Rightarrow P\big(\Sigma c_n g_n > \lambda \big) \le \frac{\mathbb{E}[e^{t\Sigma c_n g_n}]}{e^{t\lambda}} \le \frac{e^{\alpha^2 t^2 \Sigma c_n^2}}{e^{t\lambda}}$$

choose $t = \frac{\lambda}{2\alpha^2 \Sigma c_n^2}$

$$= e^{-\frac{\lambda^2}{4\alpha^2 \Sigma c_n^2}},$$

Similarly, $P\big(\Sigma c_n g_n < -\lambda \big) \le C e^{-c\lambda^2 / \|c_n\|^2_{\ell^2_n}}$.

$$\Rightarrow \left\| \sum c_n g_n \right\|_{L^p(\Omega)}^p \le p \int_0^\infty \lambda^{p-1} e^{-c \lambda^2 / \|c_n\|_{\ell_n^2}^2} \, d\lambda$$

$$\overset{\text{ch. of var}}{\lesssim} p \left(\sum c_n^2 \right)^{p/2} \underbrace{\int_0^\infty \lambda^{p-1} e^{-\lambda^2/2} \, d\lambda}_{\sim (p-1)!! = \frac{(2p)!}{2^p \cdot p!}}$$

$$\sim \left(p \sum c_n^2 \right)^{p/2}.$$

$\boxtimes$

Aside $\int_{\mathbb{R}} x^k e^{-x^2/2} \, dx$, k even.

$\Leftarrow$ IBP does not help. Use the moment generating function

$$\mathbb{E}[e^{tX}] \overset{\text{ch. f var}}{=} e^{\frac{1}{2}t^2} = \sum_{n=0}^\infty \frac{1}{n! \, 2^n} t^{2n}$$

$$\mathbb{E}\left[\sum_{k=0}^\infty \frac{(tX)^k}{k!} \right] = \sum_{k=0}^\infty \frac{\mathbb{E}[X^k]}{k!} t^k. \quad \text{Compare the coefficients}$$

Pf of Lemma 2.1 : By Minkowski's integral inequality.

$$\forall\, p \geq 2 \quad \Big\| \|f^\omega\|_{H^s} \Big\|_{L^p(\Omega)} \leq \Big\| \Big\| \sum_n \underbrace{\langle n\rangle^s \hat{f}(n)\, e^{in\cdot x}}_{c_n(x)} g_n \Big\|_{L^p(\Omega)} \Big\|_{L^2_x}$$

$$\| \langle\nabla\rangle^s f^\omega \|_{L^2_x} \overset{\text{Lem 2.2}}{\lesssim} \sqrt{p} \Big\| \underbrace{\| \langle n\rangle^s \hat{f}(n) \|_{\ell^2_n}}_{= \|f\|_{H^s}} \Big\|_{L^2_x(\mathbb{T}^d)}$$

$$\sim \sqrt{p}\, \|f\|_{H^s}.$$

By Chebyshev's ineq,

$$P(\|f^\omega\|_{H^s} > \lambda) < \left(\frac{C_0\, p^{1/2} \|f\|_{H^s}}{\lambda} \right)^p, \quad \forall\, p \geq 2.$$

Let $p = \left(\dfrac{\lambda}{C_0\, e \|f\|_{H^s}} \right)^2$. if $p \geq 2$, then we have

$$P(\|f^\omega\|_{H^s} > \lambda) < e^{-p} = e^{-c\, \lambda^2 / \|f\|_{H^s}^2}$$

If $p = \left(\frac{\lambda}{C_0 e \|f\|_{H^s}}\right)^2 \leq 2$, then

choose C s.t. $Ce^{-2} \geq 1$. Then, we have

$$P(\|f^\omega\|_{H^s} > \lambda) \leq 1 \leq Ce^{-2} \leq Ce^{-c\lambda^2/\|f\|_{H^s}^2}$$

(Altenative way: Establish $\mathbb{E}\left[e^{\alpha \|f^\omega\|_{H^s}^2}\right] < \infty$

by Taylor expansion + $\big\| \|f^\omega\|_{H^s} \big\|_{L^p(\Omega)} \lesssim \sqrt{p}\, \|f\|_{H^s}$

(ii) Given $p \geq 2$, let $q \geq p$.

$$\big\| \|f^\omega\|_{L^p_x} \big\|_{L^q(\Omega)} \leq \Big\| \big\| \sum_n \underbrace{\hat{f}(n) e^{in\cdot x}}_{c_n} \cdot g_n \big\|_{L^q(\Omega)} \Big\|_{L^p_x}$$

$$\lesssim \sqrt{q}\, \Big\| \underbrace{\|\hat{f}(n)\|_{\ell^2_n}}_{=\|f\|_{L^2}} \Big\|_{L^p_x}$$

$$\sim \sqrt{q}\, \|f\|_{L^2}. \quad \forall q \geq p.$$

$\square$

Lemma 2.3 : Suppose $f = \sum \widehat{f}(n) e^{in\cdot x} \in H^s \setminus H^{s+\varepsilon}$ for some $\varepsilon > 0$.

Suppose $\exists\, c > 0$ s.t.

$$\limsup_{|n|\to\infty} \underbrace{P(\{|g_n| \le c\})}_{\mu_n(-c,\,c)} \le 1 - \delta < 1$$

(e.g. satisfied if g_n is i.i.d. and $g_n \not\equiv 0$)

Then, $\quad P(f^\omega \in H^{s+\varepsilon}) = 0$

Pf : For simplicity, assume g_n is real-valued.

$$\int e^{-\|f^\omega\|^2_{H^{s+\varepsilon}}} dP \overset{\text{indep}}{=} \prod_n \int e^{-\langle n\rangle^{2(s+\varepsilon)} |\widehat{f}(n)|^2 |g_n|^2} d\mu_n$$

$$\int_{-c}^{c} + \int_{|g_n| \ge c}$$

$$\le \prod_n \Big(\mu_n(-c,c) + \underbrace{e^{-c^2 \langle n\rangle^{2(s+\varepsilon)} |\widehat{f}(n)|^2}}_{=:\alpha_n} (1 - \mu_n(-c,c)) \Big)$$

$$= \prod_n \big(\mu_n(-c,c)(1-\alpha_n) + \alpha_n \big)$$

$$\le \prod_n \big((1-\delta)(1-\alpha_n) + \alpha_n \big) = \prod_n \big(1 - \delta(1-\alpha_n) \big) \overset{(**)}{=} 0$$

$$\Rightarrow \quad \|f^\omega\|_{H^{s+\varepsilon}} = \infty, \quad a.s.$$

• Proof of (**): By assumption, $\sum \langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 = \infty$.

Then, we claim $\sum (1-\alpha_n) = \infty$

$$\sum \left(1 - e^{-c^2 \langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2}\right)$$

• If $\langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 \not\to 0$ as $|n| \to \infty$, then $\sum (1-\alpha_n) \geq \sum (1-\varepsilon) = \infty$.

• otherwise, $\exists N$ s.t. $c^2 \langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 \leq 1$, $\forall |n| \geq N$

$$\Rightarrow \sum_{|n| \geq N} (1-\alpha_n) \leq C \sum_{|n| \geq N} \langle n \rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 = \infty$$

(b/c $e^{-x} \leq 1 - C' x$ for $0 \leq x \leq 1$ (for some $C' \ll 1$)

Then, $\left(\prod_n (1 - \delta(1-\alpha_n))\right)^{-1} = \prod_n \left(1 + \frac{\delta(1-\alpha_n)}{1-\delta(1-\alpha_n)}\right)$

$$\leq \prod_n (1 + 2\delta(1-\alpha_n)) \quad \text{by choosing } \delta \leq \tfrac{1}{2}.$$

$$= \infty.$$

$$\Big(\prod_n (1 + \underset{>0}{a_n}) < \infty \iff \sum a_n < \infty. \Big)$$

Lec 10 : 08/02/17 (Wed)

· Probabilistic Strichartz estimate

$S(t) = e^{it\Delta}$ linear Schrödinger semigroup

· Strichartz estimates on $\mathbb{R}^d$

$$\| S(t) f \|_{L^q_t L^r_x (\mathbb{R} \times \mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$$

$2 \le q, r \le \infty$, $(q, r, d) \ne (2, \infty, 2)$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

Prop 2.4: $f \in L^2(\mathbb{T}^d)$

Given $2 \le q, r < \infty$, we have

$$P\left(\| S(t) f^\omega \|_{L^q_t L^r_x ([-T,T] \times \mathbb{T}^d)} > \lambda \right) \le C \exp\left(- c \frac{\lambda^2}{T^{\frac{2}{q}} \|f\|^2_{L^2}} \right)$$

for all $\lambda > 0$ and all $T > 0$.

$(\to 0$ as $\lambda \to \infty$ or $T \to 0)$

Rmk: • Set $\lambda = T^{\theta}\|f\|_{L^2}$, $\theta < \frac{1}{q}$.

Prop 2.4 $\Rightarrow$ $\|S(t)f^{\omega}\|_{L^q_T L^r_x} \le T^{\theta}\|f\|_{L^2}$

outside a set of prob $\le c\, e^{-cT^{2(\theta-\frac{1}{q})}}$ $(\to 0 \text{ as } T \to 0)$

• Fix $T > 0$. Given $\varepsilon > 0$,

$$\|S(t)f^{\omega}\|_{L^q_T L^r_x} \le C_T\left(\log\frac{1}{\varepsilon}\right)^{1/2}\|f\|_{L^2}$$

outside a set of prob $< \varepsilon$

$\left(\text{or } \le C_T K \|f\|_{L^2},\ \text{prob} < e^{-cK^2}\right)$

Pf: Let $p \ge \max(q, r)$.

$$\Big\| \|S(t)f^{\omega}\|_{L^q_T L^r_x} \Big\|_{L^p(\Omega)} \overset{\text{Mink}}{\le} \Big\| \|S(t)f^{\omega}\|_{L^p(\Omega)} \Big\|_{L^q_T L^r_x}$$

$$S(t)f^{\omega} = \sum \underbrace{e^{-it|n|^2}\hat{f}(n)e^{in\cdot x}}_{=C_n(t,x)} g_n$$

$$\lesssim \sqrt{p}\,\Big\| \|\hat{f}(n)\|_{\ell^2_n} \Big\|_{L^q_T L^r_x}$$

$$= \sqrt{p}\, T^{\frac{1}{q}}\|f\|_{L^2_x} \Rightarrow \text{Apply Chebyshev} \quad \square$$

• The same estimate holds for the half wave operator $e^{\pm it|\nabla|}$ (3)

$\Rightarrow$ Also for $\cos t|\nabla| = \dfrac{e^{it|\nabla|} + e^{-it|\nabla|}}{2}$

• Next, we consider $\dfrac{\sin t|\nabla|}{|\nabla|}$.

<u>Prop 2.5</u>: (probabilistic Strichartz estimate for wave eqn)

Let $(f_0, f_1) \in \mathcal{H}^0 = L^2 \times H^{-1}$.

$$P\Big(\| \underbrace{S(t)}_{= \left(\cos t|\nabla|,\ \frac{\sin t|\nabla|}{|\nabla|}\right)} (f_0^\omega, f_1^\omega) \|_{L^q_t L^r_x (I \times \mathbb{T}^d)} > \lambda \Big)$$

$I = [a, b]$

$$\le C \exp\left(-c \frac{\lambda^2}{\max(1, a^2, b^2)|I|^{2/q} \|(f_0, f_1)\|^2_{\mathcal{H}^0}} \right)$$

<u>Pf</u>: Repeat the proof of Prop 2.4, noting

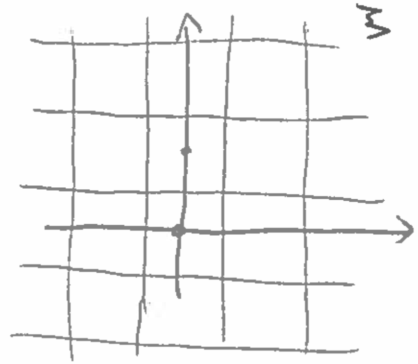
$$\left| \frac{\sin t|n|}{|n|} \right|_{n=0} = |t| \le \max(1, |a|, |b|)$$

See O. Pocovnicu arXiv '15

· $\mathbb{R}^d$?

$$f(x) = \int \hat{f}(\xi) e^{ix\cdot\xi} d\xi$$

$$= \sum_{n\in\mathbb{Z}^d} \int_{Q_n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi$$



$$Q_n = n + \left(-\tfrac{1}{2}, \tfrac{1}{2}\right]^d$$

$$\Rightarrow \quad \mathbb{R}^d_\xi = \bigcup_{n\in\mathbb{Z}^d} Q_n \quad \text{Wiener decomposition}$$

For practical purpose, let $\psi \in C_c^\infty(\mathbb{R}^d; [0,1])$ s.t

$$\operatorname{supp} \psi \subset [-1,1]^d$$

and $$\sum_{n\in\mathbb{Z}^d} \underbrace{\psi(\xi - n)}_{\text{smoothed version of } 1_{Q_n}} \equiv 1$$

$$\Rightarrow \quad f(x) = \sum_{n\in\mathbb{Z}^d} \underbrace{\psi(D-n) f}_{= \int \psi(\xi - n)\hat{f}(\xi) e^{ix\cdot\xi} d\xi}$$

Randomization of a function on $\mathbb{R}^d$

$$f^\omega = \sum_{n\in\mathbb{Z}^d} g_n \psi(D-n) f.$$

Rmk: · Why unit scale? $\Leftarrow$ a matter of choice.

Can introduce a similar randomization based on dilated cubes. (e.g. Bényi-O-P '15)

· translation on $\mathbb{R}^d_\xi \iff$ modulation $e^{ix\cdot\xi_0}$ on $\mathbb{R}^d_x$

$\rightsquigarrow$ modulation space $\|f\|_{M^{p,q}_s} = \big\| \langle n\rangle^s \|\psi(D-n)f\|_{L^p_x} \big\|_{\ell^q_n}$

Wiener amalgam space

$$\|f\|_{W^{p,q}} = \big\| \|\psi(D-n)f\|_{\ell^q_n} \big\|_{L^p_x}$$

(Rmk: On $\mathbb{T}^d$,

$$\|f\|_{M^{p,q}} = \|f\|_{W^{p,q}} = \|f\|_{\mathcal{F}L^q} = \|\hat{f}(n)\|_{\ell^q_n}$$)

· What about randomization based on the LP-decomposition?

$$f = \sum_{\substack{N\geq 1\\ \text{dyadic}}} P_N f, \quad P_N \overset{LP}{=} \text{proj onto } \{|\xi|\sim N\}.$$

Let's consider

$$f^\omega = \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \pm P_N f.$$

$1 < p < \infty$

$$\| f^\omega \|_{L^p} \overset{\text{LP theory}}{\sim} \Big\| \Big(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} |P_N f|^2 \Big)^{1/2} \Big\|_{L^p}$$

$$\sim \|f\|_{L^p} \Rightarrow \text{No gain of integrability!}$$

· Back to Wiener randomization on $\mathbb{R}^d$.

We can not use the finiteness of the domain.

$\Rightarrow$ Use Bernstein's inequality

$$\| P_N f \|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| f \|_{L^p}$$

In fact, really smooth cutoff

$$\| \mathcal{F}^{-1} (\mathbb{1}_Q f) \|_{L^q} \lesssim |Q|^{\frac{1}{p} - \frac{1}{q}} \| f \|_{L^p}.$$

$\xi_0 =$ center of Q

$$\| e^{ix \cdot \xi_0} \mathcal{F}^{-1} (\mathbb{1}_Q f) \|_{L^q}$$

centered at 0 on $\mathbb{R}^d_\xi$

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$$\Rightarrow \quad \| \Psi(D-n) f \|_{L^r_x} \lesssim \| \Psi(D-n) f \|_{L^2_x}$$

$\Rightarrow$ For $p \geq \max(q, r)$

$$\big\| \| S(t) f^\omega \|_{L^q_T L^r_x} \big\|_{L^p(\Omega)} \overset{\text{Mink}}{\leq} \Big\| \big\| \sum_n \underbrace{(\Psi(D-n) S(t) f)}_{= c_n} g_n \big\|_{L^p(\Omega)} \Big\|_{L^q_T L^r_x}$$

$$\leq \sqrt{p} \, \big\| \| \Psi(D-n) S(t) f \|_{\ell^2_n} \big\|_{L^q_T L^r_x}$$

$$\overset{\text{Mink}}{\leq} \sqrt{p} \, \big\| \underbrace{\| \Psi(D-n) S(t) f \|_{L^r_x}}_{\lesssim_{\text{Bernstein}} \| \Psi(D-n) \cancel{S(t)} f \|_{L^2_x}} \big\|_{L^q_T \ell^2_n}$$

unitarity

$$\lesssim \sqrt{p} \, \big\| \| f \|_{L^2(\mathbb{R}^d)} \big\|_{L^q_T}$$

$$\lesssim \sqrt{p} \, T^{\frac{1}{q}} \| f \|_{L^2(\mathbb{R}^d)} \Rightarrow \text{Prob. Strichartz estimate on } \mathbb{R}^d.$$

Prop 2.6: (Global-in-time prob. Strichartz estimate on $\mathbb{R}^d$)

Let (q, r) be Schrödinger admissible.

$q, r < \infty$.

Let $\tilde{r} \geq r$. Then,

finite

$$P\left(\| S(t) f^\omega \|_{L^q_t L^{\tilde{r}}_x (\mathbb{R} \times \mathbb{R}^d)} > \lambda \right) \leq C \exp\left(-c \frac{\lambda^2}{\|f\|^2_{L^2}} \right)$$

$\Leftarrow$ Proof uses deteministic Strichartz estimate.

Lec 11 : 13 / 02 / 17 (Mon)

Prop 2.7 : $f \in H^{\varepsilon}(\mathbb{T}^d)$, $\varepsilon > 0$.

Given $2 \le q < \infty$, we have

$$P\left(\| S(t) f^{\omega} \|_{L^q_t L^\infty_x([-T,T] \times \mathbb{T}^d)} > \lambda \right) \le C \exp\left(- c \frac{\lambda^2}{T^{2/q} \|f\|^2_{H^\varepsilon}} \right)$$

Pf: By Sobolev embedding,

$$\| S(t) f^{\omega} \|_{L^q_T L^\infty_x} \lesssim \| \langle \nabla \rangle^{\varepsilon} S(t) f^{\omega} \|_{L^q_T L^r_x} \qquad \varepsilon r > d$$

$$= \| S(t) (\langle \nabla \rangle^{\varepsilon} f^{\omega}) \|_{L^q_T L^r_x}$$

$\Rightarrow$ Apply Prop 2.4.

$\boxtimes$

- The same estimate holds if $q = \infty$.

By Sobolev in time.

$$\| S(t) f^\omega \|_{L^\infty_T L^r_x} \lesssim \| \langle \partial_t \rangle^\varepsilon \| S(t) f^\omega \|_{L^r_x} \|_{L^q_T} \qquad \varepsilon q > 1$$

(*)

See also Oh-Pocovnicu '16 JMPA.

If we can place $\langle \partial_t \rangle^\varepsilon$ inside L^r_x-norm, we can replace $\langle \partial_t \rangle^\varepsilon$ by $\langle \nabla \rangle^{2\varepsilon}$ (Schrödinger case)

ex: $\varepsilon = 1$. consider

$$\| \partial_t \| F(t) \|_{L^r_x} \|_{L^q_T} = \Big\| \lim_{h \to 0} \frac{\| F(t+h) \|_{L^r_x} - \| F(t) \|_{L^r_x}}{h} \Big\|_{L^q_T}$$

$$\leq \Big\| \lim_{h \to 0} \Big\| \frac{F(t+h) - F(t)}{h} \Big\|_{L^r_x} \Big\|_{L^q_T}$$

$$= \Big\| \Big\| \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} \Big\|_{L^r_x} \Big\|_{L^q_T}$$

for nice F.

Justify (*) by interpolation.

2.3 Probabilistic well-posedness of NLW on $\mathbb{T}^3 / \mathbb{R}^3$

- Cubic NLW : $\begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \end{cases}$ defocusing (needed only for the global argument)

Recall $s_{crit} = 1/2$.

Assume $(u_0, u_1) \in \mathcal{H}^s \setminus \mathcal{H}^{1/2}$, $0 \le s < 1/2$

$\Rightarrow$ Randomize (u_0, u_1) and consider

$$(*) \quad \begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ (u, \partial_t u)\big|_{t=0} = (u_0^\omega, u_1^\omega) \end{cases}$$

Thm 2.8: The Cauchy problem $(*)$ is almost surely locally well-posed. More precisely, given small $T > 0$, $\exists\, \Omega_T$ (also depends on the fixed (u_0, u_1)) s.t. (1) $P(\Omega_T^c) \le C e^{-c/T^\gamma}$ for some $\gamma > 0$.

(2) For each $\omega \in \Omega_T$, $\exists!$ soln $\underset{u^\omega}{u}$ to $(*)$ with $(u, \partial_t u)\big|_{t=0} = (u_0^\omega, u_1^\omega)$

Rmk: • $\Sigma = \bigcup_{0<T\ll 1} \Omega_T$. Then, $P(\Sigma) = 1$.

• For each $\omega \in \Sigma$, u exists on $[-T_\omega, T_\omega]$.

• u lies in

$$S(t)(u_0^\omega, u_1^\omega) + C_T H^1$$

↑ smoother

Pf: Write $u = \underbrace{S(t)(u_0^\omega, u_1^\omega)}_{=: z^\omega} + v$

$v \leftarrow \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} u^3(t')\,dt'$

$\Rightarrow$ v satisfies

$$\begin{cases} -\partial_t^2 v + \Delta v = (v+z)^3. \\ (v, \partial_t v)|_{t=0} = 0 \end{cases}$$

Let $\Gamma v = -\int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} (v+z)^3(t')\,dt'$.

$$\Rightarrow \|\Gamma v\|_{L_T^\infty H^1} \le \left\| \int_0^t \|(v+z)^3(t')\|_{L_x^2}\,dt' \right\|_{L_T^\infty}$$

$$= \|(v+z)^3\|_{L_T^1 L_x^2} = \|v+z\|_{L_T^3 L_x^6}^3$$

$$\lesssim \|v\|^3_{L^3_T L^6_x} + \|z\|^3_{L^3_T L^6_x}$$

$$\overset{\text{Sob in } x}{\lesssim} T\|v\|^3_{L^\infty_T \dot H^1} + \underbrace{\|z\|^3_{L^3_T L^6_x}}_{\substack{\le 1 \text{ outside a set of} \\ \text{prob.} \le C e^{-\frac{c}{T^\delta \|(u_0,u_1)\|^2_{L^2}}}}}$$

- A difference estimate follows in a similar manner:

$$\|\Gamma v_1 - \Gamma v_2\|_{L^\infty_T \dot H^1} \lesssim T^{1/3}\Big(T^{2/3}\|v_1\|^2_{L^\infty_T \dot H^1} + T^{2/3}\|v_2\|^2_{L^\infty_T \dot H^1} + \|z\|^2_{L^3_T L^6_x}\Big)\|v_1 - v_2\|_{L^\infty_T \dot H^1}.$$

$\Rightarrow$ Γ^ω is a contraction on $B_{O(1)}$ in $L^\infty \dot H^1$ for every $\omega \in \Omega_T$

- Also, $\|v(t)\|_{L^2_x} \overset{\text{Mink}}{\le} \int_0^t \underbrace{\|\partial_t v(t')\|_{L^2}}_{\text{controlled}} dt'$

$\overset{\text{FTC}}{=} \int_0^t \partial_t v\, dt'$

$$\partial_t v = -\int_0^t \cos\big((t-t')|\nabla|\big)(v+z)^3(t')\,dt' \in C_T L^2_x$$

$\boxtimes$

The same proof works for

$$\begin{cases} -\partial_t^2 v + \Delta v = (v + z^\omega)^3 \\ (v, \partial_t v)\big|_{t=0} = (v_0, v_1) \in \mathcal{H}^1 \end{cases}$$

by performing a fixed pt argument in $B_{O(\|(v_0,v_1)\|_{\mathcal{H}^1})}$

$$\Gamma v = S(t)(v_0, v_1) - \int_0^t \frac{\sin(t-t')|\nabla|}{|\nabla|} (v+z)^3(t')\,dt'.$$

Rmk: In terms of the original NLW, we have a.s. LWP for

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ (u, \partial_t u) = \underset{\mathcal{H}^1}{\underset{\Uparrow}{(v_0, v_1)}} + \underbrace{(u_0^\omega, u_1^\omega)}_{\uparrow \text{ rough \& random}} \end{cases}$$

$0 < s \leq 1$ **Thm 2.9:** The defocusing cubic NLW is a.s. GWP (with respect to (u_0^ω, u_1^ω).)

For $s=0$, see Burq-Tzvetkov JEMS '14

Pf: $E(u, \partial_t u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int (\partial_t u)^2 + \frac{1}{4}\int u^4$

- Estimate the growth of $E(v(t))$ ← in $\mathcal{H}^1$

$$\Rightarrow \quad \partial_t E(v) = \int \partial_t v \big(\underbrace{\partial_t^2 v - \Delta v}_{= -(v+z)^3} + v^3\big)\, dx$$

No v^3 term

$$\overset{C-S}{\leq} \left(\int (\partial_t v)^2 dx\right)^{1/2} \left(\int (v^2|z| + |v|z^2 + |z|^3)^2 dx\right)^{1/2}$$

$|v|z^2 = z^{1/2} z^{3/2} \lesssim v^2|z| + |z|^3$

$v^4 z^2$

$$\lesssim E(v)\left(1 + \|z\|^2_{L^\infty_{T,x}}\right) + \|z\|^6_{L^\infty_T L^6_x}$$

Gronwall $\Rightarrow$

$$E(v)(t) \lesssim \|z\|^6_{L^6_{T,x}}\, e^{c\int_0^t 1 + \|z\|^2_{L^\infty_{T,x}} dt'}, \quad |t| \leq T.$$

Lec 12: 15/02/17 (Wed)

Our goal is to prove a.s. GWP.

$\Leftarrow$ suffices to prove "almost a.s. GWP"

$\forall\, T, \varepsilon > 0 \ \exists\, \Omega_{T,\varepsilon}$ with $P(\Omega_{T,\varepsilon}^c) < \varepsilon$

s.t. $\forall\, \omega \in \Omega_{T,\varepsilon}$, $\exists !$ soln u^ω with $(u^\omega, \partial_t u^\omega)\big|_{t=0} = (u_0^\omega, u_1^\omega)$

on $[-T, T]$

Fix $T, \varepsilon > 0$.

$$A_\lambda = \{ \omega \in \Omega : \| z^\omega \|_{L^6_{T,x} \cap L^\infty_{T,x}} \le \lambda \}$$

$\Rightarrow$ Can choose $\lambda = \lambda(\|(u_0, u_1)\|_{\mathcal{H}^\varepsilon}, T) \gg 1$ s.t.

$$P(A_\lambda^c) < \frac{\varepsilon}{2}$$

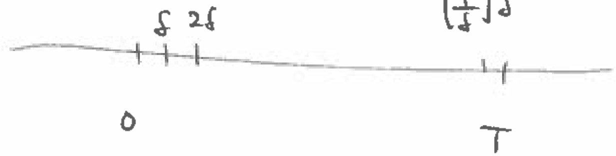
i.e. on A_λ, $E(v)(t) \le C_T$. $\quad \forall\, |t| \le T$.

① Choose local existence time

$$\delta = \delta\Big(\underbrace{\sup_{|t|\le T} \|(v,\partial_t v)(t)\|_{\mathcal{H}^1}}_{\le C_T^{1/2}}\Big) > 0$$

② By possibly making δ smaller, we have

$$P\Big(\underbrace{\|z\|_{L^3_{I_j}L^6_x} \ge 1}_{\text{set } =:\, B_j^c}\Big) \le e^{-c/\delta^{2/3}}, \qquad I_j = ((j-1)\delta, j\delta)$$



$$\Rightarrow \forall \omega \in A_\lambda \cap \bigcap_{j=-[\frac{T}{\delta}]-1}^{[T/\delta]+1} B_j,$$

$$\exists!\ u = u^\omega \text{ on } [-T, T]$$

Lastly, $P\big((\bigcap_j B_j)^c\big) \lesssim \frac{T}{\delta} P(B_j^c) \le T\delta^{-1} e^{-c/\delta^{2/3}}$

$< \varepsilon/2$ by choosing $\underset{\delta(T,\varepsilon)}{\delta} \ll 1.$

$\boxtimes$

· <u>Rmk</u>: One actually need to work with v_N ← smooth

$\Leftarrow$ soln to $-\partial_t^2 v_N + \Delta v_N = (v_N + \underbrace{z_N^\omega}_{= P_{\le N} z^\omega})^3$

to justify the switching of ∂_t and $\int dx$

in the probabilistic energy estimate (and let $N \to \infty$)

indep of N.

· <u>$3 < p \le 5$</u>: $-\partial_t^2 u + \Delta u = \underbrace{|u|^{p-1} u}_{=: \mathcal{N}(u)}$ on $\mathbb{R}^3$ or $\mathbb{T}^3$

As before, set $u = z + v$

$$\Rightarrow \quad -\partial_t^2 v + \Delta v = \mathcal{N}(v + z).$$

only need some space-time control.

1 <u>a.s. LWP</u>: Consider

$$\begin{cases} -\partial_t^2 v + \Delta v = \mathcal{N}(v + f) \\ (v, \partial_t v)\big|_{t=t_0} = (v_0, v_1) \end{cases} \quad \text{on } I \ni t_0$$

$$\|(\Gamma v, \partial_t \Gamma v)\|_{C_I \dot{\mathcal{H}}^1} + \|\Gamma v\|_{L_I^{2p/p-3} L_x^{2p}}$$

$$\lesssim \|(v_0, v_1)\|_{\dot{\mathcal{H}}^1} + \|N(v+f)\|_{L_I^1 L_x^2}$$

$$\leq |I|^{\theta} \|v+f\|^p_{L_I^{\frac{2p}{p-3}} L_x^{2p}}$$

Strichartz pair $(s=1)$

$x: \frac{1}{2} = \underbrace{\frac{1}{2p} + \cdots + \frac{1}{2p}}_{p \text{ factors}}$

$t: 1 = p \cdot \frac{p-3}{2p} + \theta$

Need $\frac{p-3}{2} \leq 1 \Leftrightarrow p \leq 5$

$\theta = 0$ when $p = 5$

$$\lesssim \|(v_0, v_1)\|_{\dot{\mathcal{H}}^1} + |I|^{\theta} \|v\|^p_{L_I^{\frac{2p}{p-3}} L_x^{2p}} + |I|^{\theta} \|f\|^p_{L_I^{\frac{2p}{p-3}} L_x^{2p}}$$

- $p < 5$: LWP on $|I| \leq \delta \sim \left(\|(v_0, v_1)\|_{\dot{\mathcal{H}}^1} + \|f\|^p_{L_I^{2p/p-3} L_x^{2p}} \right)^{-(p-1)}$
- $p = 5$: δ depends on the profile of (v_0, v_1).

---

(2) a.s. GWP: $3 < p \leq 5$: ($p = 5$: Oh-Pocovnicu '16)

side computation: $\partial_t (|v|^{p+1}) = \partial_t (v^2)^{\frac{p+1}{2}} = (p+1)|v|^{p-1} v \partial_t v$

Let $E(v) = \frac{1}{2} \int (\partial_t v)^2 + \frac{1}{2} \int |\nabla v|^2 + \frac{1}{p+1} \int |v|^{p+1}$

$$\Rightarrow \partial_t E(v) = \int \partial_t v \, (\underbrace{\partial_t^2 v - \Delta v}_{-|v+f|^{p-1}(v+f)} + |v|^{p-1}v)$$

$$F(x) = |x|^{p-1}x$$
$$F'(x) = p|x|^{p-1}$$
$$F''(x) = p(p-1)|x|^{p-3}x$$

Taylor

$$\Rightarrow |v+f|^{p-1}(v+f) - |v|^{p-1}v = p|v|^{p-1}f + \frac{p(p-1)}{2!}|v+\theta f|^{p-3}(v+\theta f)\cdot f^2.$$

$$\Rightarrow \partial_t E(v) = -p\int \partial_t v \cdot |v|^{p-1}f - \frac{p(p-1)}{2}\int \partial_t v \, \underbrace{|v+\theta f|^{p-2}(v+\theta f)\cdot f^2}_{\lesssim |v|^{p-2}f^2 + f^p.}$$

$$=: \mathrm{I} + \mathrm{II}.$$

$$\cdot\ |\mathrm{II}| \overset{C\text{-}S}{\lesssim} \left(\int (\partial_t v)^2\right)^{1/2} \left(\|f(t)\|_{L^\infty_x}^4 \int |v|^{2(p-2)} + \|f(t)\|_{L^{2p}}^{2p}\right)^{1/2}$$

$$\lesssim \left(1 + \|f\|_{L^\infty_{T,x}}^{2}\right) E(v) + \|f(t)\|_{L^{2p}}^{2p}.$$

$2(p-2) \le p+1$

$\Leftrightarrow p \le 5$

$$\cdot \int_0^t I = -\underbrace{p\int_0^t\!\!\int \partial_t v\, |v|^{p-1}}_{= \partial_t(|v|^{p-1} v)} f$$

$$\overset{\text{IBP}}{=} -\int_x |v|^{p-1} v\, f \,\Big|_0^t \qquad v(0) = 0$$

$$\overset{\text{Young}}{\lesssim} \underbrace{\varepsilon \int_x |v(t)|^{p+1}}_{\varepsilon E(t) \;\Leftarrow\; \text{hide it on LHS}} + \underbrace{\frac{1}{\varepsilon}\int_x |f(t)|^{p+1}}_{= \frac{1}{\varepsilon}\|f\|^{p+1}_{L^\infty_T L^{p+1}_x}}$$

$$+\, p \int_0^t \boxed{\int_x |v|^{p-1} v\, \underbrace{\partial_t f}_{=: \langle\nabla\rangle \tilde{f}}}$$

① place $\langle\nabla\rangle^{s-}\tilde{f}$ in $L^\infty_{T,x}$

$$\int_x \langle\nabla\rangle^{1-s+}\left(|v|^{p-1} v\right) \langle\nabla\rangle^{s-}\tilde{f}$$

② $\left(\int |v|^{(p-1)\frac{p+1}{p-1}}\right)^{\frac{p-1}{p+1}} \lesssim E^{\frac{p-1}{p+1}}$

If $f = z$, then

$\partial_t z = \langle\nabla\rangle \tilde{z}$

$\cdot\ \tilde{z} = \tilde{S}(t)(u_0^\omega, u_1^\omega)$

$$= -\frac{|\nabla|}{\langle\nabla\rangle}\sin(t|\nabla|)u_0^\omega + \frac{\cos t|\nabla|}{\langle\nabla\rangle} u_1^\omega.$$

Satisfies the same probabilistic Strichartz estimates

③ $\left(\int \left(\langle \nabla \rangle^{1-s+} v \right)^{\frac{p+1}{2}} \right)^{\frac{2}{p+1}}$ 7

low freq easy

$= \| \langle \nabla \rangle^{1-s+} v \|_{L^{\frac{p+1}{2}}}$ $\Rightarrow$ Assume $\langle \nabla \rangle \sim |\nabla|$

$\lesssim \| \nabla v \|_{L^2}^{\theta} \ \| v \|_{L^{p+1}}^{1-\theta}$

$$\frac{2}{p+1} = \frac{\theta}{2} + \frac{1-\theta}{p+1} \Rightarrow \theta = \frac{2}{p-1}$$

$$1-s+ = \theta \cdot 1 + (1-\theta) \cdot 0$$

$$\boxed{s >} 1-\theta. = 1 - \frac{2}{p-1} = \boxed{\frac{p-3}{p-1}}$$

$p=3$: $s>0$
Burq-Tzvet '14

$p=5$: $s > 1/2$
O-P '16

$3<p<5$: Xia

Lührman-Mendelson (weaker)

$$\lesssim E^{\frac{\theta}{2} + \frac{1-\theta}{p+1}}$$

$\leq \frac{2}{p+1}$

$$\Rightarrow \int_0^t I_2 \lesssim \| \langle \nabla \rangle^{s-} f \|_{L^{\infty}_{T,x}} \int_0^t E(t')\,dt'.$$

$\Rightarrow$ Apply Gronwall.

Lec 13: 20/02/17 (Mon)

· Digression: Failure of the endpt Strichartz estimate (wave)

$\circledast$ False: $\| S(t)(u_0, u_1) \|_{L^2_t L^\infty_x} \lesssim \| (u_0, u_1) \|_{\dot{H}^1}$ $\quad d = 3$

$$(q, r, d) = (2, \infty, 3).$$

(Schrödinger: fails at $(2, \infty, \underline{2})$ $\leftarrow$ ex: $\|f\|_{L^\infty_x(\mathbb{R}^2)} \not\lesssim \|f\|_{H^1(\mathbb{R}^2)}$

· Nonhomog lin wave eqn:

$$-\partial_t^2 u + \Delta u = F$$
$$(u, \partial_t u)|_{t=0} = (u_0, u_1) \qquad \text{on } \mathbb{R}^{3+1}$$

$$\Rightarrow u(t,x) = \frac{\partial}{\partial t} t \int_{\mathbb{S}^2} u_0(x + t\omega)\, d\sigma(\omega)$$

Kirchoff

$$+ \; t \int_{\mathbb{S}^2} u_1(x + t\omega)\, d\sigma(\omega)$$

$$+ \int_0^t t' \int_{\mathbb{S}^2} F(t-t', x + t'\omega)\, d\sigma(\omega)\, dt'.$$

Idea: adaptation of Stein's counterexample for the spherical maximal function. (p419, Stein)

$$f \longmapsto \sup_{t>0} |A_t f| \quad \text{on } L^p$$

$$A_t f(x) = \int_{|y|=1} f(x-ty)\, d\sigma(y) = f * d\sigma_t$$

spherical average

$$\int g(x)\, d\sigma_t(x) = \int g(tx)\, d\sigma(x)$$

- Bounded: $d \geq 2, \quad p > \frac{d}{d-1}$
- Fail: $d=1, \quad p < \infty$

  $d \geq 2; \quad p \leq \frac{d}{d-1}$

$d=1$: $A_t f(x) = \frac{1}{2}[f(x+t) + f(x-t)]$

f, positive, unbdd near 0.

$\Rightarrow \sup_t A_t f(x) = \infty$

but $f \in L^p(\mathbb{R})$

$d \geq 2$: $\quad f(x) = \begin{cases} \dfrac{|x|^{1-d}}{\log 1/|x|} & \text{near } 0 \\ 0, & |x| \gg 1 \end{cases}$ ③

$\Rightarrow \forall x$, $A_t f(x)$ is unbdd near x.

$\Rightarrow \sup_t A_t f(x) = \infty$ but $f \in L^p(\mathbb{R}^d)$, $p = \frac{d}{d-1}$.

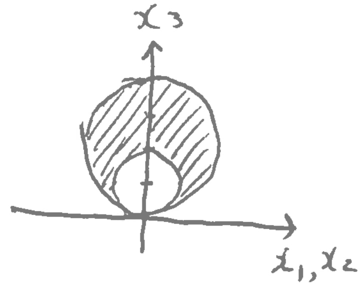
· Back to wave Strichartz.

Prop 2.10: $\quad u_0 = F = 0.$

$$u_1(x) = \frac{\mathbb{1}_A(x)}{|x|^2(1+\log|x|)^\alpha}, \quad \frac{1}{2} < \alpha \leq 1$$

$d=3$

$$A = B(2\hat{e}_3, 2) \setminus B(\hat{e}_3, 1)$$



Claim: $u_1 \in L^2$ but

$u(t, t\hat{e}_3) = \infty, \quad \forall\, 1 < t < 2$

lin soln

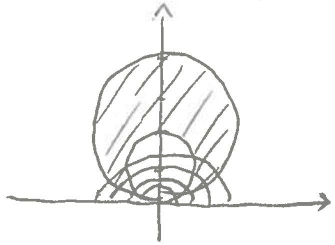
$\Rightarrow$ (LHS) of ⊛ $= \infty$. $\Rightarrow$ ⊛ fails.

④

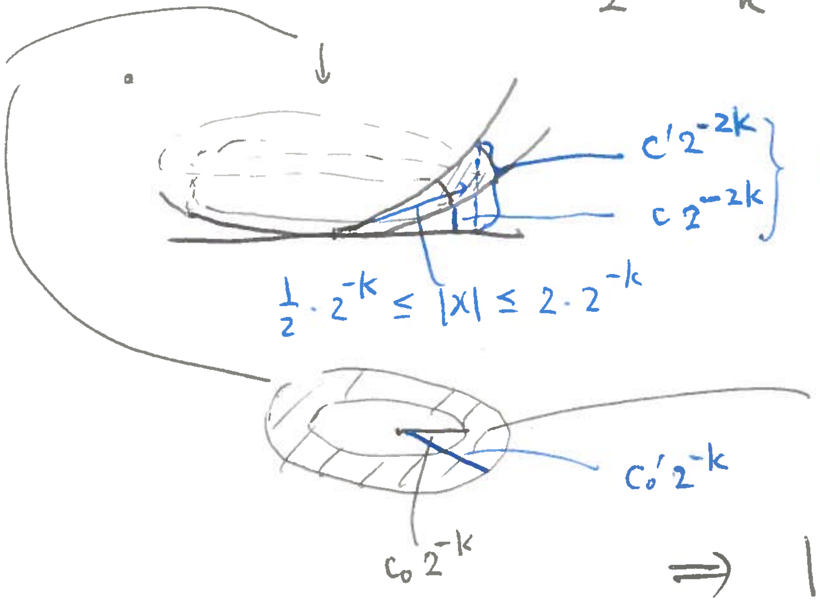
· Portion of u_1 in the dyadic shell $\{|x| \sim 2^{-k}\} \cap A = A_k$ lives in a set of volume $\sim 2^{-4k}$ and has height $\sim \dfrac{2^{2k}}{k^\alpha}$

$$\|u_1\|_{L^2} \sim \left(\sum_k \left(\frac{2^{2k}}{k^\alpha}\right)^2 \times 2^{-4k} \right)^{1/2}$$

$$\sim \left(\sum \frac{1}{k^{2\alpha}} \right)^{1/2} < \infty, \quad \alpha > 1/2$$



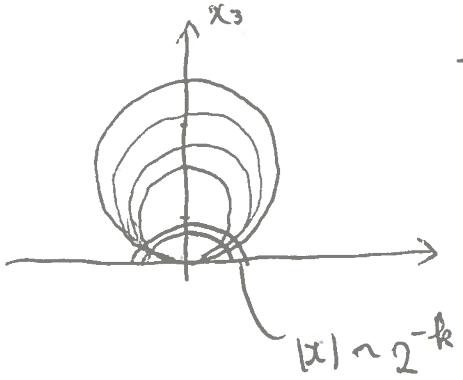
On A_k, $u_1(x) = \dfrac{1}{2^{-2k} k^\alpha}$.



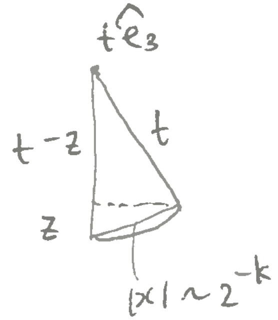
$1^2 - (1-z)^2$
$= 2^{-2k} - z^2$
$\Rightarrow z \sim 2^{-2k}$

$\Rightarrow |A_k| \sim 2^{-4k}$

$\cdot \quad u(t,x) = t \int_{S^2} u_1(x + t\omega)\, d\sigma(\omega)$

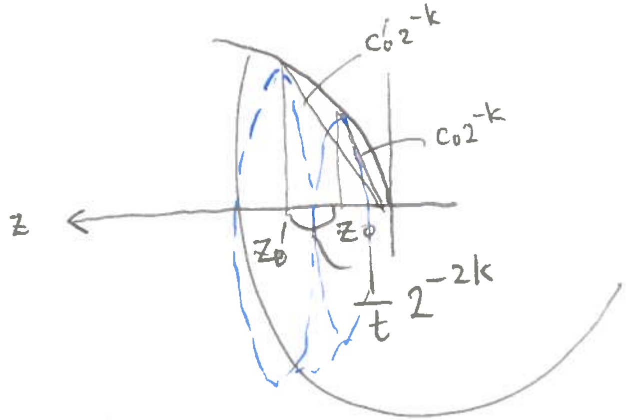


Take $x = t\hat{e}_3$, $\quad 1 < t < 2$



$$t^2 - (t-z)^2 = |x|^2 - z^2$$

$$\Rightarrow z \sim \frac{1}{t} 2^{-2k}$$



$f(z) = \sqrt{t^2 - (z-t)^2} \Leftarrow$ rotate around the z-axis

$$\text{surface area} = \int_{z_0}^{z_0'} 2\pi \cancel{f(z)} \underbrace{\sqrt{1 + (f'(z))^2}}_{\frac{t-z}{\cancel{\sqrt{t^2-(z-t)^2}}}}\, dz \sim 2^{-2k}$$

$(\because \ 1 < t < 2)$

$\Rightarrow$ contribution from A_k

$$2^{-2k} \times \frac{2^{2k}}{k^\alpha} = \frac{1}{k^\alpha} \notin l^1_k \qquad \alpha \le 1$$

$$\Rightarrow \quad u(t, t\hat{e}_3) = \infty, \qquad 1 < t < 2.$$

· <u>Rmk</u>: · localization in time does <u>NOT</u> help

· Assuming a higher regularity does NOT help

$$\|u\|_{L^2_t L^\infty_x} \not\lesssim \|u_1\|_{H^s}, \quad s \gg 1.$$

$\Leftarrow$ The estimate is L^2-scaling invariant.

· But if one assumes a higher regularity AND localization in time.

$$\Rightarrow \|u\|_{L^2_T L^\infty_x} \lesssim \underbrace{C(\varepsilon, T)}\|u_1\|_{H^\varepsilon}.$$

$$\begin{cases} C_\varepsilon (\log T)^{1/2}, & T \gg 1 \leftarrow TT^* \text{ argument.} \\ C_\varepsilon T^\varepsilon, & T \lesssim 1 \leftarrow \text{scaling the } T \gg 1 \text{ case.} \end{cases}$$

<u>Q</u>: What about replacing L^∞_x by the BMO norm?

or localize $\operatorname{supp} \widehat{u_1} \subset \{|\xi| \lesssim 1\}$?

<u>ANS</u>: Both fail!!

BMO estimate $\Longrightarrow$ freq localized estimate.

(if true)

Suppose $\| S(t)(u_0, u_1) \|_{L^2_t \mathrm{BMO}_x} \lesssim \| (u_0, u_1) \|_{\dot{\mathcal{H}}^1}$

Suppose u_0, u_1 are frequency localized.

$\Rightarrow$ so is $u = S(t)(u_0, u_1)$

$\Rightarrow$ $\| u(t, \cdot) \|_{\mathrm{BMO}_x} \sim \| u(t, \cdot) \|_{L^\infty_x}$

i.e. LP proj of a BMO function is in L^∞.

Goal: Disprove freq localized estimate. (based Montgomery-Smith '98 Duke

Disprove

$$\left\| \int e^{it\sqrt{-\Delta}} F(t)\, dt \right\|_{L^2_x} \lesssim \| F \|_{L^2_t L^1_x}$$

$\uparrow$ loc in spatial freq $|\xi| \sim 1$

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$$\| e^{-it\sqrt{-\Delta}} f \|_{L^2_t L^\infty_x} \overset{?}{\lesssim} \|f\|_{L^2_x}$$

$$\| \quad \text{(equality)}$$

$$\sup_{\substack{F \in L^2_t L^1_x \\ \|F\|_{L^2_t L^1_x} = 1}} \left| \langle e^{-it\sqrt{-\Delta}} f, F \rangle_{L^2_{t,x}} \right|$$

$$= \left| \int \langle f, e^{it\sqrt{-\Delta}} F \rangle \, dt \right|$$

$$= \left| \langle f, \int e^{it\sqrt{-\Delta}} F \, dt \rangle_{L^2_x} \right|$$

$$\le \|f\|_{L^2_x} \underbrace{\left\| \int e^{it\sqrt{-\Delta}} F \, dt \right\|_{L^2_x}}_{\overset{?}{\lesssim} \|F\|_{L^2_t L^1_x} \le 1}$$

$\mathrm{supp}\, \hat{f}(\xi) \subset \{|\xi| \sim 1\}$

$\Downarrow$

Can assume

$\mathrm{supp}\, \hat{F}(t,\xi) \subset \{|\xi| \sim 1\}$

$\forall t$.

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Goal: Disprove the freq loc. dual estimate:

$$\circledast \qquad \left\| \int e^{it\sqrt{\Delta}} F(t)\, dt \right\|_{L^2_x} \lesssim \| F \|_{L^2_t L^1_x}$$

β_t = Brownian motion

$\beta_t - \beta_s$ = mean 0 Gaussian r.v. with Var $= t-s$

$$\boxed{\mathbb{E}\left[e^{\pm 2\pi i (\beta_t - \beta_s)\xi} \right] = e^{-2\pi |t-s| \xi^2}} \qquad \forall\, \xi \in \mathbb{R}$$

Let $F(t,x) = \eta\left(\frac{t}{T}\right) \psi(x - t\hat{e}_3 - \beta_t \hat{e}_1)$

η: non-neg. smooth cutoff supp on $[-1,1]$

$\hat{\psi}(\xi)$ supported on $\{|\xi| \sim 1\}$

Suppose $\circledast$ holds

With $\hat{F}(t,\xi) = \eta\left(\frac{t}{T}\right) e^{-2\pi i t \xi_3} e^{-2\pi i \beta_t \xi_1} \hat{\psi}(\xi)$

(2)

$(*) \Rightarrow \left\| \int e^{2\pi i t|\xi|} e^{-2\pi i t \xi_3} e^{-2\pi i \beta_t \xi_1} \eta\left(\frac{t}{T}\right) dt \cdot \hat{\psi}(\xi) \right\|_{L^2_\xi}$

$$\lesssim \|F\|_{L^2_t L^1_x} \sim \sqrt{T}.$$

$(**)$ $$\int_{|\xi| \sim 1} \left| \int_t \cdots dt \right|^2 d\xi \lesssim T$$

Let $A = \{ \xi \in \mathbb{R}^3 : \xi_3 \sim 1, |\xi_2| \ll |\xi_1| \ll 1 \}$.

$\Rightarrow$ We have the following Taylor approx:

$$|\xi| - \xi_3 \sim \xi_1^2, \quad \forall \xi \in A.$$

$$\left(|\xi| = \sqrt{\xi_3^2 + \xi_1^2 + \xi_2^2} = \xi_3 + \frac{1}{2|\xi_3|}(\xi_1^2 + \xi_2^2) + \text{error} \right.$$

- Restrict the domain to A and expand the square.

(LHS) of **

$$\sim \int_A \iint_{t\,t'} e^{2\pi i(t-t')(|\xi|-\xi_3)} \underline{e^{-2\pi i(\beta_t-\beta_{t'})\xi_1}} \eta\left(\frac{t}{T}\right)\eta\left(\frac{t'}{T}\right) dt'dt\,d\xi$$

Take $\mathbb{E}$

$$\int_A \iint e^{2\pi i(t-t')(|\xi|-\xi_3)} e^{-2\pi|t-t'|\xi_1^2} \eta\left(\frac{\overset{r+t'}{t}}{T}\right)\eta\left(\frac{t'}{T}\right) dt'dt\,d\xi$$

$$\lesssim T.$$

Let $r = t-t'$.

$$\int_A \int e^{2\pi i r(|\xi|-\xi_3)} e^{-2\pi|r|\xi_1^2} \eta*\eta\left(\frac{r}{T}\right) dr\,d\xi \lesssim 1.$$

· Let $T\to\infty$

$$\int_A \int e^{2\pi i r\underbrace{(|\xi|-\xi_3)}_{x}} e^{-2\pi|r|\underbrace{\xi_1^2}_{y}} dr\,d\xi \lesssim 1$$

$$\wr$$

$$\int_A \underbrace{\frac{1}{\xi_1^2}}_{|\xi_2|\ll|\xi_1|\ll 1} d\xi_1 d\xi_2 d\overset{\xi_3\sim 1}{\xi_3} \sim \infty$$

$$\to \int \frac{1}{\xi_2} d\xi_2 = \infty$$

$$\left(\int e^{2\pi i r x} e^{-2\pi|r|y} dr = \frac{y}{\pi(x^2+y^2)} \right) \Rightarrow \text{X}$$

contradiction.

<u>Rmk</u>: Without the BM factor, there would be NO $e^{-2\pi|t|\xi_1^2}$ in the last integral.

$$\Rightarrow \int_A \delta_0(\xi_1^2)\, d\xi \lesssim 1 \quad (\Leftarrow \text{No contradiction})$$

Recall $\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q|$, $\quad f_Q = \operatorname{ave}_Q f$.

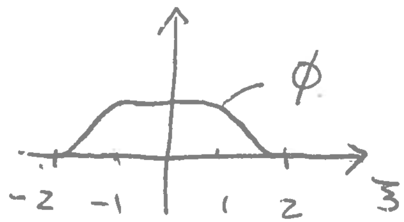
<u>Lemma 2.11</u>: $\operatorname{supp} \hat{f} \subset \{|\xi| \sim 2^k\}$, $k \in \mathbb{Z}$ (homog LP localization)

$$\Rightarrow \|f\|_{BMO} \sim \|f\|_{L^\infty}$$

<u>Pf</u>: By scaling, we may assume $k = 0$.

- Fix x.

$j \gg 1$. $\quad f(x) = P_{\le j} f(x) = \int f(x-y)\, 2^{jd}\, \check{\phi}(2^j y)\, dy$



$$\phi(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| \ge 2 \end{cases}.$$

We also have

$$\int f(x-y)\,2^{-jd}\check{\Phi}(2^{-j}y)\,dy = \underset{\substack{\uparrow \\ \text{proj onto } \{|\xi|\lesssim 2^{-j}\}}}{P_{\leq -j}} f(x) = 0.$$

$$\Rightarrow f(x) = \int f(x-y)\left(2^{jd}\check{\Phi}(2^{j}y) - 2^{-jd}\check{\Phi}(2^{-j}y)\right)dy$$

Let $Q = \{y : |y| \leq 2^{\alpha}\}$. and $Q_x = x + Q$.

Note that

$$\int f_{Q_x}\left(2^{jd}\check{\Phi}(2^{j}y) - 2^{-jd}\check{\Phi}(2^{-j}y)\right)dy$$

$$= f_{Q_x}\Big[\underbrace{\int 2^{jd}\check{\Phi}(2^{j}y)\,dy}_{=\int \check{\Phi}\,dy} - \underbrace{\int 2^{-jd}\check{\Phi}(2^{-j}y)\,dy}_{=\int \check{\Phi}\,dy}\Big] = 0$$

$$\Rightarrow f(x) = \int \left(f(x-y) - f_{Q_x}\right)\left(2^{jd}\check{\Phi}(2^{j}y) - 2^{-jd}\check{\Phi}(2^{-j}y)\right)dy$$

- If $y \in Q$, then $|f(x)| \lesssim 2^{d\alpha}\|f\|_{BMO}$.
- If $y \notin Q$, then $|f(x)| \leq 2\|f\|_{L^\infty} \underbrace{\int_{|y|\geq 2^{\alpha}} |2^{jd}\check{\Phi}(2^{j}y)| + |2^{-jd}\check{\Phi}(2^{-j}y)|\,dy}_{<\frac{1}{4} \text{ by choosing } \alpha \gg 1 \text{ (indep of } x\text{)}}$

(j is fixed)

$$\Rightarrow |f(x)| \leq C_\alpha \|f\|_{BMO} + \frac{1}{2}\|f\|_{L^\infty}$$

$\Rightarrow$ Take sup in x. $\boxtimes$

Chap 3: Invariant Gibbs measure, Part 2

We'll focus on $\mathbb{T}^2$.

Sec 3.1: Gibbs meas on $\mathbb{T}^2$ (for NLS, NLW)

$$H(u) = \frac{1}{2}\int |\nabla u|^2 \pm \frac{1}{p+1}\int |u|^{p+1}.$$

$$\Rightarrow \text{"} d\mu = Z^{-1} e^{-H(u)} du$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}} d\rho_1 \text{"}$$

where $d\rho_1 = Z^{-1} e^{-\frac{1}{2}\|u\|^2_{H^1}} du \Leftarrow$ supported on $H^s(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$, $s<0$.

$$\Rightarrow \int |u|^{p+1} dx = \infty, \quad \text{a.s.}$$

- We need to renormalize $\int |u|^{p+1}$.

We only consider the defocusing case.

(Focusing case: Brydges – Slade '96.
We can not construct Gibbs measure for the focusing cubic NLS on $\mathbb{T}^2$
even if we consider the Wick ordered nonlinearity (with the Wick ordered

In the following, we focus on the real-valued setting (for simplicity)

- Under ρ_1, we have

$$u = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{in\cdot x}, \qquad g_{-n} = \overline{g_n}, \quad \mathrm{Var}(g_n) = 1$$

indep, std Gaussian r.v

Given $N \in \mathbb{N}$, let $u_N = P_{\leq N} u$.

$\Rightarrow$ For each $x \in \mathbb{T}^2$, $u_N(x)$ is a mean 0 $\mathbb{R}$-valued Gaussian r.v. with variance:

$$\sigma_N := \mathbb{E}\left[u_N^2(x)\right] = \sum_{|n| \leq N} \frac{1}{\langle n \rangle^2} \sim \log N \nearrow \infty.$$

- Similarly,

$$\mathbb{E}\left[\int_{\mathbb{T}^2} u_N^2(x)\, dx\right] = \sum_{|n| \leq N} \frac{1}{\langle n \rangle^2} \sim \log N$$

but defining $\underbrace{:u_N^2(x):}_{\text{Wick ordered monomial}} \overset{\text{def}}{=} u_N^2(x) - \sigma_N$

$$= H_2(u_N(x); \sigma_N)$$

Hermite poly $H_2(x;\sigma) = x^2 - \sigma$

We have

$$\int_{\mathbb{T}^2} :u_N^2(x): = \sum_{|n|\le N} \frac{|g_n|^2 - 1}{\langle n \rangle^2}$$

- $\mathbb{E}\left(\int_{\mathbb{T}^2} :u_N^2: dx\right] = 0$
- $\mathbb{E}\left[\left(\cdots \right)^2\right] = \mathbb{E}\left[(g_0^2 - 1)^2\right]$

$$+ 2\,\mathbb{E} \sum_{\substack{n \in I \\ |n| \le N}} \frac{(|g_n|^2 - 1)^2}{\langle n \rangle^4}$$

" $I = \mathbb{Z}^2/2$ "

$$\lesssim 1 + \sum_{|n|\le N} \frac{1}{\langle n \rangle^4} \lesssim 1 \quad \text{indep of } N$$

$\mathbb{E}\left[(|g_n|^2 - 1)(|g_m|^2 - 1)\right] = 0$ unless $n = m$.

$\Rightarrow$ $\int_{\mathbb{T}^2} :u_N^2: dx$ is a well-defined r.v.

$\int_{\mathbb{T}^2} :u^2: dx = \lim_{N \to \infty} \int :u_N^2: dx$ exists in $L^q(\Omega)$ for any $q < \infty$.

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Sec 3.2: Hermite polynomials, white noise functional, and Wick ordering

- Hermite polynomials $H_k(x;\sigma)$ defined through the generating function:

$$G(t,x;\sigma) = e^{tx-\frac{1}{2}\sigma t^2}$$
$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma)$$

$H_0(x;\sigma)=1$, $H_1(x;\sigma) = x$,
$H_2(x;\sigma) = x^2-\sigma$, $H_3(x;\sigma) = x^3-3\sigma x$,
$H_4(x;\sigma) = x^4-6\sigma x^2+3\sigma^2$.

- $\partial_x H_k(x;\sigma) = k H_{k-1}(x;\sigma)$

- White noise functional:

$$d\rho_0 = Z^{-1} e^{-\frac{1}{2}\|u\|_{L^2}^2} du \quad \text{on} \quad H^{-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$$

$$\leftrightsquigarrow \quad u_w(x) = \sum_{n \in \mathbb{Z}^d} g_n(\omega) e^{inx} \Leftarrow \text{distribution.}$$

- $W_{(\cdot)} : L^2(\mathbb{T}^d) \longrightarrow L^2(\Omega)$

by $W_f(\omega) = \langle f, u_w(\omega) \rangle_{L^2} = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) \overline{g_n}(\omega)$

$\Rightarrow$ W_f is a Gaussian r.v. with mean 0.

and

$$Var(W_f) = \mathbb{E}\Big[\sum_n \hat{f}(n) \overline{g_n} \; \overline{\sum_m \hat{f}(m) \overline{g_m}} \Big] \qquad g_{-n} = \overline{g_n}.$$

$$= \|f\|_{L^2}^2$$

Moreover, $\mathbb{E}[W_f \overline{W_h}] = \langle f, h \rangle_{L^2}$.

$W_{(\cdot)}$ is unitary. (~ Wiener integral $\int \underset{\text{deterministic}}{f} \, dB$)

Lemma 3.1: $f, h \in L^2(\mathbb{T}^2)$ with $\|f\|_{L^2} = \|h\|_{L^2} = 1$

Then, $\mathbb{E}\left[H_k(W_f)\, H_m(W_h) \right] = \delta_{km}\, k! \left[\langle f, h \rangle_{L^2_x} \right]^k$

Rmk: In the complex-valued setting, we use the (generalized) Laguerre polynomials (Oh - Thomann '15)

Pf: First, recall

$$\int_\Omega e^{W_f}\, dP = e^{\frac{1}{2}\|f\|^2_{L^2}}$$

$\mathcal{I}$ = index set
= "$\mathbb{Z}^2/2$"
$= \mathbb{Z} \times \mathbb{N} \cup \mathbb{N} \times \{0\}$
$\cup \{(0,0)\}$
$\mathcal{I}^* = \mathcal{I} \setminus \{(0,0)\}$

$$(\text{LHS}) = \prod_{n \in \mathcal{I}^*} \frac{1}{\pi} \int_{\mathbb{C}} e^{2\,\mathrm{Re}(\widehat{f}(n)\, \overline{g_n}) - |g_n|^2}\, dg_n$$

$$\times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\widehat{f}(0)\, g_0}\, e^{-\frac{1}{2} g_0^2}\, dg_0$$

$$= \prod_{n \in \mathcal{I}^*} \frac{1}{\pi} \int_{\mathbb{R}} e^{2\,\mathrm{Re}\,\widehat{f}(n)\,\mathrm{Re}\, g_n \,-}\, e^{-(\mathrm{Re}\, g_n)^2}\, d\,\mathrm{Re}\, g_n$$

$$\times \int_{\mathbb{R}} e^{2\,\mathrm{Im}\,\widehat{f}(n)\,\mathrm{Im}\, g_n - (\mathrm{Im}\, g_n)^2}\, d\,\mathrm{Im}\, g_n$$

$$\times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cdots \, dg_0$$

$$= e^{\sum_{n \in I^*} |\hat{f}(n)|^2} e^{\frac{1}{2} (\hat{f}(0))^2} = e^{\frac{1}{2} \|f\|_{L^2}^2}$$

$$\Longrightarrow \int_\Omega G(t, W_f)\, G(s, W_h)\, dP$$

$$= e^{-\frac{1}{2}(t^2+s^2)} \underbrace{\int_\Omega e^{W_{tf+sh}} \, dP}_{= e^{\frac{1}{2}\|tf+sh\|_{L^2}^2}}$$

by linearity of $W_{(\cdot)}$

$$= e^{ts\langle f, h\rangle_{L^2}} . = \sum_{l=0}^{\infty} \frac{(ts)^l}{l\,!} (\langle f, h\rangle)^l .$$

On the other hand,

$$(\mathrm{LHS}) = \sum_{k,m=0}^{\infty} \frac{t^k s^m}{k!\, m!} \int_\Omega H_k(W_f)\, H_m(W_h)\, dP$$

$\boxtimes$

Under ρ_1, we have

$$u = \sum_{n \in \mathbb{Z}^2} \frac{g_n}{\langle n \rangle} e^{in\cdot x}.$$

Let $u_N = P_{\leq N} u$.

$\Rightarrow$ $u_N(x)$ is a Gaussian r.v. with mean 0 and $\text{var} \sim \log N$ $\forall x \in \mathbb{T}^2$.

$$\underset{\sigma_N}{\overset{\shortparallel}{\text{var}}}$$

$\Rightarrow$ Define the Wick ordered monomial

$$:u_N^k(x): \; = H_k(u_N(x); \sigma_N)$$

Let $G_N(u) = \frac{1}{2m} \int_{\mathbb{T}^2} :(P_{\leq N} u)^{2m}: \, dx$

Prop 3.2: $\{G_N(u)\}$ is a Cauchy seq in $L^q(\rho_1)$ for any finite $q \geq 2$. Moreover,

$$\| G_M(u) - G_N(u) \|_{L^q(\rho_1)} \leq C_m (q-1)^m \frac{1}{N^{1/2}}$$

for any $q \geq 1$, $M \geq N \geq 1$.

Rmk: Prop 3.2 allows us to define

$$G(u) = G_\infty(u)$$
$$= \frac{1}{2m}\int_{\mathbb{T}^2} :u^{2m}: dx = \lim_{N\to\infty} G_N(u) \text{ in } L^q(\rho_1)$$

and the Wick ordered Hamiltonian

$$H_{Wick}(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 + \frac{1}{2m}\int_{\mathbb{T}^2} :u^{2m}:$$

pf ($q=2$): $e_n(y) = e^{in\cdot y}$

$$\eta_N(x)(\cdot) = \frac{1}{\sigma_N^{1/2}} \sum_{|n|\le N} \frac{\overline{e_n(x)}}{\langle n\rangle} e_n(\cdot)$$

$$\gamma_N(\cdot) = \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} e_n(\cdot)$$

$$\sigma_N = \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} \sim \log N$$

Note that $\|\eta_N(x)\|_{L^2(\mathbb{T}^2)} = 1$ for each fixed $x \in \mathbb{T}^2$.

and

(*) $\langle \eta_M(x), \eta_N(y)\rangle_{L^2(\mathbb{T}^2)} = \frac{1}{\sigma_M^{1/2}\sigma_N^{1/2}} \gamma_N(y-x)$, $M \ge N \ge 1$.

$$\cdot\ u_N(x) = \sigma_N^{1/2} \frac{u_N(x)}{\sigma_N^{1/2}} = \sigma_N^{1/2}\, \overline{W_{\eta_N(x)}}$$

$$= \sigma_N^{1/2}\, W_{\eta_N(x)}$$

$$\Rightarrow\ :u_N^{2m}(x): = H_{2m}(u_N(x);\, \sigma_N) \qquad H_k(x;\sigma) = \sigma^{k/2} H_k(x;1)$$

$$= \sigma_N^m H_{2m}\left(\frac{u_N(x)}{\sigma_N^{1/2}};\, 1\right)$$

$$= \sigma_N^m H_{2m}(\underline{W_{\eta_N(x)}})$$

↑ L_x^2-norm is 1.

$$\Rightarrow\ (2m)^2 \| G_M(u) - G_N(u) \|^2_{L^2(\rho_1)}$$

$$= \int_{\mathbb{T}_x^2 \times \mathbb{T}_y^2} \int_\Omega \sigma_M^{2m} H_{2m}(W_{\eta_M(x)}) H_{2m}(W_{\eta_M(y)})$$

$$- \sigma_M^m \sigma_N^m H_{2m}(W_{\eta_M(x)}) H_{2m}(W_{\eta_N(y)})$$

$$- \sigma_N^m \sigma_M^m H_{2m}(W_{\eta_N(x)}) H_{2m}(W_{\eta_M(y)})$$

$$+ \sigma_N^{2m} H_{2m}(W_{\eta_N(x)}) H_{2m}(W_{\eta_N(y)})\, dP\, dx\, dy$$

Lemma 3.1 & ⊛

$$= C_m \int_{\mathbb{T}^2_x \times \mathbb{T}^2_y} (\gamma_M(y-x))^{2m} - (\gamma_N(y-x))^{2m} \, dx\, dy$$

$$= C_m \int_{\mathbb{T}^2} (\gamma_M(x))^{2m} - (\gamma_N(x))^{2m} \, dx$$

$$\lesssim \int_{\mathbb{T}^2} |\gamma_M(x) - \gamma_N(x)| \left(|\gamma_M(x)|^{2m-1} + |\gamma_N(x)|^{2m-1} \right) dx$$

$$\underset{C-S}{\leq} \underbrace{\|\gamma_M - \gamma_N\|_{L^2}} \left(\|\gamma_M^{2m-1}\|_{L^2} + \|\gamma_N^{2m-1}\|_{L^2} \right)$$

$$= \left(\sum_{N \leq |n| \leq M} \frac{1}{\langle n \rangle^4} \right)^{1/2} \lesssim \frac{1}{N}$$

$$\cdot \; \|\gamma_M^{2m-1}\|_{L^2} = \|\gamma_M\|_{L^{4m-2}}^{2m-1} \underset{H-Y}{\leq} \Big(\sum_{|n| \leq M} \frac{1}{\langle n \rangle^{\boxed{2 \cdot \frac{4m-2}{4m-3}}}} \Big)^{\frac{4m-3}{2}} \leq C_m < \infty$$

>2 indep of M.

☐

01/03/17 (Wed)

Sec 3.3: Hypercontractivity of the Ornstein-Uhlenbeck semigroup and Wiener chaos estimate

$H = L^2(\mathbb{R}^d, \mu_d)$

standard Gaussian meas on $\mathbb{R}^d$.

$L = \Delta - x \cdot \nabla$

= Hartree-Fock operator (OU operator)

Consider

$$\partial_t u = Lu \quad \text{on } \mathbb{R}^d.$$

Lemma 3.3 (hypercontractivity of OU semigroup) Nelson '65

$p > 1$, $q \geq 1$. Then, we have

$$\| e^{tL} f \|_{L^q(\mathbb{R}^d, \mu_d)} \leq \| f \|_{L^p(\mathbb{R}^d, \mu_d)}$$

for $t \geq \frac{1}{2} \log\left(\frac{q-1}{p-1}\right)$

Rmk: • The estimate holds indep of dim $d \geq 1$ (even $d = \infty$.)

• This is equivalent to Log Sobolev inequality.

Define a homog Wiener chaos of order k

$$H_{\vec{k}}(x) = \prod_{j=1}^{d} H_{k_j}(x_j) \qquad x = (x_1, \cdots, x_d)$$

$$|\vec{k}| = k_1 + \cdots + k_d = k$$

• $\mathcal{H}_k$ = closure of homog Wiener chaoses of order k under $L^2(\mathbb{R}^d, \mu_d)$ (or any $L^p(\mathbb{R}^d, \mu_d)$.)

Ito – Wiener decomposition,

$$L^2(\mathbb{R}^d, \mu_d) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

- $F \in \mathcal{H}_k$ is an eigenfunction of L with eigenvalue $-k$.

$\Rightarrow$ Cor 3.4 : $F \in \mathcal{H}_k$. Then for $q \geq 2$, we have

$$\|F\|_{L^q(\mathbb{R}^d, \mu_d)} \leq (q-1)^{k/2} \|F\|_{L^2(\mathbb{R}^d, \mu_d)}$$

$\left(\Leftarrow \; p = 2, \; t = \frac{1}{2} \log (q-1) \text{ in Lemma } 3.3 \right.$

Lemma 3.5 : $S_k(\omega) = \sum_{\Gamma(k,d)} C(n_1, \cdots, n_k)\, g_{n_1}(\omega) \cdots g_{n_k}(\omega) \in \bigoplus_{j=0}^{k} \mathcal{H}_j$

$\{g_n\}$ std indep Gaussian r.v's

where $\Gamma(k,d) = \{ (n_1, \cdots, n_k) \in \mathbb{Z}^k : |n_j| \leq d \}$

$\Rightarrow$ Then, for $q \geq 2$

$$\|S_k\|_{L^q(\Omega)} \leq C\sqrt{k+1}\, (q-1)^{k/2} \|S_k\|_{L^2(\Omega)}$$

$\uparrow$ can be removed

Thomann–Tzvetkov

$\left(\Leftarrow \text{Cor } 2.4 \text{ and } x^j = \sum_{m=0}^{[j/2]} c_{m,j}\, \sigma^m H_{j-2m}(x;\sigma) \right.$

ex: $n \neq m$

$$g_n^2 \, g_m^3 = (g_n^2 - 1 + 1)(g_m^3 - 3g_m + 3g_m)$$

$$\Rightarrow g_n^2 g_m^3 = \underbrace{H_2(g_n) H_3(g_m)}_{\in \mathcal{H}_5} + \underbrace{H_0(g_n) H_3(g_m)}_{\in \mathcal{H}_3} + \underbrace{3H_2(g_n) H_1(g_m)}_{\in \mathcal{H}_3}$$

$$+ \underbrace{3H_0(g_n) H_1(g_m)}_{\in \mathcal{H}_1}$$

· Bach to Prop 3.2: $\quad u_N = \sum_{|n| \le N} \frac{g_n}{\langle n \rangle} e^{in\cdot x}$

$$u_N^{2m} = \sum_{|n| \le N} e^{in\cdot x} \left(\sum_{\substack{n = n_1 + \cdots + n_{2m} \\ |n_j| \le N}} \prod_{j=1}^{2m} \left(g_{n_j} / \langle n_j \rangle \right) \right)$$

← 2m fold product of Gaussian

$$\underbrace{G_N(u) = \frac{1}{2m} \int \underbrace{: u_N^{2m} :}\, dx}_{\text{projection onto } \mathcal{H}_{2m}} \in \bigoplus_{k=0}^{2m} \mathcal{H}_k \quad (\text{actually in } \mathcal{H}_{2m})$$

$\Rightarrow$ Use Lemma 3.5 and Prop 3.2 with $q = 2$.

Namely,

$$\| G_M(u) - G_N(u) \|_{L^q(\rho_1)} \lesssim C_m (q-1)^m \underbrace{\| G_M(u) - G_N(u) \|_{L^2(\rho_1)}}_{\lesssim \frac{1}{N^{1/2}}}$$

$\boxtimes$

Sec 3.4 : Construction of the Gibbs measure on $\mathbb{T}^2$

Goal : Construct

$$d\mu = Z^{-1} e^{-H_{Wick}(u)} du$$
$$= Z^{-1} e^{-\frac{1}{2m} \int_{\mathbb{T}^2} :u^{2m}: dx} d\rho_1$$

(Φ_2^{2m} or φ_2^m Euclidean QFT)

(deg: $2m$; dim: 2)

Let $R_N(u) = e^{-G_N(u)} = e^{-\frac{1}{2m} \int :u_N^{2m}: dx}$.

Prop 3.6 : · $R_N(u) \in L^q(\rho_1)$, for any finite $q \geq 1$, with a unif bd in N (depends on q)

· $R_N(u)$ converges to some $R(u)$ in $L^q(\rho_1)$

6

- This allows us to define the Gibbs meas

$$d\mu = Z^{-1} R(u)\, d\rho_1 = Z^{-1} e^{-\frac{1}{2m}\int :u^{2m}:\,dx}\, d\rho_1$$

$$= \underset{\substack{N\to\infty \\ \uparrow \\ \text{"unif" conv.}}}{\lim} \underbrace{Z_N^{-1} R_N(u)\, d\rho_1}_{= d\mu_N}$$

Pf: Nelson's estimate. Next time.

Lec 17 06/03/17 (Mon) ①

Pf of Prop 3.6: $G_N(u)$ is not sign definite

Nonetheless, the defocusing nature plays an important role

Main observation: - $G_N(u)$ has a logarithmic upper bound.

$$-G_N(u) = -\frac{1}{2m}\int_{\mathbb{T}^2} \underbrace{H_{2m}(u_N;\sigma_N)}_{= \sigma_N^m H_{2m}\left(\frac{u_N}{\sigma_N^{1/2}}\right)} dx$$

$$\le b_m (\log N)^m \qquad H_{2m}(x;1) \ge -\underbrace{a_m}_{>0}$$

- $$\underset{e^{-G_N(u)}}{\overset{\shortparallel}{\|R_N(u)\|}}{}^q_{L^q(\rho_1)} = \int_0^\infty \rho_1(e^{-qG_N(u)} > \alpha)\,d\alpha$$

$$\le 1 + \int_1^\infty \rho_1(-qG_N(u) > \log\alpha)\,d\alpha$$

(*) $\Rightarrow$ Suffices to show $\rho_1(-qG_N(u) > \log\alpha) \le C\alpha^{-(1+\delta)}$, $\forall \alpha \ge 1, N \in \mathbb{N}$

- Given $\lambda = \log \alpha$,

Choose $N_0 \in \mathbb{R}$ s.t. $\lambda = 2q\, b_m (\log N_0)^m$.

Then, by the log upper bound,

$$\rho_1 (\underbrace{- q\, G_N(u)}_{\le\; q\, b_m (\log N)^m \;\le\; \frac{1}{2}\lambda} > \lambda) = 0 \qquad \text{for all } N < N_0$$

- For $N \geq N_0$,

$$\rho_1 (- q\, G_N(u) > \lambda)$$

$$\leq \rho_1 (- q\, G_N(u) + q\, G_{N_0}(u) > \overbrace{\lambda - q\, b_m (\log N_0)^m}^{= \frac{1}{2}\lambda})$$

$$- q\, b_m (\log N_0)^m - q\, G_{N_0}(u) \le 0$$

$$\leq \rho_1 (q\, | G_N(u) - G_{N_0}(u) | \geq \tfrac{1}{2}\lambda)$$

Prop 3.2 & Chebyshev

$$\leq C_{m,q}\, \underbrace{e^{- c N_0^{1/2m} \lambda^{1/m}}}_{\boxed{e^{-c\lambda^{1/m+1}} \times e^{c\lambda^{1/m}}}} \qquad N \geq N_0$$

~~(**)~~

$$\ll e^{-(1+\delta)\lambda} \sim \alpha^{-(1+\delta)} \Longrightarrow (*)$$

In (**), we used

$$\rho_1\left(|G_M(u) - G_N(u)| > \tilde{\lambda} \right) \le C_m\, e^{-c_m N^{1/2m} \tilde{\lambda}^{\frac{1}{m}}}$$

$\Rightarrow$ $G_N(u)$ converges in measure

$\Rightarrow$ $R_N = e^{-G_N(u)}$ converges in measure

By repeating the argument in Cor 1.4,

$$R_N \to R = e^{-G(u)} \quad \text{in } L^q(\rho_1), \quad \forall q \ge 1$$

$\boxtimes$

Sec 3.5: On the dynamical problem.

- Wick ordered NLS (WNLS):

$$i\partial_t u + \Delta u = \, : |u|^{2(m-1)} u : \quad \text{on } \mathbb{T}^2$$

- Wick ordered NLW (NLKG)

$$-\partial_t^2 u + \Delta u - u = \, : u^{2m-1} : \quad \text{on } \mathbb{T}^2.$$

- NLS: Scaling critical regularity $s_{crit} = 1 - \frac{1}{m-1}$ $(m \geq 2)$

cubic: $s_{crit} = 0$

quintic: $s_{crit} = 1/2$

but ρ_1 and μ are supported on $H^{-\varepsilon}(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$

$u = z + v$ $\Longrightarrow$

- Bourgain '96: The cubic WNLS is almost locally locally well-posed w.r.t. ρ_1 (and hence μ)

(Also, see Colliander–Oh, Duke '12)

"inv. meas argument" in Chap 1

$\Longrightarrow$ a.s. GWP & invariance of μ.

When $m \geq 3$, i.e. (super) quintic ⑤

$\Rightarrow$ regularity gap $> 1/2$ $\Leftarrow$ too large!!

· <u>Compactness argument</u>. (on measures on spacetime functions)

(FWNLS) $\quad i\partial_t u^N + \Delta u^N = F_N(u^N)$,

where $F_N(u) = P_{\leq N}\left(: |P_{\leq N} u|^{2(m-1)} P_{\leq N} u : \right)$

$$= (-1)^{m-1} (m-1)! \, \sigma_N^{m-1} \cdot P_{\leq N}\left(L_{m-1}^{(1)} \left(\frac{|u_N|^2}{\sigma_N} \right) u_N \right)$$

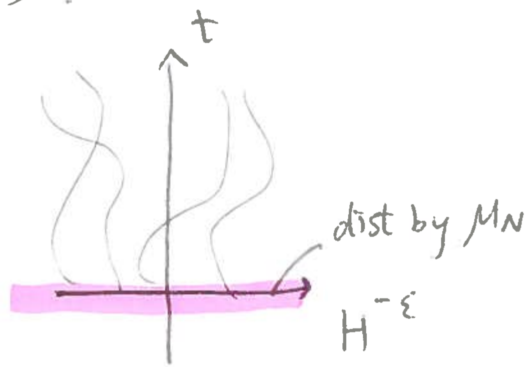
generalized Laguerre poly

· $d\mu_N = Z_N^{-1} R_N(u) \, d\rho$,

$=$ invariant Gibbs meas for (FWNLS).

① Extend μ_N = prob. meas on initial data

to ν_N = prob. meas on space-time functions



- (FWNLS) is $\overbrace{}^{}$ a.s. globally well-posed

$$\Rightarrow \exists\, \Phi_N : u_0 \in H^{-\varepsilon} \longmapsto u^N \in C(\mathbb{R}; H^{-\varepsilon})$$

- Define a prob. meas ν_N on $C(\mathbb{R}; H^{-\varepsilon})$

  by $\nu_N = \mu_N \circ \Phi_N^{-1}$

  $=$ induced prob meas under Φ_N

- Note that by invariance of μ_N,

$$\underbrace{\mathcal{L}}_{\text{law}}(u^N(t)) = \mu_N \quad \text{for any } t \in \mathbb{R}$$

② (By a soft argument), show $\{\nu_N\}$ is tight (= precompact)

$$\overset{\text{Prokhorov}}{\Longrightarrow} \quad \nu_{N_j} \rightharpoonup \nu$$

③ Skorokhod's Thm : $\exists$ another prob space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$

$\widetilde{u}^{N_j}$, $C(\mathbb{R}; H^{-\varepsilon})$-valued r.v's

u, "

s.t. $\mathcal{L}(\tilde{u}^{N_j}) = \mathcal{L}(u^{N_j}) = \nu_{N_j}$

$\mathcal{L}(u) = \nu$

and $\tilde{u}^{N_j}$ converges to u in $C(\mathbb{R}; H^{-\varepsilon})$ almost surely.

We "upgrade" weak conv to a.s. conv.

Rmk: $\mathcal{L}(\tilde{u}^{N_j}(t)) = \mathcal{L}(u^{N_j}(t)) = \mu_{N_j}$.

- By $\mu_N \underset{\text{"unif"}}{\longrightarrow} \mu$, $\mathcal{L}(u(t)) = \mu$. = Gibbs meas

Oh-Thomann '15

Thm 3.7: $\exists$ a set Σ of full probability s.t.

$\forall \phi \in \Sigma$, $\exists$ global soln $u \in C(\mathbb{R}; H^{-\varepsilon})$ to WNLS with $u|_{t=0} = \phi$.

Moreover, $\mathcal{L}(u(t)) = \mu$, $\forall t \in \mathbb{R}$

Rmk: ① only a.s. global existence (NO uniqueness) ← wptness

"energy solns"

② only invariance of μ in mild sense

Lec 18   08/03/17 (Wed)

Idea of the proof of Thm 3.7

Let $X_j = i\partial_t \tilde{u}^{N_j} + \Delta \tilde{u}^{N_j} - F_{N_j}(\tilde{u}^{N_j})$

$\quad$ ↳ $\mathcal{D}'_{t,x}$-valued r.v.

$\Longrightarrow \quad \mathcal{L}(X_j) = \delta_0$

$\Longrightarrow \quad \mathcal{L}(X_\infty) = \delta_0$, i.e.

$\quad u = \lim_{j\to\infty} \tilde{u}^{N_j}$ is a distributional soln to (WNLS)

- $i\partial_t \tilde{u}^{N_j} + \Delta \tilde{u}^{N_j} \longrightarrow i\partial_t u + \Delta u$, easy
- $:|u|^{2(m-1)}u: \; := \lim_{j\to\infty} F_{N_j}(\tilde{u}^{N_j})$ exists in $L^q(\rho_1; H^{-\varepsilon}(\mathbb{T}^2))$, $\varepsilon > 0$.

$\quad \Leftarrow$ analogous to Prop 3.2

$\quad$ First, study $\langle F_{N_j}(\tilde{u}^{N_j}), e_n \rangle_{L^2_x}$

- (defocusing) WNLW on $\mathbb{T}^2$

Oh-Thomann '15 **Thm 3.8:** WNLW is almost surely locally well-posed w.r.t. μ.

$\Longrightarrow$ a.s. GWP & invariance of μ.

- Key ingredient

(1) $Z_N = P_{\leq N} S(t)(u_0^\omega, u_1^\omega)$

$$\Longrightarrow P\left(\| \langle \nabla \rangle^{-\varepsilon} :Z_N^k: \|_{L_T^q L_x^r} > \lambda \right) \leq C \exp\left(-c \frac{\lambda^{2/k}}{T^{2/qk}} \right)$$

(2) $$:u^{2m-1}: = \lim_{N\to\infty} :u_N^{2m-1}: \qquad u_N = Z_N + v_N$$

$$= \lim_{N\to\infty} \sum_{k=0}^{2m-1} \binom{2m-1}{k} :Z_N^k: \; v_N^{2m-1-k}$$

$v_N \in H^s$ for some $s>0$

Sec 3.6 : Further topics

① μ, invariant. $\mathcal{L}(u(t)) = \mu = \mathcal{L}(u(0))$

How are they related?

Difficult question: study space-time covariance

$$\mathbb{E}\left[u(t,x)\,\overline{u(s,y)}\right]$$

· Weakly nonlinear, and large box limit.
$(\varepsilon \to 0)$ $(|\Omega| \to \infty)$

· Dynamical property?

· recurrence, ✓

· ergodicity? (difficult)

- Known for some SPDEs

$\Leftarrow$ uniqueness of an invariant meas

$\uparrow$

NOT true for (deterministic) nonlin Hamil PDEs

e.g. δ_0.

$$d\rho_0 = Z^{-1} e^{-\frac{1}{2}\|u\|_{L^2}^2} du \leftarrow$$ white noise inv for KdV. Quastel-Valkó, Oh OQV.

② $\mathbb{T}^3$.

- Gibbs meas on $\mathbb{T}^3$ is renormalizable
  only for cubic & defocusing
  i.e. Φ^4_3 — 4: deg of nonlin in Hamil; 3: dim

- Wick ordering is NOT enough.

  $\sigma_N = \mathbb{E}_{\rho_1}[u^2(x)] \sim N$ when $d=3$.

  but need to replace it by $C_1 N + C_2 \log N$.

- Stochastic quantization eqn

  space-time white noise

  $$\partial_t u = \Delta u - u - u^3 + \infty \cdot u + \xi$$

  "LWP": Hairer '14 (regularity structure)
  Catellier-Chouk '14 (paracontrolled distribution à la Gubinelli-Imkeller-Perkowski
  Kupiainen '14 (renormalization group method)

  Invariance: Hairer-Matetski '15
  GWP: Mourrat-Weber '16.

Q: NLS / NLW on $\mathbb{T}^3$?

Chap 4: Introduction to stochastic dispersive PDEs

Stochastic NLS (SNLS):

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u + \underline{\phi}\,\xi$$

ϕ: bdd op on L^2 (smoothing) (actually, Hilbert-Schmidt: $L^2 \to H^s$)

Stochastic NLW (SNLW):

$$-\partial_t^2 u + \Delta u = u^p + \xi$$

- ξ = space time white noise = "$\dfrac{\partial^2 B}{\partial x \partial t}$".

- Ito's formulation:

$$i\,du = (-\Delta u \pm |u|^{p-1}u)\,dt + \phi\,\underline{\underline{dW}}$$

$W(t)$ = L^2-cylindrical Wiener process

$$= \sum_{n\in\mathbb{Z}^d} \beta_n(t)\, e^{in\cdot x}, \quad \{\beta_n\} = \text{indep } \mathbb{C}\text{-valued Brownian motions.}$$

$$\beta_n = \beta_n^{(r)} + i\,\beta_n^{(i)}$$

indep $\mathbb{R}$-valued BM

<u>Note:</u> $X_t(\omega) = X(t,\omega)$

<u>Sec 4.1 Basic stochastic calculus</u>

$(\Omega, \mathcal{F}, P)$, prob space.

A BM. on $\mathbb{R}_+$ $(t \geq 0)$ is a stochastic process, satisfying

(i) $B(0) = 0$, a.s.

(ii) $B(t) - B(s) \sim \underset{\text{normal distri}}{N}(0, \overset{\text{var}}{t-s})$, $t > s$

(iii) $B(t_1) - B(s_1)$, $B(t_2) - B(s_2)$, $t_1 \geq s_1 \geq t_2 \geq s_2$

are indep (i.e. indep increment over disjoint intervals)

<u>Properties</u>: ① BM is a.s. conti. (i.e. $\exists$ "conti version".)

$$\mathbb{E}\left(|B(t) - B(s)|^{2k}\right) = \underbrace{\frac{(2k)!}{2^k k!}}_{=(2k-1)!!} |t-s|^k$$

i.e. $\mathbb{E}\left(|B(t) - B(s)|^p\right) \sim |t-s|^{p/2}$

Kolmogorov continuity criterion (Bass, EX 8.2)

$\{X_t\}$ with values in a metric space S. (separable?)

Suppose $p, \varepsilon > 0$ $\mathbb{E}\left(d(X_s, X_t)^p \right) \le C_0 |t-s|^{1+\varepsilon}$, $\forall\, t, s$

Then, $\forall\, 0 < \gamma < \frac{\varepsilon}{p}$

$$P\left(\sup_{s \neq t} \frac{d(X_s, X_t)}{|t-s|^{\varepsilon/p - \gamma}} \ge \lambda \right) \le \frac{C_1}{\lambda^p}.$$

$\Rightarrow X_t$ is unif conti a.s.

Back to BM. $\frac{\varepsilon}{p} - = \frac{\frac{p}{2} - 1}{p} - = \frac{1}{2} - \frac{1}{p} - \to \frac{1}{2} -$ as $p \to \infty$.

- $B \in \dot{W}^{b,p}_{loc}$, $b < \frac{1}{2}$, $p \le \infty$

$\underline{p < \infty}$:
$$\mathbb{E}\left[\| B(t) \|^p_{\dot{W}^{b,p}(I)} \right] = \int_I \int_I \frac{\mathbb{E}\left(|B(t) - B(t')|^p \right)}{|t-t'|^{1+bp}} dt'dt$$

$$\sim \iint_{I\,I} |t-t'|^{-1-bp+p/2} dt'dt < \infty$$

iff $b < \frac{1}{2}$

$B \in B^{1/2}_{p,\infty}$, $p < \infty$ $\left(\text{but } B \in B^{1/2-}_{\infty,\infty} \right)$

Roynette '93 SS rep

Bényi - Oh '11 Ad. Math

2 Covariance

$$\mathbb{E}[B(t)B(s)] = t \wedge s = \min(t,s)$$

$$\| \quad \mathbb{E}\big[B(s)(B(t)-B(s))\big] + \mathbb{E}(B^2(s)) = s, \quad t > s$$

$$\underbrace{}_{\text{indep}} \quad = 0$$

<u>White noise</u> : $\xi = dB$

$$\xi \rightsquigarrow \mathbb{E}[\xi(t)\xi(s)] = \delta(t-s), \quad \text{"}\xi \text{ scales like } \sqrt{\delta}\text{"}$$

$h > 0$

$$\phi_h(s) = \mathbb{E}\left[\frac{B(t+h)-B(t)}{h} \cdot \frac{B(s+h)-B(s)}{h} \right]$$

$$= \cdots = \frac{1}{h^2}\Big((t+h)\wedge(s+h) - (t+h)\wedge s - t\wedge(s+h) + t \wedge s \Big)$$

- $\int \phi_h(s)\,ds = 1$

$$\phi_h(s) \to \delta(t-s) \quad \text{as } h \to 0$$

Lec 19 13 / 03 / 17 (Mon)

Back to L^2-cylindrical Wiener process

$$W(t) = \sum_{n \in \mathbb{Z}^d} \beta_n(t) e^{inx} \in H^{-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$$

$$W^{-\frac{d}{2}-\varepsilon, \infty}(\mathbb{T}^d)$$

$$W \in W^{\frac{1}{2}-, \infty}_{t, loc} W^{-\frac{d}{2}-\varepsilon, \infty}_x(\mathbb{T}^d)$$

· (SNLS) $\Leftarrow$ study it in the mild formulation:

$$u(t) = S(t) u_0 \mp i \int_0^t S(t-t') |u|^{p-1} u \, dt'$$

$$- i \underbrace{\int_0^t S(t-t') \phi \, dW(t')}_{}$$

$$= \text{stochastic convolution} = \Psi$$

$$\Psi = \int_0^t S(t-t') \phi \, dW(t') = \sum_{n \in \mathbb{Z}^d} \int_0^t e^{-in^2(t-t')} \phi(e^{in \cdot x}) \, d\beta_n(t')$$

For simplicity, we only consider $\phi = \text{diag}(\phi_n)$ i.e. $\phi(e_n) = \phi_n e_n$

translation invariant.

· **Wiener integral** (real-valued case)

$$\int_a^b f(t)\, dB(t) \quad \sim \quad \langle f, dB \rangle_{L^2_t(a,b)}$$

f: deterministic

Step 1: Step func. $f(t) = \sum_{j=1}^{n} a_{j-1} \mathbb{1}_{[t_{j-1}, t_j)}(t)$

$\Rightarrow$ Define $I(f) = \sum_{j=1}^{n} a_{j-1} \left(B(t_j) - B(t_{j-1}) \right)$

(left endpt Riemann sum)

Then, ① $\mathbb{E}[I(f)] = 0$

② $\mathbb{E}\left[(I(f))^2\right] \underset{\text{indep}}{=} \sum_{j=1}^{n} a_{j-1}^2 (t_j - t_{j-1})$

$$= \int_a^b f^2(t)\, dt$$

$$= \|f\|^2_{L^2(a,b)}$$

step2: Given $f \in L^2(a,b)$,

approximate f by step functions f_n in $L^2(a,b)$

and define $I(f) = \lim_{n\to\infty} I(f_n)$ ← Gaussian

$\Longrightarrow$ ① & ② hold.

$\mathcal{H}^1$ — Wiener homog. chaoses of order 1

Namely, $I : L^2(a,b) \longrightarrow L^2(\Omega)$ is an isometry.

(just like the white noise functional in Chap 3.)

Rmk: If $f \in C^1$, we can define it as

Paley-Wiener-Zygmund integral:

$$I(f) = \int_a^b f\,dB = -\int_a^b f'(t)\,B(t)\,dt \quad (\text{if } f(b) = 0)$$

Rmk: If B is $\mathbb{C}$-valued,

$$\mathbb{E}\left[(I(f))^2\right] = 2\,\|f\|^2_{L^2([a,b])}$$

· **Ito integral**:

σ - field

filtration $\{\mathcal{F}_t\}_{t \geq 0}$ s.t. $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \subset \mathcal{F}$, $t_1 \leq t_2$

· We say $X(t)$ is adapted (non-anticipating)

if $X(t)$ is $\mathcal{F}_t$-measurable, $\forall t \geq 0$.

- progressively meas: $\forall T$,

$$[0,T] \times \Omega \ni (t, \omega) \longmapsto X(t, \omega) \text{ is } \mathcal{B}_{[0,T]} \otimes \mathcal{F}_t\text{-meas}$$

- p.m. $\rightarrow$ adapted
- adapted & left (or right) conti $\rightarrow$ p.m.

  e.g. adapted & càdlàg func $\rightarrow$ p.m.

  càdlàg: continue à droite limite à gauche

"Assume" (i) $B(t)$ is $\mathcal{F}_t$ - meas

(ii) $B(t) - B(s)$ is indep of $\{\mathcal{F}_s\}_{s<t}$.

e.g. simply take $\mathcal{F}_t^B = \sigma(\{B(s) : s \le t\})$

· We set

$$L^2_{ad}([a,b] \times \Omega) = \{ f(t,\omega) :$$

① f is adapted to $\{\mathcal{F}_t\}$

② $\int_a^b \mathbb{E}(f^2(t))\, dt < \infty \}$

Define the Ito integral here.

Rmk: can define Ito integral on a larger class of stochastic processes.

e.g. $\mathcal{L}_{ad}(\Omega ; L^2([a,b]))$

① f, adapted

② $\int_a^b |f(t,\omega)|^2 dt < \infty$, a.s.

Step 1: Step stoch process

"does not peek in the future"

$$f(t,\omega) = \sum_{j=1}^{n} a_{j-1}(\omega)\, \mathbb{1}_{[t_{j-1}, t_j)}(t)$$

$$a_j,\ \mathcal{F}_{t_j}\text{-meas}$$

$$\sum \mathbb{E}(a_j^2) < \infty$$

· Define Ito integral:

$$I(f)(\omega) = \sum_{j=1}^{n} a_{j-1}(\omega)\,\big(B(t_j) - B(t_{j-1})\big)$$

(left endpt Riemann sum)

Lemma 4.1: ① $\mathbb{E}[I(f)] = 0$

② $\mathbb{E}[(I(f))^2] = \int_a^b \mathbb{E}[f^2]\, dt$. (Ito isometry)

Recall: Conditional expectation.

If $X \in L^1(\Omega; \mathcal{F})$ and $\mathcal{G} \subset \mathcal{F}$,
$\uparrow$ sub σ-field,

then define the conditional expectation of X given $\mathcal{G}$ by the unique r.v. Y s.t.

① Y is $\mathcal{G}$-meas

② $\int_A X \, dP = \int_A Y \, dP, \quad \forall A \in \mathcal{G}$.

Rmk: ① Y is given by Radon-Nikodym Thm:

$$\mu(A) = \int_A X \, dP, \qquad A \in \mathcal{G}.$$

$$\Rightarrow \mu \ll P|_{\mathcal{G}}$$

$$\overset{\text{R-N}}{\Rightarrow} \quad d\mu = \underset{\underset{L^1}{\Uparrow}}{Y} \, dP|_{\mathcal{G}}.$$

Denote it by $Y = \mathbb{E}[X|\mathcal{G}]$

② $X \in L^2(\Omega; \mathcal{F})$

$$\Rightarrow \mathbb{E}[X \mid \mathcal{G}] = P_{L^2(\Omega;\mathcal{G})}(X)$$ $\leftarrow$ closed

---

<u>Properties</u>: (i) $\mathbb{E}\big[\mathbb{E}[X \mid \mathcal{G}]\big] = \mathbb{E}(X)$

(In general, more useful to read it as

$$\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X \mid \mathcal{G}]\big]$$

i.e. compute expectation by conditioning)

(ii) If X is $\mathcal{G}$-meas,

$$\mathbb{E}[X \mid \mathcal{G}] = X$$

(iii) If X & $\mathcal{G}$ are indep, ($\{X \in U\}$ and $A \in \mathcal{G}$ are indep $\forall, U \in \mathcal{B}_{\mathbb{R}}, A \in \mathcal{G}$)

$$\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$$

(iv) If Y is $\mathcal{G}$-meas and $\mathbb{E}[XY] < \infty$,

then $\mathbb{E}[XY \mid \mathcal{G}] = Y\,\mathbb{E}[X \mid \mathcal{G}]$

· Pf of Lemma 4.1 :

1. $$\mathbb{E}\left[a_{j-1}\left(B(t_j)-B(t_{j-1})\right)\right]$$
$$\overset{(i)}{=}\mathbb{E}\left[\mathbb{E}\left[a_{j-1}\left(B(t_j)-B(t_{j-1})\right)\middle|\mathcal{F}_{t_{j-1}}\right]\right]$$
$$\overset{(iv)}{=}\mathbb{E}\Big[a_{j-1}\underbrace{\mathbb{E}\left[B(t_j)-B(t_{j-1})\middle|\mathcal{F}_{t_{j-1}}\right]}_{\overset{(iii)}{=}\mathbb{E}[B(t_j)-B(t_{j-1})]=0}\Big]=0$$

2. $i<j$.
$$\mathbb{E}\Big[\underbrace{a_{i-1}a_{j-1}\left(B(t_i)-B(t_{i-1})\right)}_{\mathcal{F}_{t_{j-1}}\text{-meas}}\left(B(t_j)-B(t_{j-1})\right)\Big]$$
$$=\cdots\underset{(i),(iv)}{=}\mathbb{E}\Big[a_{i-1}a_{j-1}\left(B(t_i)-B(t_{i-1})\right)\underbrace{\mathbb{E}\left[B(t_j)-B(t_{j-1})\middle|\mathcal{F}_{t_{j-1}}\right]}_{\underset{(iii)}{=}0}\Big]$$
$$=0$$

$\underline{i=j}$

$$\mathbb{E}\left[\, a_{j-1}^2 \left(B(t_j) - B(t_{j-1})\right)^2 \right]$$

$$\overset{(i),(iv),(iii)}{=} \mathbb{E}\left[a_{j-1}^2\right] (t_j - t_{j-1})$$

$\boxed{\diagup}$

<u>Step 2:</u> FACT: Given $f \in L^2_{ad}([a,b] \times \Omega)$,

$\exists \{f_n\}$ of step stoch processes

converging to f.

Define Ito integral:

$$I(f) = \int_a^b f(t,\omega)\, dB(t,\omega)$$

$$\overset{\text{def}}{=} \lim_{n\to\infty} I(f_n).$$

<u>Properties</u>: ① I, linear

② $\mathbb{E}(I(f)) = 0$

③ $\mathbb{E}\left[(I(f))^2\right] = \int_a^b \mathbb{E}[f^2]\,dt$ (Ito isometry)

④ $\mathbb{E}\left[\int_a^b f(t)\,dB \int_a^b h(t)\,dB\right]$

$$= \int_a^b \mathbb{E}[f(t)\,h(t)]\,dt.$$

$I : L^2_{ad}([a,b] \times \Omega) \longrightarrow L^2(\Omega)$ is an isometry

Sec 4.2: Local well-posedness of SNLS

$$(\text{SNLS}) \begin{cases} i\,du = (-\Delta u \pm |u|^2 u)\,dt + \phi\, dW & \text{on } \mathbb{T} \\ u|_{t=0} = u_0 \end{cases}$$

$\phi = \operatorname{diag} \phi_n$, HS on $L^2(\mathbb{T})$

$$\phi(e_n) = \phi_n e_n$$

Mild formulation: $S(t) = e^{it\Delta}$

$$u(t) = S(t)u_0 \mp i \int_0^t S(t-t')\,|u|^2 u(t')\,dt'$$
$$\underbrace{- i \int_0^t S(t-t')\,\phi\, dW(t')}_{\text{stochastic convolution } \Psi}$$

· Review of deterministic LWP of cubic NLS on $\mathbb{T}$

(Bourgain '93)

· Fourier restriction norm method

$$\|u\|_{X^{s,b}} = \| \langle n \rangle^s \langle \tau + n^2 \rangle^b \widehat{u}(n,\tau) \|_{\ell^2_n L^2_\tau}$$

(lin Schrödinger : $i\partial_t u + \Delta u = 0$

$\underset{\text{F.T in spacetime}}{\Longrightarrow}$ $-(\tau + n^2)\, \widehat{u}(n,\tau) = 0$

· local-in-time version : $T > 0$

$$\|u\|_{X^{s,b}_T} = \inf \{ \|v\|_{X^{s,b}} : v|_{[0,T]} = u \}.$$

· Duhamel formula:

$$u(t) = \Gamma_{u_0}(u)(t) = \eta(t) S(t) u_0 \mp i\, \eta\left(\tfrac{t}{T}\right) \int_0^t S(t-t') |u|^2 u(t')\, dt'.$$

$$\eta(t) \equiv \begin{cases} 1 & \text{on } [0,1] \\ 0 & \text{on } [-\tfrac{1}{2}, \tfrac{3}{2}]^c \end{cases}$$

$\Rightarrow$ If $u = \Gamma_{u_0}(u)$, then u is a soln on $[0,T]$, $0 < T \leq 1$.

- Deterministic estimates

1. homog lin $\| \eta(t) S(t) u_0 \|_{X^{s,b}} \le C_b \| u_0 \|_{H^s}$.

$$\mathcal{F}_{x,t}(\eta(t) S(t) u_0) = \hat{\eta}(\tau + n^2)\, \hat{u}_0(n)$$

2. nonhomog lin :

$$\left\| \eta\left(\tfrac{t}{T}\right) \int_0^t S(t-t') F(t')\, dt' \right\|_{X^{s,b}} \lesssim T^{\theta} \| F \|_{X^{s,b-1+\theta}}$$

· $b > 1/2$, $\theta > 0$ small.

3. L^4-Strichatz estimate

$$\| u \|_{L^4_{x,t}(\mathbb{T} \times \mathbb{R})} \lesssim \| u \|_{X^{0,3/8}}$$

4. $\| u \|_{C_T H^s} \lesssim \| u \|_{X^{s,b}_T}$, $b > 1/2$.

($\Leftarrow$ 1-d Sobolev in time.)

$$\Rightarrow \|\Gamma_{u_0} u\|_{X_T^{0,\frac{1}{2}+}}$$

$$\leq \|\eta(t) S(t) u_0\|_{X^{0,\frac{1}{2}+}} + \left\|\eta\left(\tfrac{t}{T}\right)\int_0^t S(t-t')|v|^2 v(t')\,dt'\right\|_{X^{0,\frac{1}{2}+}}$$

(for <u>any</u> extension v of u)

$$\overset{①,②}{\lesssim} \|u_0\|_{L^2} + T^\theta \underbrace{\| |v|^2 v \|_{X^{0,-\frac{1}{2}+\theta+}}}$$

$$(X^{s,b})^* = X^{-s,-b}$$

$$\| \quad \sup_{\|w\|_{X^{0,\frac{1}{2}-\theta-}}=1} \underbrace{\left| \iint w \cdot |v|^2 v \, dx\, dt \right|}$$

$$\leq \|w\|_{L^4_{x,t}} \|v\|^3_{L^4_{x,t}}$$

$$\overset{③}{\lesssim} \|w\|_{X^{0,\frac{3}{8}}} \|v\|^3_{X^{0,3/8}}$$

$$\leq \|v\|^3_{X^{0,3/8}}$$

Taking inf over v.

$$\Longrightarrow \|\Gamma_{u_0} u\|_{X_T^{0,1/2+}} \lesssim \|u_0\|_{L^2} + T^\theta \|u\|^3_{X_{\underline{T}}^{0,1/2+}}.$$

- A difference estimate holds in a similar manner

$\Longrightarrow$ contraction in $B_R \subset X_T^{0,\frac{1}{2}+}$ $\quad R = C\|u_0\|_{L^2}$

$0 < T = T(\|u_0\|_{L^2}) \ll 1$

Back to SNLS.

- <u>method 1:</u> Show

$$\mathbb{E}\left[\|\Psi^\omega\|^2_{X_T^{0,\frac{1}{2}-}}\right] \lesssim 1 < \infty.$$

$\Longrightarrow$ fixed pt argument in $X_T^{0,\frac{1}{2}-}$

$$R^\omega \sim \|u_0\|_{L^2} + \|\Psi^\omega\|_{X_T^{0,\frac{1}{2}-}}$$

$$\Rightarrow T = T^\omega > 0.$$

- Then, show the continuity in time.

$$u(t) = \underset{\text{conti}}{\text{lin}} + \underbrace{\text{nonlin}}_{\substack{\text{estimate} \\ \text{in } X_T^{0,\frac{1}{2}+}}} + \Psi$$

$\uparrow$ Kolmogorov conti criterion.

- method 2: write $u = \Psi + v$. (6)

Need to estimate $\|\Psi\|_{L^q_T L^r_x}$

$p \geq q \vee r$

$$\left[\mathbb{E}\left(\|\Psi\|^p_{L^q_T L^r_x}\right)\right]^{1/p}$$

$\mathcal{H}^1$ = homog Wiener chaoses of order 1

$$\overset{\text{Minkowski}}{\leq} \Big\| \, \|\Psi(x,t)\|_{L^p(\Omega)} \Big\|_{L^q_T L^r_x}$$

$$\overset{\substack{\text{Wiener chaos}\\ \text{est}}}{\lesssim} p^{1/2} \Big\| \underbrace{\|\Psi(x,t)\|_{L^2(\Omega)}}_{\overset{\text{indep}}{=} \left(\sum_n \underbrace{\mathbb{E}\left[\left|\int_0^t e^{-in^2(t-t')}\phi_n e^{inx}\, d\beta_n(t')\right|^2\right]}_{\overset{\text{Ito isometry}}{=} 2|\phi_n|^2 t}\right)^{1/2}} \Big\|_{L^q_T L^r_x}$$

$$\lesssim p^{1/2} T^{\frac{1}{2}+\frac{1}{q}} \underbrace{\left(\sum |\phi_n|^2\right)^{1/2}}_{= \|\Phi\|_{HS(L^2 : L^2)}} \Rightarrow \text{chebyshev.}$$

$\left(\|\Phi\|_{HS(X:Y)} = \left[\sum \|\Phi(e_n)\|_Y^2\right)^{1/2}. \quad \{e_n\} \text{ ONB in } X \right.$

- Fixed pt problem for v:

$$v(t) = \eta(t)\, S(t) u_0 \mp i\, \eta\!\left(\tfrac{t}{T}\right) \int_0^t S(t-t')\, |v+\Psi|^2 (v+\Psi)(t')\, dt'.$$

$\uparrow \mathbb{1}_{[0,T]}$

$\Rightarrow$ contraction in $X_T^{0,\frac{1}{2}+}$ (a bit smoother)

only change:

$$\|v+\Psi\|_{L^4_{x,T}} \le \|v\|_{L^4_{x,t}} + \underbrace{\|\Psi^\omega\|_{L^4_{x,T}}}_{<\infty,\ \text{a.s.}}$$

Thm 4.2: Stochastic cubic NLS on $\mathbb{T}$ (with $\phi \in HS(L^2; L^2)$) is pathwise locally well-posed in $L^2(\mathbb{T})$

Sec 4.3 : Global well-posedness in $L^2(\mathbb{T})$

<u>Idea</u>: Control $M(u) = \sum_{n\in\mathbb{Z}} |\hat{u}_n|^2 = \sum p_n^2 + q_n^2$

$p_n = \mathrm{Re}\,\hat{u}_n$

$q_n = \mathrm{Im}\,\hat{u}_n$.

$$d\hat{u}_n = \left(-i n^2 \hat{u}_n - i\, \mathcal{F}_x(|u|^2 u)(n)\right) dt - i \phi_n \, d\beta_n$$

$$\Rightarrow \quad dp_n = \left(n^2 q_n + \mathrm{Im}\,\mathcal{F}_x(|u|^2 u)(n)\right) dt + \mathrm{Im}\left(\phi_n \, d\beta_n\right)$$

$$dq_n = \left(-n^2 p_n - \mathrm{Re}\,\mathcal{F}_x(|u|^2 u)(n)\right) dt \boxed{- \mathrm{Re}\left(\phi_n \, d\beta_n\right)}$$

$$\mathrm{Im}\,\phi_n \, d(\mathrm{Re}\,\beta_n) + \mathrm{Re}\,\phi_n \, d(\mathrm{Im}\,\beta_n)$$

$$-\mathrm{Re}\,\phi_n \, d(\mathrm{Re}\,\beta_n) + \mathrm{Im}\,\phi_n \, d(\mathrm{Im}\,\beta_n)$$

· <u>Ito's lemma</u>: $dX = f\,dt + g\,dB$

Consider $F(X)$

Then, $dF = \partial_x F\, dX + \frac{1}{2}\partial_x^2 F (dX)^2$

$= \partial_x F\,(f\,dt + g\,dB) + \frac{1}{2}\partial_x^2 F\cdot g^2 dt.$

$(dt^2) = 0$

$dt\,dB = dB\,dt = 0$

$(dB)^2 = dt$

$$\Rightarrow dM \overset{\text{Ito}}{=} \underbrace{2\sum_n (p_n\,dp_n + q_n\,dq_n)}_{= 2\sum_n p_n \operatorname{Im}(\Phi_n d\beta_n) - q_n \operatorname{Re}(\Phi_n d\beta_n)} + \underbrace{\sum\left((dp_n)^2 + (dq_n)^2\right)}_{= 2\sum|\Phi_n|^2 dt = 2\|\Phi\|^2_{HS(L^2;L^2)} dt.}$$

Integrate from 0 to t (large), sup over $t \in [0,T]$, & take expecatation

$\Longrightarrow$ Apply Burkholder-Davis-Gundy ineq.

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|^2_{L^2}\right] \le \|u_0\|^2_{L^2} + C(T, \|\Phi\|_{HS(L^2;L^2)}).$$

$\Longrightarrow$ a.s. GWP in $L^2(\mathbb{T})$ by iterating the local argument.

B-D-G ineq: X, local martingale

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|X(t)|^p\Big) \sim \mathbb{E}\Big[\langle X(t)\rangle_{[0,T]}^{p/2}\Big], \quad 1\le p<\infty$$

quadradic variation

Rmk: In order to justify the application of Ito's lemma, one should first consider the finite dim'l dynamics:

$$\begin{cases} i\,du^N = \left(-\Delta u^N \pm P_{\le N}(|u^N|^2 u^N)\right)dt + P_{\le N}(\phi\, dW) \\ u^N|_{t=0} = u_0^N := P_{\le N} u_0 \end{cases}$$

and obtain an a priori bound (indep of N)

Then, proceed with an approximation argument to obtain the same a priori bound for solutions to SNLS.

- Another alternative is to apply the infinite dim'l version of Ito's lemma (such as the one in Da Prato - Zabczyk.)