# Stochastic dispersive dynamics
## Chapter 3: On the transport property of Gaussian measures under Hamiltonian PDE dynamics

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Hausdorff School: Dispersive Equations, Solitons, and Blow-up

## Gaussian measures $\mu_s$:

$$\text{“}d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}^d} e^{-\frac{1}{2}\langle n \rangle^{2s} |\widehat{u}_n|^2} d\widehat{u}_n\text{”}$$

- $\mu_s$ is a Gaussian probability measure on $H^\sigma(\mathbb{T}^d)$ for $\sigma < s - \frac{d}{2}$
- Under $\mu_s$, a random function $u$ is represented by the random Fourier series:

$$u(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle^s} e^{2\pi i n \cdot x} \in H^\sigma(\mathbb{T}^d) \setminus H^{s-\frac{d}{2}}(\mathbb{T}^d), \text{ almost surely}$$

where $\{g_n(\omega)\}_{n \in \mathbb{Z}^d}$ = independent standard $\mathbb{C}$-valued Gaussian r.v.'s

Let $\vec{\mu}_s = \mu_s \otimes \mu_{s-1}$

$\Longrightarrow$ $\vec{\mu}_s$ is **invariant** under the linear wave dynamics for any $s \in \mathbb{R}$

**Goal:**

Study transport properties of Gaussian measures $\mu_s$ (or $\vec{\mu}_s$) under *nonlinear* Hamiltonian PDE dynamics

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$(H, B, \mu)$, abstract Wiener space, $H$ = Cameron-Martin space

**Cameron-Martin Theorem:** Consider the following translation map:

$$T_h : u \mapsto u + h \qquad \text{for some } h \in B$$

**Q:** What is the relation between the original Gaussian measure $\mu$ on $B$ and the translated measure $\mu^h(\,\cdot\,) = (T_h)_* \mu(\,\cdot\,) = \mu(\,\cdot - h)$?

**Theorem:** Cameron-Martin '44

- If $h \in H$, $\mu$ and $\mu^h$ are equivalent (= mutually absolutely continuous). Namely, $\mu$ is quasi-invariant under $T_h$
- Otherwise, they are mutually singular

- This allows us to take a derivative of $\mu$ in the direction of $h \in H$ (= $H$-derivative) $\Longrightarrow$ starting point of Malliavin calculus
- For $\mu_s$ on $\mathcal{D}'(\mathbb{T}^d)$, $\mu_s$ and $\mu_s^h$ are equivalent if and only if $h \in H^s(\mathbb{T}^d)$. Namely, $h$ is $(\frac{d}{2} + \varepsilon)$-*smoother* than typical elements $u \in H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$

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## Ramer's generalization of Cameron-Martin Theorem:

$$T : u \mapsto u + F(u)$$

- We now allow the shift $F(u)$ to depend on a random element $u \in B$

**Theorem:** Ramer '74 (rough formulation)

$\mu$ is quasi-invariant under $T$ if the $H$-derivative of $F$ at $u$:

$$DF(u) : H \to H$$

is a Hilbert-Schmidt map for every $u \in B$

- For $\mu_s$ on $\mathcal{D}'(\mathbb{T}^d)$, (roughly speaking)

  $\mu_s$ is quasi-invariant under $T$ if $F(u)$ is $(d+\varepsilon)$-*smoother* than $u \in H^\sigma(\mathbb{T}^d)$

  (More smoothing than Cameron-Martin Theorem, now that the shift is random)

There are also works on quasi-invariance of $\mu$ under flows generated by vector fields: Cruzeiro '83, Peters '95, Bogachev and Mayer-Wolf '99, Ambrosio-Figalli '09, etc.

**Duhamel formulation (for NLW):** With $\vec{S}(t) = \begin{pmatrix} \cos(tD) & \frac{\sin(tD)}{D} \\ -D\sin(tD) & \cos(tD) \end{pmatrix}$,

$$\begin{aligned} \vec{u} = \Phi(t)\vec{u}(0) &= \vec{S}(t)\vec{u}(0) - \int_0^t \vec{S}(t-t') \begin{pmatrix} 0 \\ u^k(t') \end{pmatrix} dt' \\ &= \vec{S}(t)\Big\{ \vec{u}(0) + \underbrace{\int_0^t \vec{S}(-t') \begin{pmatrix} 0 \\ u^k(t') \end{pmatrix} dt'}_{= F(\vec{u}(0))} \Big\} \end{aligned}$$

- Gaussian measure $\vec{\mu}_s$ is *invariant* under the linear solution map $\vec{S}(t)$
  $\Longrightarrow$ The solution map $\Phi(t)$ is of the form "$\vec{u}(0) + F(\vec{u}(0))$"

**Q:** Can we study transport properties (such as invariance, quasi-invariance, singularity) of $\mu_s$ under nonlinear dispersive Hamiltonian PDEs?

**Goal:** Investigate transport properties of Gaussian measure $\mu_s$:

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du \quad \text{on } H^\sigma(\mathbb{T}^d), \ \sigma < s - \tfrac{d}{2}$$

- $s = 0$: **White noise** on $\mathbb{T}$: very rough

$$u(x) = \sum_{n\in\mathbb{Z}} g_n(\omega) e^{inx} \in H^\sigma(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T}), \quad \sigma < -\tfrac{1}{2}$$

Invariance of white noise

- KdV: Quastel-Valkó '08, Oh '09, '11, Oh-Quastel-Valkó '12
- (renormalized) cubic fourth order NLS (4NLS): Oh-Tzvetkov-Wang '17

**Q:** Is white noise $\mu_0$ invariant under (renormalized) cubic NLS on $\mathbb{T}$?

- **Very** difficult
- $\mu_0$ is a limit of invariant measures for cubic NLS (Oh-Quastel-Valkó '12) but no well-posedness...

**Q:** Can we study transport properties of $\mu_s$ for general (non-small) $s$?

- When $s$ is large, this question is *not* about rough solutions

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# Quasi-invariance of Gaussian measures $\mu_s$

- $(s = 0)$ white noise $\mu_0$ on $\mathbb{T}$: KdV and (Wick ordered) 4NLS
  invariance $\Longrightarrow$ quasi-invariance
- $(s = 1)$ invariant Gibbs measure (= Gaussian measure $\mu_1$ with weight)
  $\Longrightarrow$ $\mu_1$ is quasi-invariant
- completely integrable PDEs with infinitely many conservation laws
  $\Longrightarrow$ invariant measures $\rho_k$ (= $\mu_k$ with weight) for every integer $k \geq 2$
  - cubic NLS on $\mathbb{T}$, KdV on $\mathbb{T}$, Benjamin-Ono equation on $\mathbb{T}$
    (Zhidkov '01, Tzvetkov-Visciglia '14-15, Deng-Tz-V '15)
  - derivative NLS on $\mathbb{T}$: open (only construction)

**Q**: Gel'fand '96: Can we directly prove quasi-invariance of $\mu_s$ (at least for $s$ large) for (non-integrable) PDEs?

Remark:

- Gibbs measure problem: study of rough solutions
- When $s$ is large, this question is *not* about rough solutions

# Quasi-invariance of Gaussian measures $\mu_s$

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**Benjamin-Bona-Mahony equation (BBM)** on $\mathbb{T}$: small amplitude long surface waves

$$\partial_t u + \partial_x u - \partial_t \partial_x^2 u + \partial_x (u^2) = 0$$

$$\Longrightarrow \quad \partial_t u + (1-\partial_x^2)^{-1}\partial_x u + (1-\partial_x^2)^{-1}\partial_x (u^2) = 0$$

- **Ramer's result:** $\mu_s$ on $\mathcal{D}'(\mathbb{T})$ is quasi-invariant under the map

$$T : u_0 \mapsto u_0 + F(u_0)$$

if $F(u_0)$ is $(d+\varepsilon)$-*smoothing* $\Longrightarrow$ not sufficient for BBM

Tzvetkov '15: For $s \in \mathbb{N}$, $\mu_s$ is quasi-invariant under BBM

- A similar result holds for generalized BBM model with less smoothing
- introduced a new method to establish quasi-invariance of $\mu_s$ beyond Ramer
- uses the explicit smoothing in the nonlinearity but *not* dispersive effect

**Q**: Can we find a good model to prove quasi-invariance via *dispersive effect*?

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## **Cubic fourth order NLS (4NLS)** on $\mathbb{T}$:

$$i\partial_t u - \partial_x^4 u = |u|^2 u$$

- Globally well-posed in $H^\sigma(\mathbb{T})$, $\sigma \geq 0$
- Strongly ill-posed for $\sigma < 0$ (Oh-Wang '17: non-existence in negative Sobolev spaces)

**Theorem:** Oh-Tzvetkov '16, Oh-Sosoe-Tzvetkov '17

Let $s > \frac{1}{2}$. Then, the Gaussian measure $\mu_s$ is quasi-invariant under 4NLS

- This theorem is *optimal*: $\mu_s$ is supported on $H^\sigma(\mathbb{T})$, $\sigma < s - \frac{1}{2}$
- Unlike BBM, there is *no* apparent smoothing in 4NLS. We exhibit smoothing effects *via dispersion* after using some *gauge transform* and *normal form reductions*
- The proof consists of *local & global analysis* (in the phase space $H^\sigma(\mathbb{T})$)
  - local PDE analysis (normal form reductions, energy estimates)
  - global phase space analysis (gauge transform, a change-of-variable formula)

# Key role of dispersion

**Q**: Is *dispersion* essential for quasi-invariance of $\mu_s$?

**Yes.** Consider the dispersionless model on $\mathbb{T}$:

$$i\partial_t u = |u|^2 u$$

- Explicit solution formula $u(t,x) = e^{-it|u(0,x)|^2} u(0,x)$
- Globally well-posed in $H^\sigma(\mathbb{T}), \sigma > \frac{1}{2}$

  Note: our random data $u$ is a.s. continous for $s > \frac{1}{2} \implies \sigma > 0$

**Theorem:** Oh-Sosoe-Tzvetkov '17

Let $s > \frac{1}{2}$. Then, $\mu_s$ is *not* quasi-invariant under the dispersionless model

- The proof uses law of iterated logarithms, a fine criterion to measure the regularity of a typical function w.r.t. $\mu_s$ (= fractional Brownian loop). This property regularity property is destroyed by the flow of the dispersionless model

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# Rough idea

**Goal :** Compute $\frac{d}{dt}\mu_s(\Phi(t)(A))$

1. **Energy estimate** (local analysis): $\frac{d}{dt}\|\Phi(t)(u)\|^2_{H^s} \leq \underbrace{C(\|u\|_{L^2})}_{\text{conserved}} \underbrace{\|u\|^{2-\theta}_{H^{s-\frac{1}{2}-\varepsilon}}}_{\text{supp}(\mu_s)}$

2. **A change-of-variable formula** (global analysis):

$$\mu_s(\Phi(t)(A)) = Z_s^{-1}\int_{\Phi(t)A} e^{-\frac{1}{2}\|u\|^2_{H^s}}\,du \text{ “}=\text{” } Z_s^{-1}\int_A e^{-\frac{1}{2}\|\Phi(t)(u)\|^2_{H^s}}\,du$$

$\Longrightarrow$ (Yudovich) Given $t \in \mathbb{R}$ and $\delta > 0$, there exists $C = C(t,\delta) > 0$ such that

$$\mu_s(\Phi(t)(A)) \leq C(t,\delta)\{\mu_s(A)\}^{1-\delta}$$

$\Longrightarrow$ quasi-invariance!!

- In Step 1, we need to apply two transformations on the phase space. Then, perform (an infinite iteration of) normal form reductions to prove the energy estimate on a *modified* energy $E = \|u\|^2_{H^s} + R$
- In Step 2, we need to insert the frequency truncation $\mathbf{P}_{\leq N}$. Moreover, we need to consider a *modified* measure associated to the modified energy

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# Global analysis, Part 1

**Decomposition of solution map:**

$$\Phi(t) = \mathcal{G}_{-t} \circ S(t) \circ \Psi(t)$$

1. **Gauge transform:** Given $t \in \mathbb{R}$, define $\mathcal{G}_t$ on $L^2(\mathbb{T})$ by setting
$$\mathcal{G}_t[f] := e^{it \fint_{\mathbb{T}} |f|^2} f$$
2. **Interaction representation:** $v(t) = S(-t)\mathcal{G}_t[u(t)]$, where $S(t) = e^{-it\partial_x^4}$

- $\Phi(t)$ = solution map of the original 4NLS
- $\Psi(t)$ = solution map of $v(0) \mapsto v(t) = S(-t)\mathcal{G}_t[u(t)]$

**Proposition**

Let $s > \frac{1}{2}$. For every $t \in \mathbb{R}$, the Gaussian measure $\mu_s$ is invariant under $S(-t)$ and $\mathcal{G}_t$

$\Longrightarrow$ Suffices to prove quasi-invariance of $\mu_s$ under $\Psi(t)$

# Local analysis: modified energy and energy estimate

- $v = S(-t) \circ \mathcal{G}_t[u(t)]$ satisfies

$$\partial_t \widehat{v}_n = -i \sum_{\{\phi(\bar{n}) \neq 0\}} e^{-i\phi(\bar{n})t} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} + i|\widehat{v}_n|^2 \widehat{v}_n$$

On $\Gamma(n) \overset{\text{def}}{=} \{\phi(\bar{n}) \neq 0\}$, we have $|\phi(\bar{n})| \gtrsim n_{\max}^2 \quad \Longleftarrow \quad$ rapid oscillation

- **Modified energy:** $E(v) = \|v\|_{H^s}^2 + R(v) \longleftarrow$ correction term

## Proposition: energy estimate with smoothing

Let $s > \frac{3}{4}$. Then, for any small $\varepsilon > 0$, there exist $\theta > 0$ and $C > 0$ such that

$$\left| \frac{d}{dt} E(\mathbf{P}_{\leq N} v) \right| \leq \underbrace{C(\|v\|_{L^2})}_{\text{conserved}} \underbrace{\|v\|^{2-\theta}_{H^{s-\frac{1}{2}-\varepsilon}}}_{\text{supp}(\mu_s)}$$

- (*Infinite* iteration of) **normal form reductions** $\longrightarrow$ correction term $R$
- Standard (deterministic) PDE analysis
- The proof relies on elementary number theory (divisor counting argument)

By normal form reduction (IBP in time), we have

$$\frac{d}{dt}\|v(t)\|_{H^s}^2 = -2\,\mathrm{Re}\, i \sum_{n\in\mathbb{Z}} \sum_{\Gamma(n)} e^{-i\phi(\bar{n})t} \langle n\rangle^{2s} v_{n_1}\overline{v_{n_2}} v_{n_3}\overline{v_n}$$

$$= -2i\,\mathrm{Re}\,\frac{d}{dt}\Bigg[\underbrace{\sum_{n\in\mathbb{Z}} \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n\rangle^{2s} v_{n_1}\overline{v_{n_2}} v_{n_3}\overline{v_n}}_{=:-R(v)}\Bigg]$$

$$+ 2i\,\mathrm{Re} \sum_{n\in\mathbb{Z}} \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n\rangle^{2s} \underbrace{\partial_t (v_{n_1}\overline{v_{n_2}} v_{n_3}\overline{v_n})}_{=6\text{-linear}}$$

When $s \in (\frac{1}{2}, \frac{3}{4}]$, iterate this process ***infinitely many times***:

$$\frac{d}{dt}\|v(t)\|_{H^s}^2 = \frac{d}{dt}\underbrace{\Bigg[\sum_{j=2}^{\infty}\mathcal{N}_0^{(j)}(v)\Bigg]}_{=:-R(v)} + \sum_{j=2}^{\infty}\mathcal{N}_1^{(j)}(v) + \sum_{j=2}^{\infty}\mathcal{R}^{(j)}(v)$$

$$\Longrightarrow \quad \left|\frac{d}{dt}E(\mathbf{P}_{\leq N} v)\right| \leq C(\|v\|_{L^2}), \qquad \text{where } E(v) = \|v\|_{H^s}^2 + R(v)$$

- Guo-Kwon-Oh '13: infinite iteration of NF reductions for cubic NLS on $\mathbb{T}$ (i.e. on the equation) in the context of low regularity uniqueness problem

# Global analysis, Part2

1. **Weighted Gaussian measures:** $E(v) = \|v\|_{H^s}^2 + R(v)$
   - Construct a weighted Gaussian measure $\rho_{s,N,r}$ of the form:

$$
\begin{aligned}
d\rho_{s,N,r} &= Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \le r\}} e^{-\frac{1}{2}E(\mathbf{P}_{\le N} v)} dv \\
&= Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \le r\}} e^{-\frac{1}{2}R(\mathbf{P}_{\le N} v)} \underbrace{e^{-\frac{1}{2}\|v\|_{H^s}^2} dv}_{d\mu_s}
\end{aligned}
$$

2. **A change-of-variable formula:**

$$
\rho_{s,N,r}(\Psi_N(t)(A)) = \hat{Z}_{s,N,r}^{-1} \int_A \mathbf{1}_{\{\|v\|_{L^2} \le r\}} e^{-\frac{1}{2}E(\mathbf{P}_{\le N} \Psi_N(t)(v))} d(\mathbf{P}_{\le N} v) \otimes d\mu_{s,N}^{\perp}
$$

3. Study **measure evolution** & take limits ($N \to \infty$, then $r \to \infty$)
   - compute time derivative (**energy estimate**)

   $\Longrightarrow$ quasi-invariance of $\rho_{s,N,r}$ under $\Psi_N(t)$

   $\overset{N\to\infty}{\Longrightarrow}$ quasi-invariance of $\rho_{s,r}$ (and $\mu_{s,r}$) under $\Psi(t)$

   $\overset{r\to\infty}{\Longrightarrow}$ quasi-invariance of $\mu_s$ under $\Psi(t)$!!

**Nonlinear wave equation:** Duhamel part enjoys 1-smoothing:

$$u(t) = S(t)(u_0, u_1) + \int_0^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} |u|^{p-1} u(t')dt'$$

Gaussian measure on $(u, \partial_t u)$: $\vec{\mu}_{s+1}(u, \partial_t u) = \mu_{s+1} \otimes \mu_s(u, \partial_t u)$

- $d = 1$: Tzvetkov '15 (implicit in a remark)

**Theorem:** Oh-Tzvetkov '17 ($d = 2$, defocusing cubic NLW)

Let $s \geq 2$ be an even integer. Then, $\vec{\mu}_{s+1}$ is quasi-invariant under the defocusing cubic NLW on $\mathbb{T}^2$

- A typical element $(u, v)$ under $\vec{\mu}_{s+1}$ lives in $\mathcal{H}^\sigma = H^\sigma \times H^{\sigma-1}$, $\sigma < s$.
  Given a fixed $(h_1, h_2) \in \mathcal{H}^{\sigma+1}$, consider $T_h : (u, v) \mapsto (u, v) + (h_1, h_2)$
  Cameron-Martin $\Longrightarrow$ $\vec{\mu}_{s+1}$ and its transported measure are singular
- Given $(u_0, u_1) \in \mathcal{H}^\sigma$, we only have the nonlinear part for NLW in $\mathcal{H}^{\sigma+1}$

**Main difficulty:** energy estimate: $\partial_t \|(u, \partial_t u)\|^2_{\mathcal{H}^{s+1}}$

- *renormalized* energy (but ***no*** renormalization for the equation)

$\Longleftarrow$ We achieve this by introducing a *simultaneous* renormalization on both the $\mathcal{H}^{s+1}$-energy functional and its time derivative

- We establish a renormalized energy estimate in the *probabilistic* setting
- We need to renormalize the energy even if $s \gg 1$!!

In the following, we consider defocusing NLKG (for simplicity):

$$\partial_t^2 u + (1 - \Delta)u = -u^3$$

with Hamiltonian $E(u) = \frac{1}{2}\int(\partial_t u)^2 + \frac{1}{2}\int(Ju)^2 + \frac{1}{4}\int u^4, \quad J = \sqrt{1 - \Delta}$

**Goal:** Define a renormalized energy $E_{s,\infty} \sim \|(u, \partial_t u)\|^2_{\mathcal{H}^{s+1}}$ with a good $\partial_t$-estimate

**Ans:** $E_{s,\infty} = \frac{1}{2}\int (J^s \partial_t u)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \underbrace{\frac{3}{2}\int (J^s u)^2 u^2}_{=\infty,\ \text{a.s.}} - \frac{3}{2}\infty \int u^2$

$\Longleftarrow$ Both $E_{s,\infty}$ and $\partial_t E_{s,\infty}$ behave "well"

Define $\sigma_N$ by

$$\sigma_N = \mathbb{E}_{\vec{\mu}_{s+1}}\left[\int (J^s \mathbf{P}_{\leq N} u)^2\right] = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \frac{1}{1 + |n|^2} \sim \log N \longrightarrow \infty$$

$\Longrightarrow$ For each $p < \infty$, we have

$$X_N(\omega) := \underbrace{\int (J^s \mathbf{P}_{\leq N} u)^2 - \sigma_N}_{\to \text{"}\infty - \infty\text{"}} = \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq N}} \frac{|g_n|^2 - 1}{1 + |n|^2} \in L^p(\Omega)$$

with uniform bounds in $N \in \mathbb{N}$.

$\Longrightarrow$ $X_N$ converges to $X_\infty$ in $L^p(\Omega)$ for any $p < \infty$, allowing us to define

$$X_\infty(\omega) = \int (J^s u)^2 - \sigma_\infty := \lim_{N \to \infty} \left\{ \int (J^s \mathbf{P}_{\leq N} u)^2 - \sigma_N \right\}$$

**Goal:** Define a renormalized energy $E_{s,\infty} \sim \|(u, \partial_t u)\|^2_{\mathcal{H}^{s+1}}$ with a good $\partial_t$-estimate

**Ans:** $E_{s,\infty} = \frac{1}{2}\int (J^s \partial_t u)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \underbrace{\frac{3}{2}\int (J^s u)^2 u^2}_{=\infty,\ \text{a.s.}} - \frac{3}{2}\infty \int u^2$

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$$X_\infty(\omega) = \int (J^s u)^2 - \sigma_\infty := \lim_{N \to \infty} \left\{ \int (J^s \mathbf{P}_{\leq N} u)^2 - \sigma_N \right\}$$

$$\frac{1}{2}\partial_t\|(u,\partial_t u)\|^2_{\mathcal{H}^{s+1}} = -3\int(\partial_t J^s u)J^s u\cdot u^2 + \text{l.o.t.}$$

$$\overset{\text{IBP}}{=} -\frac{3}{2}\partial_t\left[\int (J^s u)^2 u^2\right] + 3\int (J^s u)^2\partial_t u\cdot u + \text{l.o.t.}$$

$$= -\frac{3}{2}\partial_t\left[\int \mathbf{P}_{\neq 0}[(J^s u)^2]\cdot \mathbf{P}_{\neq 0}[u^2]\right] + 3\int \mathbf{P}_{\neq 0}[(J^s u)^2]\cdot \mathbf{P}_{\neq 0}[\partial_t u\cdot u] + \text{l.o.t.}$$

$$-\underbrace{\frac{3}{2}\partial_t\left[\int (J^s u)^2\int u^2\right]}_{=\infty} + \underbrace{3\int (J^s u)^2\int \partial_t u\cdot u}_{=\infty}$$

With $\sigma_N$, we have

$$-\frac{3}{2}\partial_t\left[\int (J^s u)^2\int u^2\right] + 3\int (J^s u)^2\int \partial_t u\cdot u$$

$$= -\frac{3}{2}\partial_t\left[\Bigg(\underbrace{\int (J^s u)^2 - \sigma_N}_{=X_N}\Bigg)\int u^2\right] + 3\Bigg(\underbrace{\int (J^s u)^2 - \sigma_N}_{=X_N}\Bigg)\int \partial_t u\cdot u.$$

Define the renormalized energy $E_{s,N}(u,\partial_t u)$ by

$$
\begin{aligned}
E_{s,N}(u,\partial_t u) &= \frac{1}{2}\int (J^s\partial_t u)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \frac{3}{2}\int (J^s u)^2 u^2 - \frac{3}{2}\sigma_N \int u^2 \\
&= \frac{1}{2}\int (J^s\partial_t u)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \frac{3}{2}\int \mathbf{P}_{\neq 0}[(J^s u)^2]\cdot \mathbf{P}_{\neq 0}[u^2] \\
&\qquad + \frac{3}{2}\left(\int (J^s u)^2 - \sigma_N\right)\int u^2
\end{aligned}
$$

$$
\Longrightarrow \partial_t E_{s,N}(u) = 3\int \mathbf{P}_{\neq 0}[(J^s u)^2]\cdot \mathbf{P}_{\neq 0}[\partial_t u \cdot u] + 3\left(\int (J^s u)^2 - \sigma_N\right)\int \partial_t u\cdot u + \text{l.o.t.}
$$

**Probabilistic renormalized energy estimate:**

$$
\left\{\int_{\{E(\mathbf{P}_{\leq N}u,\mathbf{P}_{\leq N}v)\leq r\}} \left|\partial_t E_{s,N}(\pi_N\Phi_N(t)(u,v))|_{t=0}\right|^p d\mu_s(u,v)\right\}^{\frac{1}{p}} \lesssim p
$$

$\Longrightarrow E_{s,N} \to E_{s,\infty} = \frac{1}{2}\int (J^s\partial_t u)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \frac{3}{2}\int (J^s u)^2 u^2 - \frac{3}{2}\infty\int u^2$, a.s.

and $E_{s,\infty}$ satisfies the same $\partial_t$-bound

# Remarks on Chapter 3

- Propagation of the $W^{\sigma,r}$-regularity, $\sigma < s - \frac{d}{2}$ and $r \le \infty$
- We showed mutual absolute continuity of the transported measure $\Phi(t)_* \mu_s$ and the original Gaussian measure $\mu_s$. Our argument, however, does not tell us much about the *time-dependent* Radon-Nikodym derivative (in $L^1(\mu_s)$) of $\Phi(t)_* \mu_s$ with respect to $\mu_s$. It would be interesting to study more about the resulting Radon-Nikodym derivatives
    - higher integrability in $L^p(\mu_s)$, $p > 1$?
    - compactness in time? property of its time average?
- By an argument analogous to that for invariant measure, we can obtain

  $$\|u(t)\|_{H^\sigma} \lesssim C(u_0^\omega)(1+|t|)^{\alpha(s)} \text{ for any } t \in \mathbb{R},$$

  where $\alpha(s) \to \infty$, as $s \to \infty$. It is very far from the logarithmic bound for invariant measures and may be obtained by deterministic techniques.

  **Q:** Can we establish *quantitative* versions of quasi-invariance and prove new growth bounds on higher Sobolev norms of solutions in a probabilistic manner?
- Our current understanding of the corresponding question for the (more complicated) NLS is very poor (except for 1-$d$ cubic NLS)...

# Remarks & comments on Chapters 1 and 2

$\rho$ = invariant (Gibbs) measure

Dynamical properties?

1. Recurrence property: Poincaré, Furstenberg '77

2. $\rho$ invariant $\implies$ $u(t) \overset{\mathcal{D}}{\sim} u(0)$ but how are $u(t)$ and $u(0)$ related?

   Can we say anything about the space-time covariance $\mathbb{E}_\rho[u(x,t)\overline{u(y,0)}]$?

   - Lukkarinen-Spohn '11: weakly nonlinear & large box limit of lattice NLS

3. Ergodicity and 'asymptotic stability' of $\rho$?

   - Completely open
   - These questions have been answered for some stochastic PDEs. This is mainly due to *uniqueness* of invariant measures. However, for Hamiltonian PDEs, there are more than one (formally) invariant measures and such questions are out of reach at this point...

This recent development also led to

1. **Probabilistic well-posedness** beyond deterministic analysis:
   - almost sure well-posedness with *rough & random* initial data

   - Global (*without* invariant measures): Colliander-Oh '12, Nahmod-Pavlović-Staffilani '13, Burq-Tzvetkov '14, Lührmann-Mendelson '14, Bényi-Oh-Pocovnicu '15 (conditional), Pocovnicu '15, Oh-Pocovnicu '16, Killip-Murphy-Vişan '17, Oh-Okamoto-Pocovnicu '17

2. **Singular stochastic dispersive PDEs**: *space-time white noise forcing*

   - stochastic KdV on $\mathbb{T}$: LWP (Oh '09), global dynamics (Oh-Quastel-Sosoe '17)
   - stochastic NLW on $\mathbb{T}^2$: LWP (Gubinelli-Koch-Oh '17)
     GWP (Gubinelli-Koch-Oh-Tolomeo '17)
     *time-dependent* renormalization
   - stochastic cubic NLS on $\mathbb{T}$: *completely open*
     important in fiber optics

# On $\mathbb{T}^3$?

- Gibbs measure on $\mathbb{T}^3$ is renormalizable only for $p = 3$ & defocusing
  - Wick ordering is not enough (need second order correction)
  - very rough $\sim H^{-\frac{1}{2}-}(\mathbb{T}^3)$
- Stochastic quantization equation:

$$\partial_t u = \Delta u - u^3 + \infty \cdot u + \underbrace{\xi}_{\text{space-time white noise}}$$

  - formally preserves the Gibbs measure
  - "local well-posedness": Hairer '14 (regularity structure), Kupiainen '16 (RG method), Catellier-Chouk '16 (paracontrolled distribution introduced by Gubinelli-Imkeller-Perkowski '15)
  - invariance of Gibbs measure: Hairer-Matetski '15
  - global well-posedness: Mourrat-Weber '16
- (renormalized) defocusing cubic NLS/NLW on $\mathbb{T}^3$?

Completely open