

Lec 6 27/01/2016 (Wed)

①

Recall ① Schrödinger adm.  $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ ,  $2 \leq q, r \leq \infty$   
 $(q, r, d) \neq (2, \infty, 2)$ .

$$\|S(t)f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|f\|_{L_x^2(\mathbb{R}^d)}$$

$$\left\| \int_0^t S(t-t')F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\hat{q}'} L_x^{\hat{r}'}}$$

② Local well-posedness of NLS  $\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1}u \\ u|_{t=0} = u_0 \end{cases}$

Thm: Cubic NLS (with  $p=3$ ) on  $\mathbb{R}$  ( $d=1$ ) is locally well-posed in  $H^s(\mathbb{R})$ ,  $s \geq 0$ .

•  $s=0$  case has been proven.

• Now we check the case  $s=1$  ( $d=1$ ).  $H^1(\mathbb{R})$ .

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•  $\|f\|_{H^1} = \|f\|_{L^2} + \|\partial_x f\|_{L^2}$ .      $\|f\|_{W^{1,4}} = \|f\|_{L_x^4} + \|\partial_x f\|_{L_x^4}$ .

•  $(q, r) = (8, 4)$  is Schrödinger adm

•  $X_T = C_T H^1 \cap L_T^8 W^{1,4}$

$= C_T L^2 \cap \underline{C_T \partial^{\pm 1} L^2} \cap L_T^8 L_x^4 \cap \underline{L_T^8 \partial_x^{\pm 1} L_x^4}$ .

$\|u\|_{X_T} = \|u\|_{C_T L^2} + \|u\|_{L_T^8 L_x^4} + \|\partial_x u\|_{C_T L^2} + \|\partial_x u\|_{L_T^8 L_x^4}$

•  $u(t) = \Gamma_{u_0}(u)(t) := S(t) u_0 - i \int_0^t S(t-t') |u|^2 u(t') dt'$ .

Proof of case  $s=1$  :  $\|\Gamma_{u_0}(u)\|_{C_T L^2 \cap L_T^8 L_x^4} \leq C_1 \|u_0\|_{L_x^2} + C_3 T^{1/2} \|u\|_{L_T^8 L_x^4}^3 \leq \|u_0\|_{H^1}$

$\partial_x (S(t) u_0) = S(t) (\partial_x u_0)$

$$\|\partial_x \Gamma_{u_0}(u)\|_{C_T L^2 \cap L_T^8 L_x^4} \leq C_1 \|\partial_x u_0\|_{L_x^2} + C_2 \|\partial_x (|u|^2 u)\|_{L_T^{8/7} L_x^{4/3}} \quad (3)$$

$$\leq C_1 \|\partial_x u_0\|_{L_x^2} + C_3 T^{1/2} \|\partial_x (|u|^2 u)\|_{L_T^{8/3} L_x^{4/3}}$$

Leibniz Rule (Product Rule).

$$\partial_x (fg) = \partial_x f \cdot g + f \cdot \partial_x g.$$

$$\leq C_1 \|\partial_x u_0\|_{L_x^2} + 3 C_3 T^{1/2} \underbrace{\|\partial_x u\|_{L_T^8 L_x^4}}_{\leq \|u\|_{H^1}} \underbrace{\|u\|_{L_T^8 L_x^4}^2}$$

$$\Rightarrow \|\Gamma_{u_0}(u)\|_{X_T} \leq 2 C_1 \|u_0\|_{H^1} + 4 C_3 T^{1/2} \|u\|_{X_T}^3$$

Similarly, we have.

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} \leq C_4 T^{1/2} (\|u\|_{X_T}^2 + \|v\|_{X_T}^2) \|u - v\|_{X_T}.$$

$\Rightarrow$  Conclude as before ( $s=0$  case).

Remark 1: The above argument works for  $s \in \mathbb{N}$ .

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Remark 2: For  $s > 0$ , but not integer, we need fractional Leibniz Rule.

Lemma:  $s \in (0, 1]$  and  $1 < r, p_1, p_2, q_1, q_2 < \infty$ , s.t.

$$\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i} \quad \text{for } i=1, 2, \quad \text{Then}$$

$$\| |\nabla|^s (fg) \|_{L^r} \lesssim \| f \|_{L^{p_1}} \| |\nabla|^s g \|_{L^{q_1}} + \| |\nabla|^s f \|_{L^{p_2}} \| g \|_{L^{q_2}}$$

$$\widehat{|\nabla|^s f} = |\xi|^s \widehat{f}(\xi). \quad \rightarrow \quad |\nabla|^s = (\sqrt{-\Delta})^s$$

$$\widehat{\Delta f} = -|\xi|^2 \widehat{f}(\xi).$$

$$\| f \|_{H^s} = \| f \|_{L^2} + \| |\nabla|^s f \|_{L^2}$$

Proceed as before.

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$$\| |\nabla|^s (\underline{|u|^2} \underline{u}) \|_{L_T^{8/3} L_X^{4/3}} \lesssim \| |u|^2 \|_{L_T^4 L_X^2} \| |\nabla|^s u \|_{L_T^8 L_X^4}$$

$$+ \frac{\| |\nabla|^s (|u|^2) \|_{L_T^4 L_X^2} \| u \|_{L_T^8 L_X^4}}$$

(We use FLR in  $x$ -variable, then Hölder in  $t$ .)

$$\lesssim \| u \|_{L_T^8 L_X^4}^2 \| |\nabla|^s u \|_{L_T^8 L_X^4} + 2 \| |\nabla|^s u \|_{L_T^8 L_X^4} \| u \|_{L_T^8 L_X^4}^2$$

$$\lesssim 3 \| |\nabla|^s u \|_{L_T^8 L_X^4} \| u \|_{L_T^8 L_X^4}^2.$$

Then other argument is identical to  $s=1$  case. #.

Now we turn to  $d=2$ ,  $p=3$ .

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u|_{t=0} = u_0 \end{cases}$$

$$\bullet \text{Scrit} = \frac{d}{2} - \frac{2}{p-1} = \frac{2}{2} - \frac{2}{3-1} = 0$$

$L_x^2$  - critical, mass critical.

Thm: Cubic NLS on  $\mathbb{R}^2$  is locally well-posed in  $L^2(\mathbb{R}^2)$ , and  
Global well-posed in  $L^2(\mathbb{R}^2)$  with small initial data.  
( $\|u_0\|_{L^2}$  sufficiently small).

Recall:  $(q, r) = (4, 4)$  is Schrödinger adm in 2-d.

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$

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$$u(t) = \Gamma_{u_0}(u)(t) := S(t)u_0 + i \int_0^t S(t-t') |u|^2 u(t') dt'$$

① Argue as usual. Let  $X_T = C_T L^2 \cap L_T^4 L_x^4$

$$\|\Gamma_{u_0}(u)\|_{X_T} \leq \|S(t)u_0\|_{X_T} + C_2 \| |u|^2 u \|_{L_T^{4/3} L_x^{4/3}}$$

$$\leq C_1 \|u_0\|_{L_x^2} + C_2 \|u\|_{L_T^4 L_x^4}^3$$



there is no  $T$  here to help

(Need smallness to conclude).

$$\begin{aligned} \|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} &\leq C_2 \| |u|^2 u - |v|^2 v \|_{L_T^{4/3} L_x^{4/3}} \\ &\leq C_3 \left( \|u\|_{L_T^4 L_x^4}^2 + \|v\|_{L_T^4 L_x^4}^2 \right) \|u-v\|_{L_T^4 L_x^4} \end{aligned}$$

Smallness is needed for fix point argument.

② Now define  $A_\eta = \{u \in L_T^4 L_x^4 : \|u\|_{L_T^4 L_x^4} \leq 2\eta\}$ .

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With  $\eta$  to be determined later. Let  $u_0 \in L_x^2(\mathbb{R}^2)$ .

$$\cdot \forall u \in A_\eta \quad \|\Gamma_{u_0}(u)\|_{L_T^4 L_x^4} \leq \|S(t)u_0\|_{L_T^4 L_x^4} + C_3 \| |u|^2 u \|_{L_T^{4/3} L_x^{4/3}}$$

$$\leq \|S(t)u_0\|_{L_T^4 L_x^4} + C_3 \|u\|_{L_T^4 L_x^4}^3$$

$$\leq \|S(t)u_0\|_{L_T^4 L_x^4} + C_3 (2\eta)^3.$$

• In order to put  $\Gamma_{u_0}(u)$  in  $A_\eta$ , we need some smallness condition.

a) chose  $C_3 \cdot (2\eta)^2 < \frac{1}{4}$ .

b)  $\|S(t)u_0\|_{L_T^4 L_x^4(\mathbb{R} \times \mathbb{R}^2)} \lesssim \|u_0\|_{L_x^2} < \infty$ .

MCT  $\lim_{T \rightarrow 0} \|S(t)u_0\|_{L_T^4 L_x^4} = 0$

$\Rightarrow$  We can choose  $T$  sufficiently small, s.t.  
 $\|S(t)u_0\|_{L_T^4 L_x^4} < \eta$

$$\begin{aligned} \Rightarrow \|\Gamma_{u_0}(u)\|_{L_T^4 L_x^4} &\leq \frac{\|S(t)u_0\|_{L_T^4 L_x^4} + C_3 (2\eta)^2 \cdot (2\eta)}{\downarrow (b).} \longrightarrow \text{by (a)} \\ &< \eta + \frac{1}{4}\eta \\ &< 2\eta \quad \implies \Gamma_{u_0}(u) \in A_\eta. \end{aligned}$$

•  $\forall u, v \in A_\eta$ .

$$\begin{aligned} \|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{L_T^4 L_x^4} &\leq C_2 \| |u|^2 u - |v|^2 v \|_{L_T^{4/3} L_x^{4/3}} \\ &\leq C_3 \left( \underbrace{\|u\|_{L_T^4 L_x^4}^2}_{\leq (2\eta)^2} + \underbrace{\|v\|_{L_T^4 L_x^4}^2}_{\leq (2\eta)^2} \right) \|u - v\|_{L_T^4 L_x^4} \\ &\leq C_3 \cdot 2 \cdot (2\eta)^2 \|u - v\|_{L_T^4 L_x^4} \\ &\leq \frac{1}{2} \|u - v\|_{L_T^4 L_x^4} \end{aligned}$$

$\implies$  Fix point argument in  $L_T^4 L_x^4$ .

Remark: We need to make sure

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$$\|S(t)u_0\|_{L_T^4 L_x^4} \leq \eta.$$

① Choose  $T$  small.

② Since  $\|S(t)u_0\|_{L_T^4 L_x^4(\mathbb{R} \times \mathbb{R}^2)} \leq C_0 \|u_0\|_{L_x^2}$ . We can choose  $u_0$

$$\text{s.t. } \|u_0\|_{L_x^2} \leq \frac{1}{C_0} \eta \Rightarrow \|S(t)u_0\|_{L_T^4 L_x^4(\mathbb{R} \times \mathbb{R}^2)} \leq \eta.$$

Which means that  $T = +\infty$ , global solution!!

③ Now  $u \in L_T^4 L_x^4$ . now we show  $u \in G_T L^2$ .

$$\|u\|_{G_T L^2} \leq C_1 \|u_0\|_{L^2} + C_2 \| |u|^2 u \|_{L_T^{4/3} L_x^{4/3}}$$

$$\leq C_1 \|u_0\|_{L^2} + C_2 (2\eta)^2 < \infty$$