

Lec 7 01/02/16

①

Sec 4 More on estimates

(4.1) Dispersion estimate for Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = f \end{cases}$$

$$\Rightarrow S(t)f(x) = e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} f(y) dy$$

$$\Rightarrow \|S(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L_x^1}$$

• Goal: Prove dispersive estimate WITHOUT the explicit formula.

• 1-d case:

$$S(t)f = K_t * f$$

$$\begin{aligned} K_t &= \mathcal{F}^{-1}(e^{-it|\xi|^2}) = \int_{\mathbb{R}} e^{-it|\xi|^2 + ix \cdot \xi} d\xi \\ &= \frac{1}{(4\pi i t)^{1/2}} e^{-\frac{|x|^2}{4it}}. \end{aligned}$$

$$\text{Claim: } \|K_t(x)\|_{L_x^\infty} \lesssim \frac{1}{|t|^{1/2}} \quad (t \neq 0) \quad (2)$$

Then, by Young's ineq,

$$\begin{aligned} \|S(t)f\|_{L_x^\infty} &= \|K_t * f\|_{L_x^\infty} & \frac{1}{\infty} + 1 = \frac{1}{\infty} + 1 \\ &\leq \|K_t\|_{L_x^\infty} \|f\|_{L_x^1} \\ &\lesssim \frac{1}{|t|^{1/2}} \|f\|_{L_x^1} \end{aligned}$$

Assume $t > 0$.

$$\begin{aligned} K_t(x) &= \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-i\zeta^2 + i\frac{x}{\sqrt{t}}\zeta} d\zeta & \zeta = t^{1/2}\xi \\ &= \frac{1}{\sqrt{t}} K_1\left(\frac{x}{\sqrt{t}}\right) \end{aligned}$$

$$\Rightarrow \|K_t\|_{L_x^\infty} = \frac{1}{\sqrt{t}} \|K_1\|_{L_x^\infty} \quad (\because t \text{ is fixed.})$$

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Tool: Method of stationary phase.

$$K_1(x) = \int_{\mathbb{R}} e^{-i\bar{s}^2 + ix\bar{s}} d\bar{s}$$

$$= \int_{\mathbb{R}} e^{-i\Phi(\bar{s})} d\bar{s} \quad (x \text{ is fixed})$$

where $\Phi(\bar{s}) = \bar{s}^2 - x\bar{s}$

Idea: Integration by parts.

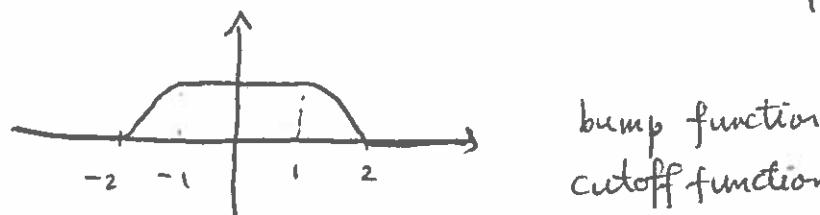
$$e^{-i\Phi(\bar{s})} = \frac{\partial_{\bar{s}} e^{-i\Phi(\bar{s})}}{-i\Phi'(\bar{s})} \quad \Phi'(\bar{s}) = 2\bar{s} - x.$$

← if $\Phi(\bar{s})$ is not small, good.

$$\text{"} \int e^{-i\Phi(\bar{s})} \cdot 1 d\bar{s} = \int \partial_{\bar{s}} e^{-i\Phi(\bar{s})} \cdot \frac{1}{-i\Phi'(\bar{s})} d\bar{s}$$

$$\stackrel{\text{IBP}}{=} \int e^{-i\Phi(\bar{s})} \left(\frac{1}{i\Phi'(\bar{s})} \right)' d\bar{s} \text{."}$$

Let $\Psi \in C^\infty(\mathbb{R}; [0, 1])$ s.t. $\Psi(\bar{s}) = \begin{cases} 1, & |\bar{s}| \leq 1 \\ 0, & |\bar{s}| \geq 2 \end{cases}$

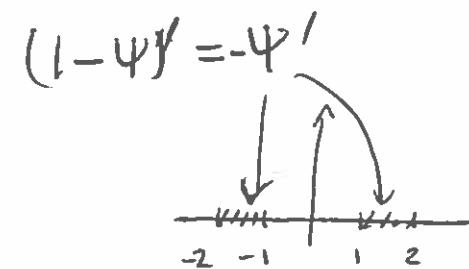


$$K_1(x) = \int_{\mathbb{R}} e^{-i\Phi(\bar{z})} \underbrace{\psi(2\bar{z}-x)}_{=\psi(\bar{z})} d\bar{z}$$

$$+ \int_{\mathbb{R}} e^{-i\Phi(\bar{z})} (1-\psi)(2\bar{z}-x) d\bar{z} =: I + II.$$

$$\cdot |I(x)| \leq \int_{-2 \leq \bar{z} \leq 2} |\psi(2\bar{z}-x)| d\bar{z} \lesssim 1$$

$$\begin{aligned} \cdot II(x) &\stackrel{IBP}{=} \int e^{-i\Phi(\bar{z})} \partial_{\bar{z}} \left(\frac{(1-\psi)(2\bar{z}-x)}{\Gamma \Phi'(\bar{z})} \right) d\bar{z} & \Phi(\bar{z}) = 2\bar{z} - x \\ &= \int e^{-i\Phi(\bar{z})} \left\{ -\frac{(1-\psi)'(2\bar{z}-x)}{i(2\bar{z}-x)^2} \Big|_2 \right\} d\bar{z} \\ &+ \int_{|2\bar{z}-x| \geq 1} e^{-i\Phi(\bar{z})} \frac{(1-\psi)'(2\bar{z}-x)}{\Gamma \Phi'(\bar{z})} d\bar{z} \end{aligned}$$



$$\begin{aligned} \Rightarrow |II(x)| &\leq C \int_{|2\bar{z}-x| \geq 1} \frac{1}{(2\bar{z}-x)^2} d\bar{z} \\ &+ \int_{2\bar{z}-x \in \text{supp } \psi'} |(1-\psi)'(2\bar{z}-x)| d\bar{z} \lesssim 1 \end{aligned}$$

□

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(4.2) Glimpse on oscillatory integrals.

$$I(\lambda) = \int_a^b e^{i\lambda\Phi(x)} \psi(x) dx$$

phase $\Phi(x)$, real-valued

$\psi(x)$, complex-valued (with cpt support)

Lemma: $\text{supp } \psi \subset [a, b]$

\subset cpt subset
 $\Phi'(x) \neq 0$ for all $x \in [a, b]$

Then, $I(\lambda) = O(\lambda^{-N})$ as $\lambda \rightarrow \infty$ (for any $N \in \mathbb{N}$)

Pf: let $Df(x) = \frac{1}{i\lambda\Phi'(x)} \frac{df}{dx}$ $\left(f = O(g^\circ) \Leftrightarrow \lim \frac{|f|}{g} \leq c \right)$

$$\Rightarrow \text{Note: } D(e^{i\lambda\Phi}) = e^{i\lambda\Phi}.$$

Transpose $D^T f(x) = -\frac{d}{dx} \left(\frac{f}{i\lambda\Phi'(x)} \right)$

$$\langle Df, g \rangle_{L^2} = \langle f, D^T g \rangle_{L^2}$$

$$\Rightarrow I(\lambda) = \int_a^b e^{i\lambda\Phi} \psi dx \stackrel{\text{IBP}}{=} \int_a^b e^{i\lambda\Phi} (D^T)^N (\psi) dx$$

$\overset{N \text{ times}}{\underset{D(e^{i\lambda\Phi})}{\parallel}}$

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$$\Rightarrow |I(\lambda)| \lesssim_{N,\psi,\phi} \lambda^{-N} \quad \square$$

Rmk: • Under change of var: $x \mapsto \phi(x)$,
 Lemma $\Leftrightarrow \mathcal{F}$ (cpt supported func) has rapid decay.

$$I(\lambda) = \int e^{i\lambda y} \underbrace{\psi \circ \phi^{-1}(y)}_{\text{cptly supp.}} J \cdot dy \quad y = \phi(x)$$

- If we do not assume that ψ vanishes at the endpoints, then the decay is much worse.

Prop (van der Corput) ϕ , real valued, smooth on (a, b)

Suppose that $\exists k$ s.t. $|\phi^{(k)}(x)| \geq 1$ for all (a, b)

$$\text{Then, } \left| \int_a^b e^{i\lambda \phi(x)} dx \right| \leq C_k \lambda^{-1/k}$$

provided. (i) $k \geq 2$
 or (ii) $\phi'(x)$ is monotonic when $k = 1$.

$$\text{Pf: } \underline{\text{(ii) }} \quad \int_a^b e^{i\lambda\phi} dx = \int_a^b D(e^{i\lambda\phi}) \cdot 1 dx$$

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$$= \underbrace{\int_a^b e^{i\lambda\phi} D^T(1) dx}_{+ \underbrace{\frac{e^{i\lambda\phi}}{i\lambda\phi'} \Big|_a^b}_{1 \cdot 1 \leq \frac{2}{\lambda}}$$

$$\cdot \left| \int_a^b e^{i\lambda\phi} D^T(1) dx \right|$$

$$= \frac{1}{\lambda} \left| \int_a^b e^{i\lambda\phi} \frac{d}{dx}\left(\frac{1}{\phi'}\right) dx \right| = \frac{1}{\lambda} \int_a^b \left| \frac{d}{dx}\left(\frac{1}{\phi'}\right) \right| dx$$

$$\stackrel{\phi' \text{ is monotonic}}{=} \frac{1}{\lambda} \left| \int_a^b \frac{d}{dx}\left(\frac{1}{\phi'}\right) dx \right| \stackrel{\text{FTC}}{=} \frac{1}{\lambda} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \approx \frac{1}{\lambda}$$

$\rightarrow \frac{1}{\phi'} \text{ is monotonic}$

(i) We proceed by induction on k .

Suppose that the case for k is known.

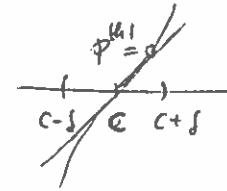
WLOG, assume $\phi^{(k+1)}(x) \geq 1$, $\forall x \in (a, b)$

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Let $x = c$ be the (unique) point in $[a, b]$

s.t. $|\phi^{(k)}(x)|$ attains its min.

If $\phi^{(k)}(c) = 0$, then $|\phi^{(k)}(x)| \geq \delta$ on $(c-\delta, c+\delta)^c$

Write $\int_a^b = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b$  $|\frac{\phi^{(k)}}{\delta}| \geq 1$.

By inductive hypothesis,

$$\left| \int_a^{c-\delta} + \int_{c+\delta}^b e^{i\lambda \phi} dx \right| \leq C_k (\lambda \delta)^{-1/k}$$

$$i\lambda \phi = i(\lambda \delta) \left[\frac{\phi}{\delta} \right]$$

On the other hand,

$$\left| \int_{c-\delta}^{c+\delta} \dots \right| \leq 2\delta. \quad \text{equate them.}$$

$$\Rightarrow \delta \sim (\lambda \delta)^{-1/k} \Leftrightarrow \lambda^{-1} \sim \delta^{k+1} \Leftrightarrow \delta \sim \lambda^{-1/k+1}.$$

If $\phi^{(k)}(c) \neq 0$, then $c = a$. Write $\int_a^b = \int_a^{a+\delta} + \int_{a+\delta}^b$

proceed as before.

□

(When $k+1=2$, $\phi^{(k+1)} \geq 1 \Rightarrow \phi'$ is monotonic.)

(9)

Cor: same assumption as the previous prop.

$$\left| \int_a^b e^{i\lambda \Phi(x)} \Psi(x) dx \right| \leq C_k \lambda^{-1/k} \left[|\Psi(b)| + \int_a^b |\Psi'(x)| dx \right].$$

Pf: let $F(x) = \int_a^x e^{i\lambda \Phi(y)} dy$

. \Rightarrow By Prop. $|F(x)| \leq C_k \lambda^{-1/k}$.

$$\int_a^b e^{i\lambda \Phi} \Psi dx = \underset{\substack{F'' \\ \text{IBP}}}{\underline{F(b)\Psi(b) - F(a)\Psi(a)}} - \int_a^b \underline{F(x)\Psi'(x)} dx.$$

□