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Lec 3 18/01/16 (Mon)

$$(NLS) \quad i\partial_t u + \Delta u = \pm |u|^{p-1}u$$

\Rightarrow Duhamel formula:

$$u(t) = \Gamma_{u_0}(u) := S(t)u_0 \mp i \int_0^t S(t-t')(|u|^{p-1}u)(t')dt'$$

where $S(t) = e^{it\Delta}$ ($S(t)f = \mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{f}(\xi))$)

LWP $\Leftrightarrow \Gamma_{u_0}$ has a fixed pt ($t \in [-T, T]$)

① $S(t)$ is unitary on $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$

$$\|S(t)f\|_{H^s} = \left(\int (1+|\xi|^2)^s |e^{-it|\xi|^2} \hat{f}(\xi)|^2 d\xi \right)^{1/2} = \|f\|_{H^s}$$

$S(t)f \in C(\mathbb{R}_+; H^s(\mathbb{R}^d_x)) \quad f \in H^s$

$$S(\cdot): f \in H^s(\mathbb{R}^d) \mapsto S(\cdot)f \in C_t H^s = C(\mathbb{R}_+; H^s(\mathbb{R}^d))$$

\uparrow
 H^s -valued function (in t)

Fix $t \in \mathbb{R}$

$$S(t_1 + t_2) = S(t_1)S(t_2)$$

$$\| S(t+h)f - S(t)f \|_{H^s}$$

$$= \| \cancel{S(t)} (S(h) - 1)f \|_{H^s}$$

= Write down on the Fourier side.

separate $|\beta| \leq N \leftarrow$ mean value theorem

$$> N \leftarrow \| P_{\{|\beta| \geq N\}} f \|_{H^s} < \frac{\varepsilon}{4}$$

Let $s > \frac{d}{2}$. Fix $u_0 \in H^s(\mathbb{R}^d)$.

$$\| \nabla(u) \|_{C_T H^s} \leq \| S(t) u_0 \|_{C_T H^s} + \left\| \int_0^t S(t-t') |u|^{p-1} u(t') dt' \right\|_{C_T H^s}$$

$$\left(C_T H^s = C([-T, T]; H^s(\mathbb{R}^d)) : \|u\|_{C_T H^s} = \| \|u(t)\|_{H^s} \|_{L_T^\infty} \right)$$

$$\leq \| u_0 \|_{H^s} + \int_0^T \| |u|^{p-1} u \|_{C_T H^s} dt$$

↑ unitarity of $S(t)$ & Minkow Int. Ineq.

$$\leq \| u_0 \|_{H^s} + T \| u \|_{C_T H^s}^p \leq 2 \| u_0 \|_{H^s} =: R$$

$\underbrace{\quad}_{\leq \| u_0 \|_{H^s}}$

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For $u \in \overline{B_R} \subset CTH^s$

$$T \|u\|_{CTH^s}^p \leq T R^p \stackrel{\text{WANT}}{\leq} \frac{R}{2}$$

$$\Rightarrow T = \frac{1}{2R^{p-1}} \sim \|u_0\|_{H^s}^{-(p-1)}$$

$$\Rightarrow \Gamma : \overline{B_R} \hookrightarrow$$

$$\| \Gamma_u(u) - \Gamma_v(v) \|_{CTH^s} \leq \int_0^T \| |u|^{p-1} u - |v|^{p-1} v \|_{CTH^s} dt$$

$u, v \in \overline{B_R}$

$$\leq CT \left(\|u\|_{CTH^s}^{p-1} + \|v\|_{CTH^s}^{p-1} \right) \|u - v\|_{CTH^s}$$

\uparrow Choose < 1

telescoping sum if $p \in \mathbb{N}$. ($\text{if } p \notin \mathbb{N}$, use MVT.)

$$\text{ex: } |u|^2 u - |v|^2 v = u \bar{u} u - v \bar{v} v$$

$$= (u - v) \bar{u} u + v(\bar{u} - \bar{v}) u + v \bar{v} (u - v)$$

- choose $T \ll 1$ s.t.

$$CT \left(\|u\|_{C_T H^s}^{p-1} + \|v\|_{C_T H^s}^{p-1} \right) < 1$$

$$\Leftarrow T \sim R^{-(p-1)}$$

↑
some small const.

\Rightarrow Banach fixed pt thm (contraction mapping principle)

A contraction on a closed ball in a complete metric space X has a unique fixed pt.

$\Rightarrow \exists ! u \in B_R$ s.t. $u = \Gamma_{u_0}(u)$.

Namely, soln to (NLS).

Rmk: ① $u \in C_T H^s$ (prove it.)

$$u(t) = S(t)u_0 + i \int_0^t S(t-t') |u|^{p-1} u(t') dt'$$

$\underbrace{\qquad\qquad\qquad}_{=: F(t)}$

$$\begin{aligned}
 F(t+h) - F(t) &= \int_0^{t+h} s(t+h-t') \dots - \int_0^t s(t-t') \dots \\
 &= \int_t^{t+h} s(t+h-t') \dots - \int_0^t (s(t+h-t') - s(t-t')) \dots
 \end{aligned}
 \tag{5}$$

② Uniqueness only in B_R .

- Gronwall's inequality.

$$u(t) \leq \alpha(t) + \int_0^t \beta(t') u(t') dt' \quad \alpha, \beta \geq 0$$

$$\Rightarrow u(t) \leq \underline{\alpha}(t) e^{\int_0^t \beta(t') dt'}$$

if $\alpha \equiv 0$ then $u \equiv 0$.

- two solns u, v with $u|_{t=0} = v|_{t=0} = u_0$

Note: $\|u_0\|_{H^s} = \frac{1}{2} R$. issue: u, v may be large

Since $u, v \in C_T H^s$, $T = T(R)$

$$\exists T_0 > 0 \text{ s.t. } \|u\|_{C_{T_0} H^s}, \|v\|_{C_{T_0} H^s} \leq \frac{3}{4} R$$

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③ same proof applies to

non-algebraic nonlinearity. $|u|^p$, $p \in \mathbb{N}, p \geq 2$.

$|u|^{p-1} u$, $p \in \mathbb{N}$.

need to use chain rule

Sec 3: Scaling, Strichartz estimates, and LWP part II

Scaling: If u is a soln to (NLS) with $u|_{t=0} = u_0$,

$$i\partial_t u + \Delta u = \pm |u|^{p-1} u,$$

then define $u^\lambda(t, x) = \frac{1}{\lambda^a} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$

$$u_0^\lambda(x) = \frac{1}{\lambda^a} u_0(\frac{x}{\lambda})$$

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$$\Rightarrow a+2 = ap \Rightarrow a = \frac{2}{p-1}$$

$$\Rightarrow u^\lambda(t, x) = \frac{1}{\lambda^{2/p-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

is also a soln to (NLS) with $u^\lambda|_{t=0} = u_0^\lambda$

\Leftarrow scaling symmetry

ex of symmetries: time translation, spatial translation,
 $u \mapsto e^{i\theta} u$, Galilean symmetry
time reversal
 $u(t, x) \mapsto e^{\square} u(t, x+tv)$

$$u(t, x) \mapsto \overline{u(-t, x)}$$

Scaling critical Sobolev index: $s_c = \text{crit.}$

$$\|f^\lambda\|_{H^{s_c}(\mathbb{R}^d)} = \|f\|_{H^{s_c}(\mathbb{R}^d)}$$

$$\|f^\lambda\|_{\dot{H}^s} = \left(\int |\tilde{z}|^{2s} |\hat{f}^\lambda(\tilde{z})|^2 d\tilde{z} \right)^{1/2} \quad (8)$$

$$\begin{aligned} f^\lambda(x) &= \frac{1}{\lambda^{d/p-1}} f\left(\frac{x}{\lambda}\right) \Rightarrow \hat{f}^\lambda(\tilde{z}) = \lambda^{d-\frac{2}{p-1}} \hat{f}(\lambda \tilde{z}) \\ &= \left(\int |\lambda \tilde{z}|^{2s} |\hat{f}(\lambda \tilde{z})|^2 d(\lambda \tilde{z}) \right)^{1/2} \cdot \lambda^{d-\frac{2}{p-1}-s-\frac{d}{2}} \\ &= \lambda^{\frac{d}{2}-\frac{2}{p-1}-s} \|f\|_{\dot{H}^s(\mathbb{R}^d)}, \quad \forall \lambda > 0. \end{aligned}$$

$$\Rightarrow \boxed{s_c = \frac{d}{2} - \frac{2}{p-1}} \quad (< \frac{d}{2})$$

$u_0 \in H^s(\mathbb{R}^d)$. The Cauchy problem (NLS) is

- subcritical (w.r.t. scaling) if $s > s_c$.

→ expect good behavior. LWP, etc.

- critical if $s = s_c$ ⇒ delicate balance between linear dispersion & nonlinear concentration

- supercritical if $s < s_c$ ⇒ expect ill-posedness.

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• subcritical case : $s > s_c$

$$u^\lambda(t, x) = \frac{1}{\lambda^{2/p-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

$$\|u_0^\lambda\|_{\dot{H}^s} = \lambda^{\frac{s_c-s}{2}} \|u_0\|_{\dot{H}^s}$$

$\lambda < 0$

u on $[0, T]$ \longleftrightarrow u^λ on $[0, \lambda^2 T]$ Think of $\lambda \gg 1$

$$u_0 \Rightarrow u_0^\lambda \text{ (in } \dot{H}^s\text{)}$$

• supercritical : $s < s_c$

u on $[0, T]$ \longleftrightarrow u^λ on $[0, \lambda^2 T]$, $\lambda \gg 1$

$$u_0 \ll u_0^\lambda \text{ (in } \dot{H}^s\text{)}$$

\Rightarrow "larger initial data, longer time of existence"

\Rightarrow Too good to be true.

• critical $s = s_c$: need more info than the \dot{H}^s -norm of u .