13/01/16 (Wed) Lec. 2

- Folland, Real Analysis (Chap 6, 8, 9)
- Grafakos, Classical Fourier Analysis (Chap 1, Chap 2, Chap 5)
  ~~Modern~~ = Modern = (Chap 6)
- Tao, Nonlinear Dispersive PDEs (Appendix)

Cazenave, Semilin. Schrödinger equations

Linares-Ponce, Intro to disp. PDEs

Fourier transform on $\mathbb{R}^d$:

$f \in \mathcal{S}(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)$ and "fast decay"

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \bigcirc \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\xi}\, dx$$

$$\check{f}(x) = \hat{f}(-x) = \square \int f(\xi)\, e^{+2\pi i x\cdot\xi}\, d\xi$$

$\bigcirc = 1$, $\square = \frac{1}{2\pi}$, $\bigcirc = \frac{1}{\sqrt{2\pi}}$, $\square = \frac{1}{\sqrt{2\pi}}$

Basic properties: • $\|f\|_{L^2} = \|\hat{f}\|_{L^2} = \|\check{f}\|_{L^2}$ (Plancherel's identity)

• $f: L^2 \to L^2$ ($f: \mathcal{S} \to \mathcal{S}$ bijection)

• $(\hat{f})^{\vee} = (\check{f})^{\wedge} = f$

• Parseval $\int f(x)\overline{g}(x)\,dx = \int \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi$

$$\langle f, g\rangle_{L^2_x} = \langle \hat{f}, \hat{g}\rangle_{L^2_\xi}$$

• Hausdorff-Young's ineq: $\|\hat{f}\|_{L^p} \le \|f\|_{L^{p'}}$, $p \ge 2$

• $(\partial^\alpha f)^{\wedge}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi) \to (\partial^\alpha \hat{f})(x) = (-2\pi i x)^\alpha f(x)$

$\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}^d_{\ge 0}$

$\partial^\alpha f = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \cdots \partial^{\alpha_d}_{x_d} f$

$\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_d^{\alpha_d}$

• $f * g(x) = \int f(x-y)\,g(y)\,dy = \int f(y)\,g(x-y)\,dy$

$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \Rightarrow \hat{f} * \hat{g} = \widehat{fg}$

- $f_\varepsilon(x) = \frac{1}{\varepsilon^d} f\left(\frac{x}{\varepsilon}\right)$ | $g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right)$

  $\Rightarrow \widehat{f_\varepsilon}(\xi) = \hat{f}(\varepsilon\xi)$ | $\widehat{g^\varepsilon}(\xi) = \varepsilon^d \hat{g}(\varepsilon\xi)$

- Riemann-Lebesgue Lemma

  $f \in L^1(\mathbb{R}^d) \Rightarrow \hat{f}(\xi) \to 0$ as $|\xi| \to \infty$

  $\mathcal{F} L^1 \subset C_0(\mathbb{R}^d)$

  $\uparrow$ decaying to 0 at ∞.

  C_c^∞ = cpt supp

- $(\mathcal{S}(\mathbb{R}^d))^* = \mathcal{S}'(\mathbb{R}^d)$ = tempered distribution.

  $T \in \mathcal{S}'(\mathbb{R}^d)$: $\mathcal{S} \to \mathbb{C}$, conti.

  $T_k \to T$ in $\mathcal{S}' \iff T_k(f) \to f \quad \forall f \in \mathcal{S}$

  weak * top

  $\Rightarrow$ can define F.T. on $\mathcal{S}'$.

$f \in \mathcal{S}'$, $g \in \mathcal{S}$ $\Rightarrow$ $\hat{f}(g)\ (= \langle \hat{f}, g \rangle)$

$\hat{f} \in \mathcal{S}'$ $\|^{\text{def}}$ $f(\hat{g})$

Sobolev space $H^s(\mathbb{R}^d)$ = completion of $\mathcal{S}$ w.r.t.

$$\|f\|_{H^s} = \left(\int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}, \quad s \in \mathbb{R}$$

$$(1+|\xi|^2)^{1/2} = \langle \xi \rangle \sim 1+|\xi| \qquad = \| \langle \xi \rangle^s \hat{f}(\xi) \|_{L^2_\xi(\mathbb{R}^d)}$$

- $H^0 = L^2$
- $H^1 = L^2 \cap \{ \partial f \in L^2 \}$

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2$$

$$|\xi|^2 |\hat{f}(\xi)|^2 = |\widehat{\nabla f}|^2$$

$$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$$

Sobolev embedding Thm: If $s > \frac{d}{2}$ $(2s > d)$,

$$H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$$

$$\left(\Leftarrow \|f\|_{L^\infty} \leq C \|f\|_{H^s} \right)$$

pf: $\|f\|_{L^\infty_x} \leq \|\hat{f}\|_{L^1_\xi} = \int \langle \xi \rangle^{-s} \langle \xi \rangle^{s} |\hat{f}(\xi)| \, d\xi$

$$\leq \left(\int \frac{1}{\langle \xi \rangle^{2s}} \, d\xi \right)^{1/2} \|f\|_{H^s} \qquad \square$$

Cauchy-Schwarz ineq $\qquad \leq C$

$A \lesssim B \iff A \leq C B$ for some $C > 0$

$A \sim B \iff A \lesssim B$ and $B \lesssim A$

$A \ll B \iff A \leq \varepsilon B$ for some small $\varepsilon > 0$

Other Sobolev spaces: ① L^p_s = Bessel potential space

$$\|f\|_{L^p_s} = \|\mathcal{F}^{-1}(\langle\xi\rangle^s \hat{f}(\xi))\|_{L^p_x}$$

$p=2$: $L^2_s = H^s$ Sobolev emb. $ps > d$

② $W^{s,p}$

$$\|f\|_{W^{s,p}} = \|f\|_{L^p} + \left(\iint \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\right)^{1/p}, \quad ps > d$$

$0 < s < 1$

↖ check

Elias Stein: Singular integral... Λ^p_s

③ H^s = inhomog Sobolev space $\leftarrow \langle\xi\rangle^s = (1+|\xi|^2)^{s/2}$

$\dot{H}^s$ = homog Sobolev space $\leftarrow |\xi|^s$

$$\dot{H}^1 = \{\nabla f \in L^2\}$$

$s \geq 0$, $H^s \subseteq \dot{H}^s$ $\quad \dot{L}^p_s$ = Riesz potential

Rmk: $\dot{H}^s$ up to polynomial. $\quad \||\xi|^s \hat{f}(\xi)\|_{L^2}$

$s \neq 0$

Sobolev inequality: $0 \le \frac{s}{d} = \frac{1}{p} - \frac{1}{q}$ $1 \le p \le q < \infty$

$$\|f\|_{L^q} \le C \|f\|_{\dot{W}^{s,p}} \qquad \dot{W}^{s,p} = \dot{L}^p_s$$

(Gagliardo - Nirenberg inequality)

Interpolation: $s = \theta s_1 + (1-\theta) s_2$ $\quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$

$\theta \in (0,1)$

$$\|f\|_{H^s} \le \|f\|^{\theta}_{H^{s_1}} \|f\|^{1-\theta}_{H^{s_2}}$$

Algebra property of H^s: $s > d/2$ $\quad H^s \subset \mathcal{F}L^1 \subset C_0$

$$f, g \in H^s \Rightarrow fg \in H^s$$

$$\left(\Leftarrow \|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s} \right)$$

Pf: $\langle \xi \rangle^s \le \langle \xi - \xi_1 \rangle^s \langle \xi_1 \rangle^s$ ($\Leftarrow$ triang ineq)

$s \ge 0$

$$\langle \xi \rangle^s \lesssim \langle \xi - \xi_1 \rangle^s + \langle \xi_1 \rangle^s$$

$$\|fg\|_{H^s} = \| \langle \xi \rangle^s \hat{f} * \hat{g} \|_{L^2_\xi}$$

$$= \| \langle \xi \rangle^s \int \hat{f}(\xi - \xi_1) \hat{g}(\xi_1) \, d\xi_1 \|_{L^2_\xi}$$

$$\lesssim \| \int \langle \xi - \xi_1 \rangle^s |\hat{f}(\xi - \xi_1)| \, |\hat{g}(\xi_1)| \, d\xi_1 \|_{L^2_\xi} + \text{--} \quad \langle \xi_1 \rangle^s \hat{g}(\xi_1)|$$

$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$\frac{1}{2} + 1 = \frac{1}{2} + 1$

$$\| \left((\langle \cdot \rangle^s |\hat{f}|) * |\hat{g}| \right)(\xi) \|_{L^2_\xi}$$

$$\le \| \langle \xi \rangle^s \hat{f}(\xi) \|_{L^2_\xi} \| \hat{g}(\xi) \|_{L^1_\xi}$$

$$\lesssim \|f\|_{H^s} \|g\|_{H^s}$$

$s > d/2$

$\square$

Sec 2: LWP in $H^s(\mathbb{R}^d)$, $s > d/2$

$$(NLS)\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1}u \\ u|_{t=0} = u_0 \end{cases} \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d$$

First, consider

$$(S)\begin{cases} i\partial_t u = -\Delta u \\ u|_{t=0} = u_0 \end{cases} \qquad \partial \sim i\xi$$

$\Rightarrow$ Take F.T. in x

$$\begin{cases} i\partial_t \hat{u}(\xi) = |\xi|^2 \hat{u}(\xi) \\ \hat{u}(\xi)|_{t=0} = \hat{u}_0(\xi) \end{cases} \qquad \xi \in \mathbb{R}^d$$

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_d}^2 = -\sum \xi_j^2 = -|\xi|^2$$

$$\downarrow$$

$$(i\xi_1)^2$$

$$\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{u}_0(\xi)$$

$$\Rightarrow \quad u(t,x) = \mathcal{F}^{-1}\left(e^{-it|\xi|^2}\hat{u}_0(\xi)\right)(x)$$

$$\overset{\text{def}}{=} \underline{\underline{e^{it\Delta}}}\, u_0(x) \qquad \partial_t u = \underline{\underline{i\Delta}}\, u$$

We'll use $S(t) = e^{it\Delta}$

nonhomog LS : $\begin{cases} i\partial_t u + \Delta u = F(t,x) \\ u|_{t=0} = u_0 \end{cases}$

$$\overset{\text{F.T in } x}{\Longrightarrow} \quad \left(\cancel{i}\,\partial_t \hat{u}(\xi) + i|\xi|^2\hat{u}(\xi) = \underset{-i}{\wedge}\hat{F}(t,\xi)\right), \quad \forall \xi \in \mathbb{R}^d$$

$$\times\, e^{it|\xi|^2}$$

$$\partial_t\left(e^{it|\xi|^2}\hat{u}(\xi)\right) = -i\, e^{it|\xi|^2}\hat{F}(t,\xi)$$

Integrate from 0 to t

$$e^{it|\xi|^2}\hat{u}(t,\xi) - \hat{u}_0(\xi) = -i\int_0^t e^{it'|\xi|^2}\hat{F}(t',\xi)\,dt'.$$

$$\Rightarrow \quad \hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{u}_0(\xi) - i\int_0^t e^{-i(t-t')|\xi|^2}\hat{F}(t',\xi)\,dt'$$

$$\Rightarrow \quad u(t) = S(t)u_0 - i\int_0^t S(t-t')F(t')\,dt'$$

Duhamel formula: $F = \pm|u|^{p-1}u$ on $[-T,T]$

We say u is a soln to (NLS) if u satisfies

$$u(t) = S(t)u_0 \mp i\int_0^t S(t-t')|u|^{p-1}u(t')\,dt' = \Gamma_{u_0}(u)$$

for $t \in [-T,T]$

Give u_0, Define $\Gamma_{u_0}(u) = \Gamma(u) = \text{RHS}$

Goal: Find u s.t. $u = \Gamma(u)$

Lec 3 18/01/16 (Mon)

$$(NLS) \quad i\partial_t u + \Delta u = \pm |u|^{p-1}u$$

$\Longrightarrow$ Duhamel formula:

$$u(t) = \Gamma_{u_0}(u) := S(t)u_0 \mp i\int_0^t S(t-t')(|u|^{p-1}u)(t')\,dt'$$

where $S(t) = e^{it\Delta}$ $\quad \left(S(t)f = \mathcal{F}^{-1}\left(e^{-it|\xi|^2}\hat{f}(\xi)\right).\right)$

LWP $\Longleftrightarrow$ Γ_{u_0} has a fixed pt $\quad (t \in [-T, T].)$

① $S(t)$ is unitary on $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$.

$$\|S(t)f\|_{H^s} = \left(\int (1+|\xi|^2)^s \left|e^{-it|\xi|^2}\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} = \|f\|_{H^s}$$

- $S(t)f \in C(\mathbb{R}_t; H^s(\mathbb{R}^d_x)) \qquad f \in H^s$

$$S(\cdot): \quad f \in H^s(\mathbb{R}^d) \longmapsto S(\cdot)f \in C_t H^s = C(\mathbb{R}_t : H^s(\mathbb{R}^d))$$

$\uparrow$ H^s-valued function (in t)

$S(t_1+t_2) = S(t_1)S(t_2)$

Fix $t \in \mathbb{R}$

$\| S(t+h)f - S(t)f \|_{H^s}$

$= \| S(t)(S(h)-1)f \|_{H^s}$

= write down on the Fourier side.

separate $|\xi| \le N$ ← mean value thm

$> N$ ← $\| P_{\{|\xi| \ge N\}} f \|_{H^s} < \frac{\varepsilon}{4}$

Let $s > \frac{d}{2}$. Fix $u_0 \in H^s(\mathbb{R}^d)$.

$$\| \Gamma(u) \|_{C_T H^s} \le \| S(t)u_0 \|_{C_T H^s} + \left\| \int_0^t S(t-t') |u|^{p-1} u(t') \, dt' \right\|_{C_T H^s}$$

($C_T H^s = C([-T,T]; H^s(\mathbb{R}^d))$: $\| u \|_{C_T H^s} = \| \| u(t) \|_{H^s} \|_{L^\infty_T}$

$$\le \| u_0 \|_{H^s} + \int_0^T \| |u|^{p-1} u \|_{C_T H^s} \, dt$$

↑ unitarity of $S(t)$ & Minkow int. ineq.

$$\le \| u_0 \|_{H^s} + T \| u \|^p_{C_T H^s} \le 2 \| u_0 \|_{H^s} =: R$$

$\le \| u_0 \|_{H^s}$

For $u \in \overline{B_R} \subset C_T H^s$

$$T \|u\|^p_{C_T H^s} \leq T R^p \overset{\text{WANT}}{\leq} \frac{R}{2}$$

$$\Rightarrow T \leq \frac{1}{2R^{p-1}} \sim \|u_0\|^{-(p-1)}_{H^s}$$

$$\Rightarrow \Gamma : \overline{B_R} \circlearrowleft$$

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{C_T H^s} \leq \int_0^T \| |u|^{p-1}u - |v|^{p-1}v \|_{C_T H^s} \, dt$$

$u, v \in \overline{B_R}$

$$\leq C T \left(\|u\|^{p-1}_{C_T H^s} + \|v\|^{p-1}_{C_T H^s} \right) \|u - v\|_{C_T H^s}$$

$\uparrow$ telescoping sum if $p \in \mathbb{N}$ (If $p \notin \mathbb{N}$, use MVT.)

Choose < 1

ex: $|u|^2 u - |v|^2 v = u\bar{u}u - v\bar{v}v$

$= (u-v)\bar{u}u + v(\bar{u}-\bar{v})u + v\bar{v}(u-v)$

- Choose $T \ll 1$ s.t.

$$C T \left(\|u\|_{C_T H^s}^{p-1} + \|v\|_{C_T H^s}^{p-1} \right) < 1$$

$$\Leftarrow \quad T \sim R^{-(p-1)}$$

↑ some small const.

$\Rightarrow$ <u>Banach fixed pt thm (contraction mapping principle)</u>

A contraction on a closed ball in a complete metric space X has a unique fixed pt.

$\Rightarrow \exists!\ u \in B_R$ s.t. $u = \Gamma_{u_0}(u)$.

Namely, soln to (NLS).

Rmk: (1) $u \in C_T H^s$ (prove it.)

$$u(t) = S(t) u_0 \mp i \underbrace{\int_0^t S(t - t') |u|^{p-1} u(t') \, dt'}_{=: F(t)}.$$

$$F(t+h) - F(t) = \int_0^{t+h} S(t+h-t')\,\cdots - \int_0^{t} S(t-t')\,\cdots \qquad H^s$$

$$= \int_t^{t+h} S(t+h-t')\,\cdots - \int_0^{t} \big(S(t+h-t') - S(t-t') \big)\,\cdots$$

(5)

(2) Uniqueness only in B_R

- Gronwall's inequality.

$$u(t) \le \alpha(t) + \int_0^t \beta(t')\, u(t')\, dt' \qquad \alpha,\ \beta \ge 0$$

$$\Rightarrow \quad u(t) \le \alpha(t)\, e^{\int_0^t \beta(t')\, dt'}$$

if $\alpha \equiv 0$ then $u \equiv 0$.

- two solns u, v with $u|_{t=0} = v|_{t=0} = u_0$

Note: $\|u_0\|_{H^s} = \frac{1}{2} R$.

Since $u, v \in C_T H^s$, (issue: u, v may be large; $T = T(R)$)

$\exists T_0 > 0$ s.t. $\|u\|_{C_{T_0} H^s}$, $\|v\|_{C_{T_0} H^s} \le \frac{3}{4} R$

(3) same proof applies to
non-algebraic nonlinearity $|u|^p$, $p \in \mathbb{N}$. $p \geq 2$.

$\longrightarrow$ $|u|^{p-1}u$, $p \in \mathbb{N}$.

need to use chain rule

Sec 3: Scaling, Strichartz estimates, and LWP part II

Scaling: If u is a soln to (NLS) with $u|_{t=0} = u_0$,

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u,$$

then define $u^\lambda(t,x) = \frac{1}{\lambda^a} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$

$$u_0^\lambda(x) = \frac{1}{\lambda^a} u_0\left(\frac{x}{\lambda}\right)$$

$$\Rightarrow \quad a+2 = ap \Rightarrow a = \frac{2}{p-1}$$

$$\Rightarrow \quad u^\lambda(t,x) = \frac{1}{\lambda^{2/p-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

is also a soln to (NLS) with $u^\lambda|_{t=0} = u_0^\lambda$

$\Leftarrow$ scaling symmetry

ex of symmetries: time translation, spatial translation, $u \mapsto e^{i\theta} u$, Galilean symmetry $u(t,x) \to e^{\square} u(t, x+tv)$

time reversal, $u(t,x) \mapsto \overline{u(-t,x)}$

Scaling critical Sobolev index: $s_c = s_{crit}$.

$$\|f^\lambda\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|f\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$$

$$\|f^\lambda\|_{\dot{H}^s} = \left(\int |\xi|^{2s} |\hat{f^\lambda}(\xi)|^2 d\xi\right)^{1/2}$$

$$\left(f^\lambda(x) = \frac{1}{\lambda^{\frac{2}{p-1}}} f\left(\frac{x}{\lambda}\right) \Rightarrow \hat{f^\lambda}(\xi) = \lambda^{d-\frac{2}{p-1}} \hat{f}(\lambda\xi)\right.$$

$$= \left(\int |\lambda\xi|^{2s} |\hat{f}(\lambda\xi)|^2 d(\lambda^d \xi)\right)^{1/2} \cdot \lambda^{d-\frac{2}{p-1}-s-\frac{d}{2}}$$

$$= \lambda^{\frac{d}{2}-\frac{2}{p-1}-s} \|f\|_{\dot{H}^s(\mathbb{R}^d)}, \quad \forall \lambda > 0.$$

$$\Rightarrow \boxed{s_c = \frac{d}{2} - \frac{2}{p-1}} \left(< \frac{d}{2}\right)$$

$u_0 \in H^s(\mathbb{R}^d)$. The Cauchy problem (NLS) is

- subcritical (w.r.t. scaling) if $s > s_c$.
  $\Rightarrow$ expect good behavior. LWP, etc.
- critical if $s = s_c$ $\Rightarrow$ delicate balance between linear dispersion & nonlinear concentration
- supercritical if $s < s_c$ $\Rightarrow$ expect ill-posedness.

- **subcritical case** : $s > s_c$

$$u^\lambda(t,x) = \frac{1}{\lambda^{2/p-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

$$\|u_0^\lambda\|_{\dot{H}^s} = \lambda^{\underbrace{s_c - s}_{<0}} \|u_0\|_{\dot{H}^s}$$

u on $[0,T]$ $\longleftrightarrow$ u^λ on $[0, \lambda^2 T]$ Think of $\lambda \gg 1$

$u_0 \Rightarrow u_0^\lambda$ (in $\dot{H}^s$)

- **supercritical** : $s < s_c$

u on $[0,T]$ $\longleftrightarrow$ u^λ on $[0, \lambda^2 T]$, $\lambda \gg 1$

$u_0 \ll u_0^\lambda$ (in $\dot{H}^s$)

$\Rightarrow$ " larger initial data, longer time of existence "

$\Rightarrow$ Too good to be true.

- **critical** $s = s_c$: need more info than the $\dot{H}^s$-norm of u_0.

Lec 4. 20/01/16 (Wed)

· Yuzhao Wang yuzhao.wang@ed.ac.uk

① Dispersive estimate:

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = f \end{cases} \Rightarrow u(t) = \mathcal{F}^{-1}\left(e^{-4\pi^2 i t|\xi|^2}\hat{f}(\xi)\right)$$

$$\overset{\text{def.}}{=} S(t) f$$

$$\Rightarrow S(t) f = \int_{\mathbb{R}^d} e^{+2\pi i x\cdot\xi} e^{-4\pi^2 i t|\xi|^2} \hat{f}(\xi)\, d\xi$$

$$= \int_{\mathbb{R}^d} e^{2\pi i \xi\cdot(x-2\pi\xi t)} \hat{f}(\xi)\, d\xi$$

$g(x-ct)$

If $\hat{f}$ is "localized" around $\xi = \xi_0 \in \mathbb{R}^d$,

$\Rightarrow$ $2\pi\xi$ phase velocity.

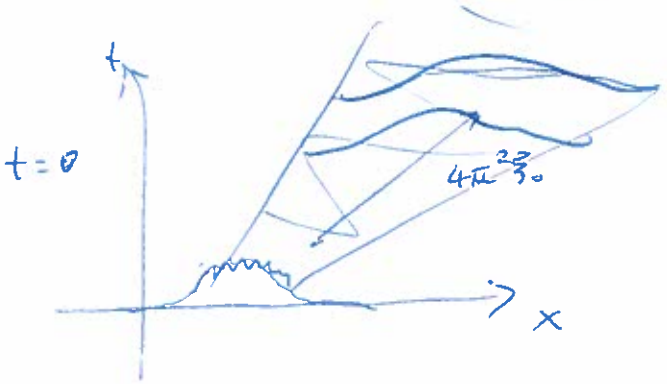
- consider $f = e^{2\pi i x \cdot \xi_0} + e^{2\pi i x \cdot (\xi_0 + \Delta\xi)}$ ($\Delta\xi$ small)

$\Rightarrow S(t) f(x) = e^{2\pi i x \cdot \xi_0 - 4\pi^2 i t |\xi_0|^2}$ ——— plane wave soln

$+ e^{2\pi i x \cdot (\xi_0 + \Delta\xi) - 4\pi^2 i t |\xi_0 + \Delta\xi|^2}$

$= e^{2\pi i x \cdot \xi_0 - 4\pi^2 i t |\xi_0|^2} \; e^{2\pi i \Delta\xi \cdot (x - 4\pi \xi_0 t)} + \underbrace{O(|\Delta\xi|^2)}_{\text{small}}$

group velocity $= 4\pi \xi_0$



· different frequency components travel at different speed

$\Rightarrow$ spread out in space

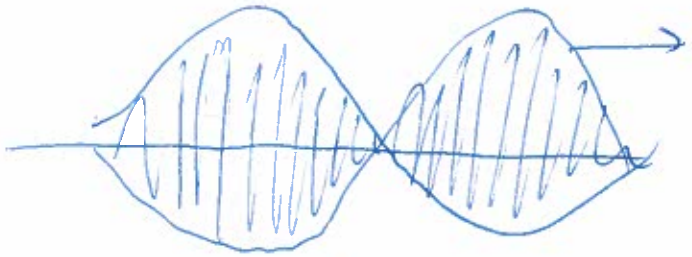
dispersion

Schrödinger: $\omega(\xi) = 2\pi |\xi|^2$ ——— dispersive relation

$\partial_t u = i \Delta u$

phase velocity: "$\omega(\xi)/\xi$"

group velocity: $\nabla \omega(\xi)$



$$S(t) f(x) = K_t * f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} f(y)\, dy$$

$$K_t(x) = \frac{1}{(4\pi i t)^{d/2}} e^{-\frac{|x|^2}{4it}} \quad (\longrightarrow \delta_0 \text{ as } t \to 0)$$

Prop (Dispersive estimate) Let $t \neq 0$.

$$\| S(t) f \|_{L^\infty_x(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \| f \|_{L^1_x}$$

Rmk: The "height" of $S(t) f \downarrow 0$ uniformly.

FACT: $g(x) = e^{-\pi|x|^2}$ | $g_\varepsilon(x) = g(\frac{x}{\varepsilon})$

$\Rightarrow \hat{g}(\xi) = e^{-\pi|\xi|^2}$ | $\Rightarrow \hat{g}_\varepsilon(\xi) = \varepsilon^d \hat{g}(\varepsilon\xi)$

$$S(t)f = \mathcal{F}^{-1}\left(\underbrace{e^{-4\pi^2 i t|\xi|^2}}_{\uparrow \hat{K}_t} \hat{f}(\xi)\right) = K_t * f$$

$$\Rightarrow K_t = \mathcal{F}^{-1}\left(e^{-4\pi^2 i t|\xi|^2}\right) = \frac{1}{(4\pi i t)^{d/2}} e^{-\frac{|x|^2}{4it}}$$

Pf of FACT: Let $F(s) = \int_{\mathbb{R}} e^{-\pi(x+is)^2} dx$, $s \in \mathbb{R}$

$$\Rightarrow \frac{d}{ds} F(s) = \int_{\mathbb{R}} -2\pi i(x+is)\, e^{-\pi(x+is)^2} dx$$

$$= \int_{-\infty}^{\infty} i \frac{d}{dx}\left(e^{-\pi(x+is)^2}\right) dx = 0 \Rightarrow F(s) \equiv \text{const.}$$

$$\cdot\ \mathcal{F}\left(e^{-\pi|x|^2}\right)(\xi) = \int_{\mathbb{R}^d} e^{-\pi|x|^2} e^{-2\pi i x\cdot\xi} dx = \prod_{j=1}^{d} \int_{\mathbb{R}} e^{-\pi(x_j+i\xi_j)^2} e^{\pi(i\xi_j)^2} dx_j$$

$$= \underbrace{\left(\int_{\mathbb{R}} e^{-\pi y^2} dy\right)^d}_{=1} e^{-\pi|\xi|^2}. \quad \square$$

$$\|S(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L^1_x}$$

$$\|S(t)f\|_{L^2_x} = \|f\|_{L^2_x}$$

interpolate ⟹

$$\|S(t)f\|_{L^p_x} \lesssim \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \|f\|_{L^{p'}_x} \quad \text{for } 2 \le p \le \infty$$

$\frac{\theta}{2}$

$$\frac{1}{p} + \frac{1}{p'} = 1$$

pf:

$$\frac{1}{p} = \frac{\theta}{\infty} + \frac{1-\theta}{2} = \frac{1}{2} - \frac{1}{2}\theta$$

$$\left(\frac{1}{p'} = \frac{\theta}{1} + \frac{1-\theta}{2} \right)$$

Riesz-Thorin

$$\|Tf\|_{L^{p_j}_x} \lesssim A_j \|f\|_{L^{q_j}_x}$$

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

$$\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

$$\Rightarrow \|Tf\|_{L^p} \le A_1^{\theta} A_2^{1-\theta} \|f\|_{L^q}$$

(2) **Prop** (Strichartz estimates)

We say (q, r) is Schrödinger admissible if $2 \le q, r \le \infty$, $(q, r, d) \neq (2, \infty, 2)$

$$\boxed{\frac{2}{q} + \frac{d}{r} = \frac{d}{2}} \quad \leftarrow \text{"dimension counting"}$$

(i) (homogeneous estimate) If (q, r) is Schr. admn, then

$$\| S(t) f \|_{L_t^q L_x^r (\mathbb{R} \times \mathbb{R}^d)} \lesssim \| f \|_{L_x^2 (\mathbb{R}^d)}$$

$\forall\, f \in L_x^2$.

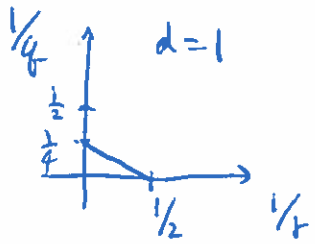
(ii) (dual homog est)

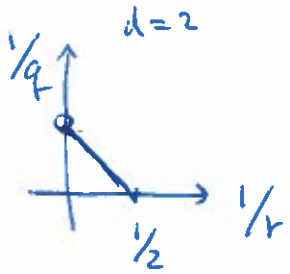
$$\| \int_{\mathbb{R}} S(-t') F(t') dt' \|_{L^2(\mathbb{R}^d)} \lesssim \| F \|_{L_t^{q'} L_x^{r'}}$$

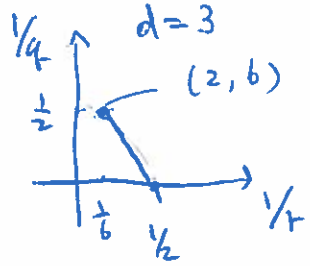
(iii) nonhomog est (retarded est)

$$\| \int_0^t S(t - t') F(t') dt' \|_{L_t^q L_x^r} \lesssim \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$(\tilde{q}, \tilde{r})$, Schr. adm.







---

pf (non-endpt case)

TT^* argument.

H, Hilbert space

B, Banach space

$T : H \to B$

$T^* : B' \to H$

$TT^* : B' \to B$

Then, $\|T\| < \infty \iff \|T^*\| < \infty \iff \|TT^*\| < \infty$

$T = S(t)$

$$\langle S(t)f, G(t,x) \rangle_{t,x} = \int_x \int_t S(t) f \, \overline{G(t,x)} \, dt\,dx$$

Parseval & Fubini

$$= \int_x f \int_t \overline{S(-t) G(t)} \, dt \, dx$$

$$= \langle f, \underbrace{\int_{\mathbb{R}} S(-t) G(t) \, dt}_{T^*G} \rangle_x$$

$$TT^* F = \int_{\mathbb{R}} S(t-t') F(t') dt' \qquad S(a+b) = S(a)S(b)$$

$$\left(= S(t) \int S(-t') F(t') dt' \right)$$

· Lemma: Hardy-Littlewood-Sobolev inequality.

$$\left(\int_{|x|>1} \frac{1}{|x|^\alpha} dx < \infty \iff \alpha > d \right)$$

$1 < p, q, r < \infty$, $\quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$$\left\| \frac{1}{|x|^{d/p}} * f \right\|_{L^r_x} \lesssim \|f\|_{L^q_x} \qquad \left(\int \frac{1}{|x|^\alpha} dx \right)^{1/p}$$

(Young's inequality would give (LHS) $\lesssim \|f\|_{L^q_x} \underbrace{\left\| \frac{1}{|x|^{d/p}} \right\|_{L^p_x}}_{=\infty}$)

$$\|TT^* F\|_{L^q_t L^r_x} \le \left\| \int \|S(t-t') F\|_{L^r_x} \right\|_{L^q_t} \qquad r \ge 2$$

$$\lesssim \left\| \int_{\mathbb{R}} \frac{1}{|t-t'|^{d(\frac{1}{2}-\frac{1}{r})}} \|F(t')\|_{L^{r'}_x} dt' \right\|_{L^q_t}$$

$q \to r$, $q' \to q$, $d(\frac{1}{2}-\frac{1}{r}) \to \frac{1}{p}$

$$\lesssim_{\text{H-L-S}} \|F\|_{L^{q'}_t L^{r'}_x} \Rightarrow_{TT^* \text{ argument}} \text{(i) \& (ii)}$$

- Checking the condition on the application of H-L-S inequality.

Note that we used H-L-S in time, i.e. $d=1$ in H-L-S ineq.

Need $\frac{1}{q}+1 = d\left(\frac{1}{2}-\frac{1}{r}\right)+\frac{1}{q'}$,

$= \frac{d}{2}-\frac{d}{r}+1-\frac{1}{q}$

$\Leftrightarrow \frac{2}{q}+\frac{d}{r}=\frac{d}{2}$ Schrödinger admissible

Also, need

$0 < \frac{1}{q}, d\left(\frac{1}{2}-\frac{1}{r}\right), \frac{1}{q'} < 1$ for H-L-S ineq.

$\Leftrightarrow 1 < q < \infty$

and $2 < r < \frac{2d}{d-2}$

missing the endpoint cases

(When $r=2$, we have $q=\infty$

$\Rightarrow$ (i) follows from the unitarity of $S(t)$

For $(q,r) = \left(2, \frac{2d}{d-2}\right)$, $d \geq 3$, see Keel-Tao Amer. J. Math '98.

Lec 5, 25/01/16

Schrödinger adm: $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$. $\quad 2 \le q, r \le \infty$

$(q, r, d) \neq (2, \infty, 2)$

endpt $(\infty, 2)$

$$\| S(t) f \|_{L_t^\infty L_x^2} = \| f \|_{L_x^2} \quad \text{(unitarity of } S(t) \text{ on } L^2)$$

endpt $(2, \underbrace{\frac{2d}{d-2}})$

$= 6$ when $d = 3$

$= 3$ when $d = 4$

$$\| f \|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim \| f \|_{H^1(\mathbb{R}^d)}$$

↑ Soboleu

$\frac{s}{d} = \frac{1}{p} - \frac{1}{q}$

$\frac{1}{d} = \frac{1}{2} - \frac{d-2}{2d}$

(homog) $\| S(t) f \|_{L_t^q L_x^r (\mathbb{R} \times \mathbb{R}^d)} \lesssim \| f \|_{L_x^2 (\mathbb{R}^d)}$

(dual) $\left\| \int_{\mathbb{R}} S(-t') F(t') dt' \right\|_{L^2(\mathbb{R}^d)} \lesssim \| F \|_{L_t^{q'} L_x^{r'}}$

(nonhomog) $\left\| \int_0^t S(t-t') F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$ $\quad (\tilde{q}, \tilde{r})$ sch adm

· TT^* argument. $T = S(t)$

$$TT^* F(t,x) = \int S(t-t') F(t') dt'$$

Goal: $\left\| \int S(t-t') F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$

· $\Leftarrow$ · dispersive estimate $\|S(t) f\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L_x^1}$

$\Rightarrow$ interpolation $\|S(t) f\|_{L_x^p} \lesssim \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \|f\|_{L_x^{p'}}$, $p \overset{VI}{\geq} \infty \geq 2$

· H-L-S inequality: $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$, $1 < p, q, r < \infty$ (Young's)

$$\left\| \frac{1}{|x|^{d/p}} * g \right\|_{L^r} \lesssim \|g\|_{L^q} \quad \left(\times \underbrace{\left\| \frac{1}{|x|^{d/p}} \right\|_{L_x^p}}_{\left(\int_0^\infty \frac{1}{r} dr \right)^{1/p}} = \infty \right)$$

$\Leftarrow$ we used H-L-S ineq in t.

$(d=1)$

$\Rightarrow$ homog & dual Strichartz (except for endpt $(2, \frac{2d}{d-2})$.)

nonhomog

$$\left\| \int_0^t S(t-t') F(t') \, dt' \right\|_{L_t^q L_x^r}$$

$$= \left\| \int_{\mathbb{R}} S(t-t') \mathbf{1}_{[0,t]}(t') F(t') \, dt' \right\|_{L_t^q L_x^r}$$

$$\lesssim \left\| \int_{\mathbb{R}} \frac{1}{|t-t'|^{d(\frac{1}{r}-\frac{1}{r'})}} \| \mathbf{1}_{[0,t]}(t') F(t') \|_{L_x^{r'}} \, dt' \right\|_{L_t^q}$$

$$\overset{\text{H-L-S}}{\lesssim} \| F \|_{L_t^{q'} L_x^{r'}}$$

If NO time cutoff,

$$\left\| \int S(t-t') F(t') \right\|_{L_t^q L_x^r} = \left\| S(t) \int S(-t') F(t') dt' \right\|_{L_t^q L_x^r}$$

$$\underset{\text{homog}}{\lesssim} \left\| \int S(-t') F(t') dt' \right\|_{L_x^2} \underset{\text{dual}}{\lesssim} \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

If $\exists$ time cutoff $\int_0^t$ — depends on t

✓ ⓐ $\left\| \int_{t'<t} S(t-t')F(t')dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$

ⓑ $\left\| \int_{t'<t} S(t-t')F(t')dt' \right\|_{L_t^\infty L_x^2} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$.

ⓐ: $\mathbb{1}_{(-\infty,t]}(t') = \mathbb{1}_{(-\infty,0)}(t') + \mathbb{1}_{[0,t]}(t')$

Pf of ⓑ: $\underset{\text{fixed } t}{\text{"(LHS)}^2\text{"}} = \left\langle \int_{t_1<t} S(t-t_1)F(t_1)dt_1 \,,\, \int_{t_2<t} S(t-t_2)F(t_2)dt_2 \right\rangle_{L_x^2}$

$S(t)S(-t_1)$ $\qquad$ $S(t)S(-t_2)$

$\left(\because \int S(t) f \overline{g}\, dx = \int f \overline{S(-t)g}\, dx \right)$

$= \int_{t_1<t} \left\langle F(t_1) \,,\, \int_{t_2<t} S(t_1-t_2)F(t_2)dt_2 \right\rangle_{L_x^2} dt_1$

$\overset{\text{Hölder}}{\leq} \int \|F(t_1)\|_{L_x^{r'}} \left\| \int_{t_2<t} S(t_1-t_2)F(t_2)dt_2 \right\|_{L_x^r} dt_1$

$\overset{\text{Hölder int}}{\leq} \|F\|_{L_t^{q'} L_x^{r'}} \left\| \int_{t_2<t} \cdots dt_2 \right\|_{L_t^q L_x^r} = (\text{RHS})^2$

$\overset{ⓐ}{\lesssim}$

☑

(*) WTS: $\left\| \int_{t'<t} S(t-t')F(t')\,dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$

(*) $\Leftarrow$ $\left| \iint_{t'<t} \langle S(t-t')F(t'), G(t) \rangle_{L_x^2} dt'\,dt \right| \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \|G\|_{L_t^{q'} L_x^{r'}}$ (**)

$\left(\sup_{\|G\|_{L_t^{q'} L_x^{r'}} = 1} \right)$

Fix $(\tilde{q}, \tilde{r})$.

(a) $\Rightarrow$ $q = \tilde{q}$, $r = \tilde{r}$ ✓

(b) $\Rightarrow$ $q = \infty$, $r = 2$

By (b), $(LHS) \le \left\| \int_{t'<t} S(t-t')F(t')\,dt' \right\|_{L_t^\infty L_x^2} \|G\|_{L_t^1 L_x^2}$

$\overset{(b)}{\lesssim} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \|G\|_{L_t^1 L_x^2}$

$\Rightarrow$ (*) $q = \infty$, $r = 2$

R-Th interpolation $\Rightarrow$ $\circledast$ for $q \geq \tilde{q}$

As for $\circledast$ with $q \leq \tilde{q}$, use the symmetry in $\circledast\circledast$

$\circledast\circledast$ with $q \geq \tilde{q}$,

$$\Rightarrow \left| \int_{t'} \left\langle F(t'), \int_{t>t'} S(t'-t) G(t)\, dt \right\rangle_{L^2_x} dt' \right|$$

$$\lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \|G\|_{L^{q'}_t L^{r'}_x}$$

$$\Rightarrow \left\| \int_{t>t'} S(t'-t) G(t)\, dt \right\|_{L^{\tilde{q}}_{t'} L^{\tilde{r}}_x} \lesssim \|G\|_{L^{q'}_t L^{r'}_x}$$

$\Rightarrow \circledast$ with $q \leq \tilde{q}$

$$\left(\mathbb{1}^{(t')}_{(t,\infty)} = \mathbb{1}^{(t')}_{\mathbb{R}} - \mathbb{1}^{(t')}_{(-\infty,t)} \right)$$

$\square$

Christ-Kiselev lemma (for inserting $\mathbb{1}^{(t')}_{[0,t]}$) See Tao's book.

Back to well-posedness of NLS (local) $\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1}u \\ u|_{t=0} = u_0 \end{cases}$

ex 1: $d=1$, $p=3$ (cubic NLS) appears in nonlinear optics

$$s_{crit} = \frac{d}{2} - \frac{2}{p-1} = -\frac{1}{2}$$

Thm: cubic NLS on $\mathbb{R}$ is locally well-posed in $H^s(\mathbb{R})$, $s \geq 0$.

Pf: $s=0$

$$(NLS) \iff u(t) = \Gamma_{u_0}(u)(t) = S(t)u_0 \mp i\int_0^t S(t-t')|u|^2 u(t')\,dt'$$

$\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$; $\mathbb{1}_{[-T,T]}(t)$

- Note: $(q,r) = (8,4)$ is Schr. adm.

Let $X_T = \underbrace{C([-T,T]; L^2_x(\mathbb{R}))}_{= C_T L^2_x} \cap L^8_T L^4_x$

$X \cap Y \Rightarrow \|u\|_{X\cap Y} = \|u\|_X + \|u\|_Y$

$$\|\Gamma_{u_0}(u)\|_{X_T} \leq C_1\|u_0\|_{L^2_x} + \underbrace{C_2\||u|^2u\|_{L_T^{8/7}L_x^{4/3}}}_{\leq C_3 T^{1/2}\||u|^2u\|_{L_T^{8/3}L_x^{4/3}}}$$

$\frac{7}{8} = \frac{3}{8} + \frac{1}{2}$

$C_3 \leq C_2\left(\int_{-T}^{T} 1\,dt\right)^{1/2}$

$$\Rightarrow \|\Gamma_{u_0}(u)\|_{X_T} \leq C_1 \|u_0\|_{L^2_x} + \underbrace{C_3 T^{1/2} \|u\|^3_{X_T}}_{\leq \frac{1}{2} R}$$

$$\leq 2 C_1 \|u_0\|_{L^2_x} =: R \quad \text{for } u \in \overline{B_R} \subset X_T$$

$\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

provided $C_3 T^{1/2} R^2 \leq \frac{1}{2}$

- We can estimate the difference in a similar manner.

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} \leq C_3 T^{1/2} \| \underbrace{|u|^2 u - |v|^2 v}_{u\bar{u}u - v\bar{v}v = (u-v)\bar{u}u + \cdots} \|_{L^{8/3}_T L^{4/3}_x}$$

$$\leq \underbrace{C_4 T^{1/2} \left(\|u\|^2_{X_T} + \|v\|^2_{X_T} \right)}_{\leq C_4 T^{1/2} R^2} \|u - v\|_{X_T}$$

Cauchy: $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$

< 1 by choosing $T^{1/2} \sim R^{-2}$

$T \sim R^{-4}$.

$\Rightarrow$ Banach's fixed pt thm on $\overline{B_R} \subset X_T$

$\Rightarrow$ unique soln in $C_T L^2_x \cap \underbrace{L^8_T L^4_x}_{\text{need this}}$ (HW1)

$T = T(\|u_0\|_{L^2}) \sim \|u_0\|_{L^2}^{-4}$.

- Show u is conti (in t) with values in L^2_x (just as before)
  see HW 1.

Lec 6 27/01/2016 (Wed)

Recall ① Schröding adm. $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$, $2\le q, r\le\infty$

$(q,r,d)\neq(2,\infty,2)$.

$$\|S(t)f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L_x^2(\mathbb{R}^d)}.$$

$$\left\|\int_0^t S(t-t')F(t')\,dt'\right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

② Local well-posedness of NLS $\begin{cases} i\partial_t u+\Delta u=\pm|u|^{p-1}u \\ u|_{t=0}=u_0 \end{cases}$

Thm: Cubic NLS (with $p=3$) on $\mathbb{R}$ ($d=1$) is locally well-posed in $H^s(\mathbb{R})$, $s\ge 0$.

- $s=0$ case has been proven.

- Now we check the case $s=1$ $(d=1)$. $H^1(\mathbb{R})$.

- $\|f\|_{H^1} = \|f\|_{L^2} + \|\partial_x f\|_{L^2}$. $\|f\|_{W^{1,4}} = \|f\|_{L^4_x} + \|\partial_x f\|_{L^4_x}$.

- $(q,r) = (8,4)$ is Schrödinger adm

- $X_T = C_T H^1 \cap L^8_T W^{1,4}$

$$= C_T L^2 \cap C_T \partial^{-1} L^2 \cap L^8_T L^4_x \cap L^8_T \partial_x^{-1} L^4_x.$$

$$\|u\|_{X_T} = \|u\|_{C_T L^2} + \|u\|_{L^8_T L^4_x} + \|\partial_x u\|_{C_T L^2} + \|\partial_x u\|_{L^8_T L^4_x}$$

- $u(t) = \Gamma_{u_0}(u)(t) := S(t)u_0 \mp i\int_0^t S(t-t')|u|^2 u(t')\,dt'$.

Proof of case $s=1$: $\|\Gamma_{u_0}(u)\|_{C_T L^2 \cap L^8_T L^4_x} \le C_1 \underbrace{\|u_0\|_{L^2_x}}_{\le \|u_0\|_{H^1}} + C_3 T^{1/2} \|u\|^3_{L^8_T L^4_x}$

$$\partial_x\left(S(t)u_0\right) = S(t)\left(\partial_x u_0\right)$$

$$\|\partial_x \Gamma_{u_0}(u)\|_{C_T L^2 \cap L_T^8 L_x^4} \le C_1 \|\partial_x u_0\|_{L_x^2} + C_2 \|\partial_x(|u|^2 u)\|_{L_T^{8/7} L_x^{4/3}}$$

$$\le C_1 \|\partial_x u_0\|_{L_x^2} + C_3 T^{1/2} \|\partial_x(|u|^2 u)\|_{L_T^{8/3} L_x^{4/3}}$$

Leibniz Rule (Product Rule).

$$\partial_x(fg) = \partial_x f \cdot g + f \cdot \partial_x g.$$

$$\le \|u_0\|_{H^1}$$

$$\le C_1 \|\partial_x u_0\|_{L_x^2} + 3 C_3 T^{1/2} \|\partial_x u\|_{L_T^8 L_x^4} \|u\|^2_{L_T^8 L_x^4}$$

$$\Rightarrow \quad \|\Gamma_{u_0}(u)\|_{X_T} \le 2 C_1 \|u_0\|_{H^1} + 4 C_3 T^{1/2} \|u\|^3_{X_T}$$

Similarly, we have.

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} \le C_4 T^{1/2} \left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right) \|u - v\|_{X_T}.$$

$\Rightarrow$ Conclude as before ($s = 0$ case).

Remark 1: The above argument works for $s \in \mathbb{N}$.

Remark 2: For $s > 0$, but not integer, we need **fractional Leibniz Rule.**

Lemma: $s \in (0, 1]$ and $1 < r, p_1, p_2, q_1, q_2 < \infty$, s.t.

$$\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i} \quad \text{for } i = 1, 2, \quad \text{Then}$$

$$\| |\nabla|^s (fg) \|_{L^r} \lesssim \|f\|_{L^{p_1}} \| |\nabla|^s g \|_{L^{q_1}} + \| |\nabla|^s f \|_{L^{p_2}} \|g\|_{L^{q_2}}$$

$$\widehat{|\nabla|^s f} = |\xi|^s \hat{f}(\xi). \longrightarrow |\nabla|^s = \left(\sqrt{-\Delta}\right)^s$$

$$\widehat{\Delta f} = -|\xi|^2 \hat{f}(\xi).$$

- $\|f\|_{H^s} = \|f\|_{L^2} + \| |\nabla|^s f \|_{L^2}$

Proceed as before.

$$\| |\nabla|^s (|u|^2 u) \|_{L_T^{8/3} L_x^{4/3}} \lesssim \| |u|^2 \|_{L_T^4 L_x^2} \| |\nabla|^s u \|_{L_T^8 L_x^4}$$

$$+ \underline{\| |\nabla|^s (|u|^2) \|_{L_T^4 L_x^2}} \| u \|_{L_T^8 L_x^4}$$

(We use FLR in x-variable, then Hölder in t.)

$$\lesssim \| u \|^2_{L_T^8 L_x^4} \| |\nabla|^s u \|_{L_T^8 L_x^4} + 2 \| |\nabla|^s u \|_{L_T^8 L_x^4} \| u \|^2_{L_T^8 L_x^4}$$

$$\lesssim 3 \| |\nabla|^s u \|_{L_T^8 L_x^4} \| u \|^2_{L_T^8 L_x^4}.$$

Then other argument is identical to $s=1$ case. #.

Now we turn to $d=2$, $p=3$.

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u|_{t=0} = u_0 \end{cases}$$

- $s_{crit} = \frac{d}{2} - \frac{2}{p-1} = \frac{2}{2} - \frac{2}{3-1} = 0$

  L^2_x - critical, mass critical.

Thm: Cubic NLS on $\mathbb{R}^2$ is locally well-posed in $L^2(\mathbb{R}^2)$, and

Global well-posed in $L^2(\mathbb{R}^2)$ with Small initial data.

($\|u_0\|_{L^2}$ sufficiently small).

Recall: $(q,r) = (4,4)$ is Schrödinger adm in 2-d.

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$

$$u(t) = \Gamma_{u_0}(u)(t) := S(t)u_0 \mp i\int_0^t S(t-t')\,|u|^2 u(t')\,dt'$$

① Argue as usual. Let $X_T = C_T L^2 \cap L_T^4 L_x^4$

$$\|\Gamma_{u_0}(u)\|_{X_T} \le \|S(t)u_0\|_{X_T} + C_2 \||u|^2 u\|_{L_T^{4/3}L_x^{4/3}}$$

$$\le C_1 \|u_0\|_{L_x^2} + C_2 \|u\|^3_{L_T^4 L_x^4}$$

there is no T here to help

(Need smallness to conclude).

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} \le C_2 \||u|^2u - |v|^2 v\|_{L_T^{4/3}L_x^{4/3}}$$

$$\le C_3\left(\|u\|^2_{L_T^4 L_x^4} + \|v\|^2_{L_T^4 L_x^4}\right)\|u-v\|_{L_T^4 L_x^4}$$

Smallness is needed for fix point argument.

② Now define $A_\eta = \{ u \in L_T^4 L_x^4 : \|u\|_{L_T^4 L_x^4} \le 2\eta \}$.

with η to be determined later. Let $u_0 \in L_x^2(\mathbb{R}^2)$.

· $\forall\ u \in A_\eta$ $\quad \|\Gamma_{u_0}(u)\|_{L_T^4 L_x^4} \le \|S(t)u_0\|_{L_T^4 L_x^4} + C_3 \||u|^2 u\|_{L_T^{4/3} L_x^{4/3}}$

$$\le \|S(t)u_0\|_{L_T^4 L_x^4} + C_3 \|u\|^3_{L_T^4 L_x^4}$$

$$\le \underline{\|S(t)u_0\|_{L_T^4 L_x^4}} + \underline{C_3 (2\eta)^3}.$$

· In order to put $\Gamma_{u_0}(u)$ in A_η, we need some smallness condition.

ⓐ Chose $\underline{C_3 \cdot (2\eta)^2 < \frac{1}{4}}$.

ⓑ $\|S(t)u_0\|_{L_t^4 L_x^4(\mathbb{R}\times\mathbb{R}^2)} \lesssim \|u_0\|_{L_x^2} < \infty$.

$\xrightarrow{\text{MCT}}$ $\lim_{T\to 0} \|S(t)u_0\|_{L_T^4 L_x^4} = 0$ $\Longrightarrow$ { We can choose T sufficiently small, s.t. $\|S(t)u_0\|_{L_T^4 L_x^4} < \eta$

$$\Rightarrow \|\Gamma_{u_0}(u)\|_{L_T^4 L_x^4} \leq \underbrace{\|S(t)u_0\|_{L_T^4 L_x^4}}_{\downarrow (b).} + \underbrace{C_3 (2\eta)^2 \cdot (2\eta)}_{\longrightarrow \text{by } (a)}$$

$$< \eta + \tfrac{1}{4}\eta$$

$$< 2\eta \qquad \Longrightarrow \qquad \Gamma_{u_0}(u) \in A_\eta.$$

- $\forall\, u, v \in A_\eta.$

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{L_T^4 L_x^4} \leq C_2 \| |u|^2 u - |v|^2 v \|_{L_T^{4/3} L_x^{4/3}}$$

$$\leq C_3 \left(\underbrace{\|u\|^2_{L_T^4 L_x^4}}_{} + \underbrace{\|v\|^2_{L_T^4 L_x^4}}_{} \right) \|u - v\|_{L_T^4 L_x^4}$$

$$\leq C_3 \cdot 2 \cdot (2\eta)^2 \, \|u - v\|_{L_T^4 L_x^4}$$

$$\leq \tfrac{1}{2} \|u - v\|_{L_T^4 L_x^4}$$

$\Longrightarrow$ Fix point argument in $L_T^4 L_x^4$.

Remark: We need to make sure

$$\|S(t)u_0\|_{L_T^4 L_x^4} \le \eta.$$

① Choose T small.

② Since $\|S(t)u_0\|_{L_t^4 L_x^4(\mathbb{R}\times\mathbb{R}^2)} \le C_0 \|u_0\|_{L_x^2}$. We can choose u_0

s.t. $\|u_0\|_{L_x^2} \le \frac{1}{C_0}\eta \Rightarrow \|S(t)u_0\|_{L_t^4 L_x^4(\mathbb{R}\times\mathbb{R}^2)} \le \eta.$

Which means that $T = +\infty$, global solution!!

③ Now $u \in L_T^4 L_x^4$. now we show $u \in C_T L^2$.

$$\|u\|_{C_T L^2} \le C_1 \|u_0\|_{L^2} + C_2 \||u|^2 u\|_{L_T^{4/3} L_x^{4/3}}$$

$$\le C_1 \|u_0\|_{L^2} + C_2 (2\eta)^2 < \infty$$

Lec 7 01/02/16

Sec 4 More on estimates

(4.1) Dispersive estimate for Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = f \end{cases}$$

$$\Rightarrow S(t)f(x) = e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} f(y)\, dy$$

$$\Rightarrow \| S(t)f \|_{L^\infty_x} \lesssim \frac{1}{|t|^{d/2}} \| f \|_{L^1_x}$$

- Goal: Prove dispersive estimate WITHOUT the explicit formula.
- 1-d case:

$$S(t)f = K_t * f$$

$$K_t = \mathcal{F}^{-1}\left(e^{-it|\xi|^2}\right) = \int_{\mathbb{R}} e^{-it|\xi|^2 + ix\cdot\xi}\, d\xi$$

$$= \frac{1}{(4\pi i t)^{1/2}} e^{-|x|^2/4it}.$$

Claim: $\| K_t(x) \|_{L^\infty_x} \lesssim \frac{1}{|t|^{1/2}} \qquad (t \neq 0)$

Then, by Young's ineq,

$$\frac{1}{\infty} + 1 = \frac{1}{\infty} + 1$$

$$\begin{aligned} \| S(t) f \|_{L^\infty_x} &= \| K_t * f \|_{L^\infty_x} \\ &\leq \| K_t \|_{L^\infty_x} \| f \|_{L^1_x} \\ &\lesssim \frac{1}{|t|^{1/2}} \| f \|_{L^1_x} \end{aligned}$$

· Assume $t > 0$.

$$\begin{aligned} K_t(x) &= \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-i\xi^2 + i\frac{x}{\sqrt{t}}\xi} \, d\xi \qquad \xi = t^{1/2}\xi \\ &= \frac{1}{\sqrt{t}} K_1\left(\underbrace{\frac{x}{\sqrt{t}}}_{y}\right) \end{aligned}$$

$$\Rightarrow \| K_t \|_{L^\infty_x} = \frac{1}{\sqrt{t}} \| K_1 \|_{L^\infty_y} \qquad (\because t \text{ is fixed})$$

Tool: Method of stationary phase.

$$K_1(x) = \int_{\mathbb{R}} e^{-i\xi^2 + ix\xi}\, d\xi$$

$$= \int_{\mathbb{R}} e^{-i\phi(\xi)}\, d\xi \quad (x \text{ is fixed})$$

where $\phi(\xi) = \xi^2 - x\xi$

idea: Integration by parts

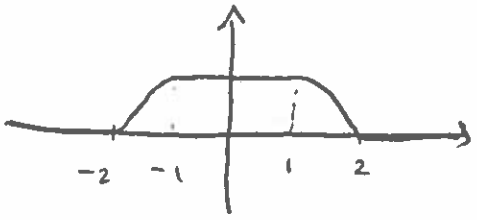
$$e^{-i\phi(\xi)} = \frac{\partial_\xi\, e^{-i\phi(\xi)}}{-i\phi'(\xi)} \qquad \phi'(\xi) = 2\xi - x$$

← if $\phi'(\xi)$ is not small, good.

$$\text{"} \int e^{-i\phi(\xi)} \cdot 1\, d\xi = \int \partial_\xi e^{-i\phi(\xi)} \cdot \frac{1}{-i\phi'(\xi)}\, d\xi$$

$$\overset{\text{IBP}}{=} \int e^{-i\phi(\xi)} \left(\frac{1}{i\phi'(\xi)}\right)' d\xi \text{."}$$

Let $\psi \in C^\infty(\mathbb{R}; [0,1])$ s.t. $\psi(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| \ge 2 \end{cases}$



bump function
cutoff function

$$K_1(x) = \int_{\mathbb{R}} e^{-i\Phi(\xi)} \psi(\underbrace{2\xi - x}_{=\Phi'(\xi)}) \, d\xi$$

$$+ \int_{\mathbb{R}} e^{-i\Phi(\xi)} (1-\psi)(2\xi - x) \, d\xi =: \mathrm{I} + \mathrm{II}.$$

- $|\mathrm{I}(x)| \le \int |\psi(\underbrace{2\xi - x}_{-2 \le \quad \le 2})| \, d\xi \lesssim 1$

- $\mathrm{II}(x) \overset{\text{IBP}}{=} \int e^{-i\Phi(\xi)} \partial_\xi \left(\frac{(1-\psi)(2\xi - x)}{i\Phi'(\xi)} \right) d\xi$ $\qquad \Phi'(\xi) = 2\xi - x.$

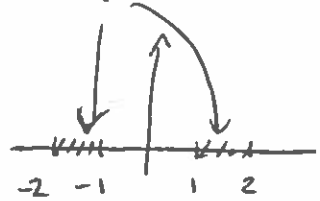
$$= \int e^{-i\Phi(\xi)} \left\{ - \frac{(1-\psi)(2\xi - x)}{i(2\xi - x)^2} \cdot 2 \right\} d\xi$$

$$+ \int_{|2\xi - x| \ge 1} e^{-i\Phi(\xi)} \frac{(1-\psi)'(2\xi - x)}{i\underbrace{\Phi'(\xi)}_{\ge 1}} \, d\xi$$

$$\Rightarrow |\mathrm{II}(x)| \le c \int_{|2\xi - x| \ge 1} \frac{1}{(2\xi - x)^2} \, d\xi$$

$$+ \int_{2\xi - x \in \operatorname{supp}\psi'} |(1-\psi)'(2\xi - x)| \, d\xi \lesssim_{\psi} 1$$

$(1-\psi)' = -\psi'$



□

(4.2) Glimpse on oscillatory integrals

$$I(\lambda) = \int_a^b e^{i\lambda\Phi(x)}\psi(x)\,dx$$

phase $\Phi(x)$, real-valued

$\psi(x)$, complex-valued (with cpt support)

Lemma: $\operatorname{supp}\psi \subset\subset [a,b]$ (↑ cpt subset)

$\Phi'(x) \neq 0$ for all $x \in [a,b]$

Then, $I(\lambda) = O(\lambda^{-N})$ as $\lambda \to \infty$ (for any $N \in \mathbb{N}$.)

Pf: let $Df(x) = \frac{1}{i\lambda\Phi'(x)}\frac{df}{dx}$

($f = O(g^0) \iff \lim \frac{|f|}{g} \leq C$.)

$\Rightarrow$ **Note:** $D(e^{i\lambda\Phi}) = e^{i\lambda\Phi}$.

$\langle Df, g\rangle_{L^2} = \langle f, D^T g\rangle_{L^2}$

Transpose $D^T f(x) = -\frac{d}{dx}\left(\frac{f}{i\lambda\Phi'(x)}\right)$

$$\Rightarrow I(\lambda) = \int_a^b \underbrace{e^{i\lambda\Phi}}_{D^N(e^{i\lambda\Phi})}\psi\,dx \underset{\text{IBP N times}}{=} \int_a^b e^{i\lambda\Phi}(D^T)^N(\psi)\,dx$$

$$\Rightarrow \quad |I(\lambda)| \lesssim_{N,\psi,\phi} \lambda^{-N} \qquad \square$$

Rmk: • Under change of var: $x \longmapsto \phi(x)$,

Lemma $\Longleftrightarrow$ $\mathcal{F}$ (cpt supported func) has rapid decay.

$$I(\lambda) = \int e^{i\lambda y} \underbrace{\psi \circ \phi^{-1}(y)}_{\text{cptly supp.}} J \cdot dy \qquad y = \phi(x)$$

• If we do not assume that ψ vanishes at the endpoints, then the decay is much worse.

Prop (van der Corput) ϕ, real valued, smooth on (a,b)

Suppose that $\exists\ k$ s.t. $|\phi^{(k)}(x)| \geq 1$ for all (a,b)

Then, $$\left| \int_a^b e^{i\lambda\phi(x)}\, dx \right| \leq C_k \lambda^{-1/k}$$

provided (i) $k \geq 2$

or (ii) $\phi'(x)$ is monotonic when $k = 1$.

Pf: (ii) $k=1$

$$\int_a^b e^{i\lambda\phi}\,dx = \int_a^b D(e^{i\lambda\phi})\cdot 1\,dx$$

$$= \int_a^b e^{i\lambda\phi} D^T(1)\,dx + \underbrace{\frac{e^{i\lambda\phi}}{i\lambda\phi'}\Big|_a^b}_{|\cdot| \le \frac{2}{\lambda}}$$

- $\left|\int_a^b e^{i\lambda\phi} D^T(1)\,dx\right|$

$$= \frac{1}{\lambda}\left|\int_a^b e^{i\lambda\phi}\frac{d}{dx}\left(\frac{1}{\phi'}\right)dx\right| \le \frac{1}{\lambda}\int_a^b \left|\frac{d}{dx}\left(\frac{1}{\phi'}\right)\right|dx$$

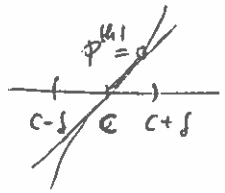
$$\underset{\phi' \text{ is monotonic}}{=} \frac{1}{\lambda}\left|\int_a^b \frac{d}{dx}\left(\frac{1}{\phi'}\right)dx\right| \overset{FTC}{=} \frac{1}{\lambda}\left|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\right| \le \frac{1}{\lambda}$$

$\to \frac{1}{\phi'}$ is monotonic

(i) We proceed by induction on k.
Suppose that the case for k is known.
WLOG, assume $\phi^{(k+1)}(x) \ge 1$, $\forall\, x \in (a,b)$

Let $x = c$ be the (unique) point in $[a,b]$ s.t. $|\phi^{(k)}(x)|$ attains its min.

- If $\phi^{(k)}(c) = 0$, then $|\phi^{(k)}(x)| \geq \delta$ on $(c-\delta, c+\delta)^c$



Write $\int_a^b = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b$

$\Rightarrow \left|\frac{\phi^{(k)}}{\delta}\right| \geq 1.$

By inductive hypothesis,

$$\left| \int_a^{c-\delta} + \int_{c+\delta}^b e^{i\lambda\phi}\, dx \right| \leq C_k (\lambda\delta)^{-1/k}$$

$i\lambda\phi = i(\lambda\delta)\boxed{\frac{\phi}{\delta}}$

On the other hand,

$$\left| \int_{c-\delta}^{c+\delta} \cdots \right| \leq 2\delta$$

equate them.

$$\Rightarrow \delta \sim (\lambda\delta)^{-1/k} \ (\Leftrightarrow \lambda^{-1} \sim \delta^{k+1} \Leftrightarrow \delta \sim \lambda^{-\frac{1}{k+1}}.$$

- If $\phi^{(k)}(c) \neq 0$, then $c = a$. Write $\int_a^b = \int_a^{a+\delta} + \int_{a+\delta}^b$

proceed as before.

☒ (When $k+1 = 2$, $\phi^{(k+1)} \geq 1 \Rightarrow \phi'$ is monotonic.)

Cor: same assumption as the previous prop.

$$\left| \int_a^b e^{i\lambda\Phi(x)} \psi(x)\, dx \right| \le C_k \lambda^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(x)|\, dx \right].$$

Pf: Let $F(x) = \int_a^x e^{i\lambda\Phi(y)}\, dy$

$\Rightarrow$ By Prop. $|F(x)| \le C_k \lambda^{-1/k}$.

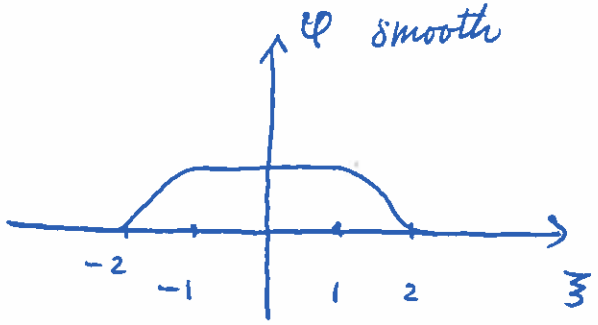
$$\int_a^b \underbrace{e^{i\lambda\Phi}}_{F'} \psi\, dx \underset{\text{IBP}}{=} F(b)\psi(b) - \overset{0}{\overbrace{F(a)}}\psi(a) - \int_a^b F(x)\psi'(x)\, dx$$

□

Lec 8 03/02/16

(4.3) Glimpse on Littlewood-Paley decomposition

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| > 2 \end{cases}$$



Given dyadic N,

($\Leftrightarrow$ $N = 2^j$ for some $j \in \mathbb{Z}$
also written as $N \in 2^{\mathbb{Z}}$

dyadic $N \ge 1$ $\Leftrightarrow$ $N \in 2^{\mathbb{N}}$

define

$$P_{\le N} f = \mathcal{F}^{-1}\left(\varphi\left(\frac{\xi}{N}\right) \hat{f}(\xi) \right)$$

$$\varphi\left(\frac{\xi}{N}\right) = \begin{cases} 1, & |\xi| \le N \\ 0, & |\xi| > 2N \end{cases}$$

Littlewood-Paley projection

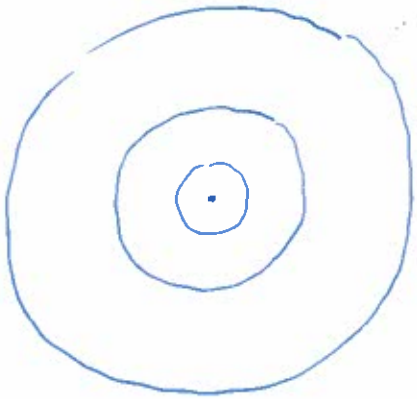
$$P_N f = P_{\le N} f - P_{\le \frac{N}{2}} f = \text{localization onto } \{|\xi| \sim N\}$$

$\frac{1}{2}N \le |\xi| \le 2N$

$P_{\le N}$: $\begin{cases} 1, & |\xi| \le N \\ 0, & |\xi| > 2N \end{cases}$ $\qquad$ $P_{\le \frac{N}{2}}$: $\begin{cases} 1, & |\xi| \le \frac{N}{2} \\ 0, & |\xi| > N \end{cases}$

- $P_{\leq N} f = \sum_{M \leq N} P_M f$
- $f = \lim_{N \to \infty} P_{\leq N} f$, i.e. $P_{\leq N} \to Id$.

$$= \sum_{\substack{M \\ \text{dyadic}}} P_M f$$



- **Thm**: (Littlewood-Paley theory)
  Let $1 < p < \infty$.

Then, $\|f\|_{L^p} \sim \| \underbrace{\left(\sum_{\substack{N \\ \text{dyadic}}} |P_N f(x)|^2 \right)^{1/2}}_{=: \text{square function of } f} \|_{L^p}$

$p=2$:

$$(RHS) = \left(\sum_N \|P_N f\|_{L^2}^2 \right)^{1/2}$$

$$\sim \left(\sum_N \int_{|\xi| \sim N} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \sim \left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} = \|f\|_{L^2}$$

· **Bernstein's inequalities**: $1 \le p \le q \le \infty$

(i) $\| P_{\le N} |\nabla|^s f \|_{L^p} \lesssim N^s \| P_{\le N} f \|_{L^p}$, $s \ge 0$

(ii) $\| P_N |\nabla|^s f \|_{L^p} \sim N^s \| P_N f \|_{L^p}$, $s \in \mathbb{R}$

(iii) $\| P_{\le N} f \|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_{\le N} f \|_{L^p}$

(iv) $\| P_N f \|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_N f \|_{L^p}$

} frequency restricted analogue of Sobolev inequality $s = \frac{d}{p} - \frac{d}{q}$

g-function

$$g(f)(x) = \left(\int_0^\infty |\nabla u(x,t)|^2 t \, dt \right)^{1/2}$$

u solves heat eqn with $u|_{t=0} = f$

$$\nabla = (\partial_{x_1}, \partial_{x_2}, \cdots \partial_{x_d}) \Rightarrow |\nabla| = \sqrt{-(\partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_d}^2)} = \sqrt{-\Delta}$$

$$|\nabla|^s f = \mathcal{F}^{-1}\left(|\xi|^s \hat{f}(\xi) \right) \quad \begin{matrix} s>0 & \text{fractional differentiation} \\ s<0 & \text{fractional integration} \end{matrix}$$

$$= c_{d,s} \int \frac{f(y)}{(x-y)^{d+s}} \, dy.$$

Pf of (ii): $P_N |\nabla|^s f = |\nabla|^s P_N f$

$\hookrightarrow |\xi|^s \psi\left(\frac{\xi}{N}\right) = N^s \left|\frac{\xi}{N}\right|^s \psi\left(\frac{\xi}{N}\right)$

where $\psi(\xi) = \varphi(\xi) - \varphi(2\xi)$.

Since $\operatorname{supp} |\xi|^s \psi(\xi)$ is away from the origin, $|\xi|^s \psi(\xi)$ is smooth.

$\Rightarrow K_s = \mathcal{F}^{-1}\left(|\xi|^s \psi(\xi)\right)$ decays rapidly.

In particular, $K_s \in L^1_x$.

($|\xi|^s \psi(\xi)$: NOT smooth at the origin, especially for $s<1$)

$\Rightarrow \| P_N |\nabla|^s f \|_{L^p} = N^s \| K_{N,s} * f \|_{L^p}$

$\leq N^s \| K_{N,s} \|_{L^1_x} \| f \|_{L^p}$ (Young)

$\| K_{N,s} \|_{L^1_x} \leq C_s < \infty$

$K_{N,s} = \mathcal{F}^{-1}\left(\left|\frac{\xi}{N}\right|^s \varphi\left(\frac{\xi}{N}\right)\right)$

$= N^d K_s(Nx)$

$\Rightarrow \| K_{N,s} \|_{L^1_x} = \| K_s \|_{L^1_x} < \infty$

(indep of N.)

(i) $P_{\leq N} |\nabla|^s f = \sum_{M \leq N} P_M |\nabla|^s f$

$\| P_{\leq N} |\nabla|^s f \|_{L^p} \overset{\text{Mink}}{\leq} \sum_{M \leq N} \| P_M |\nabla|^s f \|_{L^p} \overset{\text{(ii)}}{\leq} C_s \sum_{M \leq N} M^s \| P_M f \|_{L^p}$

$\lesssim N^s \left\| \left(\sum_{M \leq N} |P_M f|^2 \right)^{1/2} \right\|_{L^p} \overset{LP}{\sim} N^s \| P_{\leq N} f \|_{L^p}$.

Note: $\sum_{M \leq N} M^s = \sum_{j=0}^{\infty} \underbrace{(N 2^{-j})}_{=M}{}^s = N^s \sum_{j=0}^{\infty} 2^{-js} \sim N^s$ $s>0$ ☐

(4.4) Back to dispersive estimate: We'll prove

$$\oplus \qquad \| S(t) P_N f \|_{L^\infty_x(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \| P_N f \|_{L^1_x}, \quad \forall\ N \text{ dyadic}$$

($\lesssim$: implicit const indep of N.)

By unitarity, $\| S(t) P_N f \|_{L^2_x} = \| P_N f \|_{L^2_x}$

interpolation $\Rightarrow$ $\| S(t) P_N f \|_{L^p_x} \lesssim \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \| P_N f \|_{L^{p'}}, \quad 2 \le p \le \infty$

- $\| S(t) f \|_{L^p_x} \overset{\text{LP theory}}{\underset{p<\infty}{\sim}} \| \left(\sum_N |S(t) P_N f|^2 \right)^{1/2} \|_{L^p} = \| S(t) P_N f \|_{L^p_x \ell^2_N}$

$\overset{p \ge 2, \text{Mink}}{\le}$ $\left(\sum_N \| S(t) P_N f \|^2_{L^p} \right)^{1/2}, \quad p < \infty$

$\overset{(*)}{\lesssim}$ $\frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \left(\sum_N \| P_N f \|^2_{L^{p'}} \right)^{1/2}$

$\overset{p' \le 2, \text{Mink}}{\le}$ $\frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \| \left(\sum_N |P_N f|^2 \right)^{1/2} \|_{L^{p'}}$

$\overset{\text{LP theory}}{\underset{p'>1}{\sim}}$ $\frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \| f \|_{L^{p'}}$ $\Rightarrow$ Strichartz estimate.

It remains to prove ⊕.

Let $K_t(x) = \int_{\mathbb{R}^d} e^{i(-4\pi^2 t|\xi|^2 + 2\pi x\cdot\xi)}\, d\xi$

i.e. $S(t) f = K_t * f$.

$$\Rightarrow S(t) P_N f = S(t)\widetilde{P}_N P_N f$$
$$= (\widetilde{P}_N K_t) * (P_N f)$$

$$\Rightarrow \|S(t)P_N f\|_{L^\infty_x} \overset{\text{Young}}{\leq} \|\widetilde{P}_N K_t\|_{L^\infty_x} \|P_N f\|_{L^1_x}$$

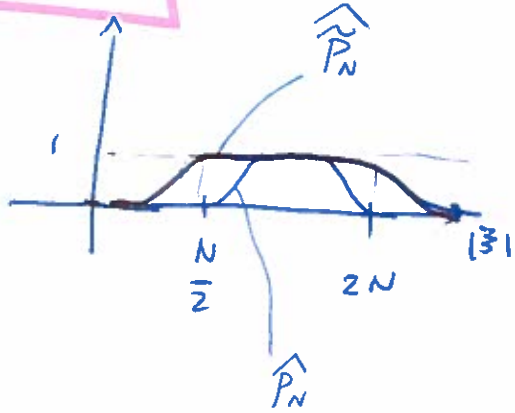
Goal: $\|\widetilde{P}_N K_t\|_{L^\infty_x} \lesssim \frac{1}{|t|^{d/2}}$

↑ implicit const indep of N.

$$\widetilde{P}_N = P_{N/2} + P_N + P_{2N}$$
$$= P_{\leq N} - P_{\leq \frac{N}{4}}$$

$\begin{cases} 1, & |\xi| \leq 2N \\ 0, & |\xi| > 2N \end{cases}$ — $\begin{cases} 1, & |\xi| \leq N/4 \\ 0, & |\xi| \geq N/2 \end{cases}$

$$\Rightarrow \widetilde{P}_N P_N = P_N$$



Rmk: $\sum_N \widetilde{P}_N = 3\cdot \mathrm{Id}$ and

$$\|f\|_{L^p} \sim \left\| \left(\sum_N |\widetilde{P}_N f|^2 \right)^{1/2} \right\|_{L^p}, \quad 1 < p < \infty$$

just like P_N

For simplicity, drop $\sim$ in $\widetilde{P_N}$.

$$P_N K_t(x) = \int_{\mathbb{R}^d} e^{i(-4\pi^2 t|\xi|^2 + 2\pi x\cdot\xi)} \psi\left(\frac{\xi}{N}\right) d\xi$$

$\psi(\cdot) = \varphi(\cdot) - \varphi(2\cdot)$

↑ smooth bump func supported $\{|\xi| \sim 1\}$

Case 1: $t \lesssim N^{-2}$

By change of var. $t^{1/2}\xi \longmapsto \xi$, $\quad \xi = t^{1/2}\xi$

$$|P_N K_t(x)| = \frac{1}{|t|^{d/2}} \left| \int_{\mathbb{R}^d} e^{i(-4\pi^2|\xi|^2 + 2\pi\left(\frac{x}{t^{1/2}}\right)\cdot\xi)} \psi\left(\frac{\xi}{t^{1/2}N}\right) d\xi \right|$$

$|\xi| \lesssim 1$

$t^{1/2} N \lesssim 1$

$$\lesssim \frac{1}{|t|^{d/2}}.$$

- $\boxed{\text{Case 2: } t \gg N^{-2}.}$

$$P_N K_t(x) = N^d \int_{\mathbb{R}^d} e^{2\pi i N x\cdot\xi}\, e^{-4\pi^2 i t N^2 |\xi|^2} \underbrace{\psi(\xi)}_{=\psi(|\xi|)} d\xi \qquad \xi = \frac{\xi}{N}$$

- subcase 2.a: $|x| \ll tN$.

By polar coordinates,

$$P_N K_t(x) = N^d \int_{S^{d-1}} \int_0^\infty e^{2\pi i N r x\cdot\omega} \times e^{-4\pi^2 i t N^2 r^2} \underbrace{\psi(r)}_{r\sim 1} r^{d-1}\, dr\, d\sigma(\omega) \qquad \xi = r\omega \quad r \geq 0,\ \omega \in S^{d-1}$$

Note $\left| 2\pi N r x\cdot\omega - 4\pi^2 \underbrace{t N^2 r^2}_{tN\cdot N r^2} \right| \sim tN^2.$

$|x| \ll tN$, $r \sim 1$

$\Longrightarrow$ By IBP k times, (No bdry term b/c $\psi(0) = \psi(\infty) = 0$)

$$|P_N K_t(x)| \lesssim \frac{N^d}{(tN^2)^k} \lesssim \frac{1}{|t|^{d/2}}$$

$k = \frac{d}{2}$ if d is even

$k = \frac{d+1}{2}$ if d is odd

$(t \gg N^{-2})$

Lec 9 08/02/16

- subcase 2.b : $|x| \geq tN$

- radial function on $\mathbb{R}^d$: $f(x) = f_0(|x|)$

FACT: Let f be radial. Then,

$$\widehat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{d-2}{2}}} \int_0^\infty f_0(r)\, J_{\frac{d-2}{2}}(2\pi|\xi| r)\, r^{\frac{d}{2}}\, dr$$

J_ν = Bessel function of ord ν

See Appendix B of Grafakos.

- $J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + \underbrace{O(r^{-3/2})}_{\text{error}}$ as $r \to \infty$

$$\Rightarrow P_N K_t(x) \sim \frac{N^{\frac{d+2}{2}}}{|x|^{\frac{d-2}{2}}} \int_{r\sim 1} e^{-4\pi^2 i t N^2 r^2} \Psi(r)\, J_{\frac{d-2}{2}}(2\pi N|x| r)\, r^{\frac{d}{2}}\, dr.$$

Ψ: $\operatorname{supp}\Psi \subset [\frac{1}{2}, 2]$

- **error term.**

$$\lesssim \frac{N^{\frac{d+2}{2}}}{|x|^{\frac{d-2}{2}}} \frac{1}{(N|x|)^{3/2}} = \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d+1}{2}}} \cdot \frac{1}{N} \overset{|x| \geq t N}{\lesssim} \frac{1}{|t|^{\frac{d+1}{2}}} \frac{1}{N}$$

$$\lesssim_{t \gg N^{-2}} \frac{1}{|t|^{d/2}}.$$

- **Main contribution:**

$$I_t^{\pm}(x) = \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} \int_{r \sim 1} e^{-4\pi^2 i t N^2 r^2} e^{\pm 2\pi i N|x| r} \underbrace{\psi(r)\, r^{\frac{d-1}{2}}}_{\text{smooth func}} dr$$

$$\overset{\text{cor to van der Caput}}{\lesssim} \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} \frac{1}{(tN^2)^{1/2}} \leq \frac{N}{|t|^{\frac{d-1}{2}}} \frac{1}{t^{1/2} N} = \frac{1}{|t|^{d/2}}.$$

□

Rmk: dispersive estimate (with P_N) for the wave eqn can be obtained in a similar but simpler argument.

only need $\widehat{d\sigma}(\xi) = \int_{S^{d-1}} e^{-2\pi i \xi \cdot \omega} d\sigma(\omega) = \frac{2\pi}{|\xi|^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(2\pi|\xi|)$

See Monica Vişan's lec note (Oberwolfach)

(4.4) Maximal function estimate

$|\nabla|^{\frac{1}{4}}$

Prop: $\|S(t)f\|_{L^4_x L^\infty_t} \lesssim \|D^{1/4} f\|_{L^2_x}$ $\quad d=1$ $\quad D=|\nabla|$

$$\left(\left\| D^{-1/2} \int_0^t S(t-t') F(t)\, dt' \right\|_{L^4_x L^\infty_t} \lesssim \|F\|_{L^{4/3}_x L^1_t} \right.$$

$$\|S(t)f(x)\|_{L^\infty_t} = \sup_{t\in\mathbb{R}} |S(t)f(x)| \leftarrow \text{maximal function}$$

Q: $u(t,x) = S(t)f(x) \longrightarrow f(x)$, a.e. ?

Carleson '80, $f \in H^s(\mathbb{R})$

Dahlberg - Kenig '82: false if $s < \frac{1}{4}$

Sjölin - Vega '85: $s > 1/2$, $\forall d$.

- Bourgain '90's, Vargas-Vega, Tao - V - V, Tao - Vargas
- Sanghyuk Lee '06: $d=2$, $s > 3/8$
- Bourgain '12: $d \geq 3$ suff. cond. $s > \frac{1}{2} - \frac{1}{4d}$
  $d \geq 4$ nec. cond $s \geq \frac{1}{2} - \frac{1}{2d}$

Pf of a.e. conv

$$\limsup_{t\downarrow 0} |S(t)f(x) - f(x)| \le \sup_t |S(t)(f(x)-g(x))| + |f(x)-g(x)| + \limsup_{t\downarrow 0} |S(t)g(x) - g(x)|$$

g smooth

$= 0.$

- Given $\alpha > 0$, let

$$A = \{x \in \mathbb{R} : \limsup_{t\downarrow 0} |S(t)f(x) - f(x)| > \alpha\}$$

$$\subset \{x \in \mathbb{R} : \sup_t |S(t)(f(x)-g(x))| > \tfrac{\alpha}{2}\} \cup \{|f(x)-g(x)| > \tfrac{\alpha}{2}\} =: B \cup C$$

$$|B| = \int_x \mathbb{1}_B \, dx \le \int_x \left(\tfrac{2}{\alpha}\sup_t |\cdot|\right)^r dx \quad (\le 1)$$

↑ Chebyshev's ineq.

$$\Rightarrow |A| \le |B| + |C| \le \left(\tfrac{2}{\alpha}\right)^4 \left\| \sup_t |S(t)(f(x)-g(x))| \right\|_{L^4_x} + \left(\tfrac{2}{\alpha}\right)^4 \|f(x)-g(x)\|_{L^4_x}$$

Max. func est.

$$\overset{\text{Sob}}{\le} C\left(\tfrac{2}{\alpha}\right)^4 \|f-g\|_{\dot{H}^{\frac{1}{4}}(\mathbb{R})} \quad (< \varepsilon)$$

$$\le \frac{\varepsilon}{\alpha^4}$$

$\dot{H}^{\frac{1}{4}} \subset L^4$ $\quad \frac{1}{4} = \frac{1}{2} - \frac{1}{4}$ $\quad$ (5/d)

true for any $\varepsilon > 0$. $\Rightarrow |A| = 0$.

Lec 10 10/02/16

(*) **Maximal function estimate** ($d=1$):

$$\| S(t)f \|_{L_x^4 L_t^\infty} \lesssim \| D^{\frac{1}{4}} f \|_{L_x^2} = \| f \|_{\dot{H}^{\frac{1}{4}}(\mathbb{R})}$$

$D = |\partial_x| \sim |\xi|$

Pf: (*) $\Longleftrightarrow$

$$\| \int_{\mathbb{R}} D^{-\frac{1}{4}} S(-t) G(t)\, dt \|_{L_x^2} \lesssim \| G \|_{L_x^{4/3} L_t^1}$$

$$(\text{LHS})^2 = \int\int \overline{G(x,t)} \left[\int_{\mathbb{R}} D^{-1/2} S(t-t') G(t')\, dt' \right] dt\, dx$$

$$\left(\Longleftarrow \int_x D^{-\frac{1}{4}} \int_{t'} S(-t') G(t')\, dt' \overline{\int_t D^{-\frac{1}{4}} S(-t) G(t)\, dt}\; dx \right.$$

Hölder

$$\leq \| G \|_{L_x^{4/3} L_t^1} \| \quad \|_{L_x^4 L_t^\infty}$$

$\Longrightarrow$ Suffices to show

(**) $$\| \int_{\mathbb{R}} D^{-1/2} S(t-t') G(t')\, dt' \|_{L_x^4 L_t^\infty} \lesssim \| G \|_{L_x^{4/3} L_t^1}$$

Lemma: $\left| \int_{\mathbb{R}} e^{i(x\xi - t\xi^2)} \frac{1}{|\xi|^{1/2}} \, d\xi \right| \lesssim \frac{1}{|x|^{1/2}}$

Assuming Lemma, we prove (**)

$S(t) f = K_t *_x f$

$$\left\| \int_{\mathbb{R}} D^{-1/2} S(t-t') G(t') \, dt' \right\|_{L_x^4 L_t^\infty} \longrightarrow \int_y \int_{t'} D^{-1/2} K(t-t', x-y) G(t', y) \, dt' \, dy$$

$t - t'$: convol in t'.

$\frac{1}{\infty} + 1 = \frac{1}{\infty} + 1$

$$\lesssim \left\| \frac{1}{|x|^{1/2}} *_x \int_{\mathbb{R}} |G(t')| \, dt' \right\|_{L_x^4}$$

Young's & Lemma

$\frac{1}{4} + 1 = \frac{3}{4} + \frac{1}{2}$

H-L-S

$$\lesssim \|G\|_{L_x^{4/3} L_t^1}.$$

$\left\| \frac{1}{|x|^{1/2}} \right\|_{L_x^2(\mathbb{R})} = \infty$ but H-L-S works!

$\square$

Pf of Lemma:

① $|\xi| \lesssim |x|^{-1}$

$$(LHS) \lesssim |\xi|^{1/2} \Big|_0^{|x|^{-1}} \lesssim \frac{1}{|x|^{1/2}}$$

② $|\xi| \gg |x|^{-1}$: We only consider $\xi \gtrsim |x|^{-1}$.

Change of var: $\xi^2 = \eta$

$$(LHS) = \int e^{ix\eta^{1/2} - it\eta} \frac{1}{\eta^{3/4}} d\eta \quad (+)$$

$\eta \gtrsim |x|^{-2}$

$i\partial_t u = D^{1/2} u$

Let $\Phi(\eta) = \eta^{1/2}$. $\Phi'(\eta) \sim \eta^{-1/2}$

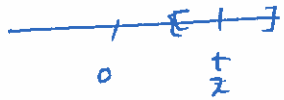
$\Phi''(\eta) \sim \eta^{-3/2}$.

$\Phi(\eta) \cdot x - t\eta$

2.a $\Omega_1 = \{ \eta \gtrsim |x|^{-2} : |\Phi'(\eta) - \frac{t}{x}| \le |\frac{t}{2x}| \}$

$\Rightarrow \Phi'(\eta) \sim \frac{t}{x} \Rightarrow \eta \sim (\frac{x}{t})^2$.

$\Phi'(\eta) \sim \eta^{-1/2}$



$$\oplus \sim \int e^{\underbrace{ix\phi - it\eta}} |\phi''|^{1/2}\, d\eta$$

$$x \min \Phi'' \cdot \boxed{\frac{\Phi}{\min \Phi''}}$$

$$\Phi(\eta) = \phi(\eta) - \frac{t}{x}\eta$$

$$\Phi'' = \phi''$$

$$\underbrace{\phi'' \sim \left(\frac{t}{x}\right)^3}_{\eta^{-3/2}}$$

$$(\quad)'' \geq 1$$

Cor to van der Corput

$$\lesssim \Big(\underbrace{(\min \phi'')\, x}_{\substack{\wr \\ \frac{x}{t^{3/2}}}}\Big)^{-1/2} \Big\{ \underbrace{\max |\phi''|^{1/2}}_{\sim \left(\frac{t}{x}\right)^{3/2}} + \underbrace{\int |\phi''|^{-1/2} |\phi'''|\, d\eta}_{\sim \left(\frac{x}{t}\right)^{3/2} \left(\frac{t}{x}\right)^5 \left(\frac{x}{t}\right)^2 = \left(\frac{t}{x}\right)^{3/2}} \Big\}$$

$$\phi''' \sim \eta^{-5/2}$$

$$\lesssim \frac{1}{|x|^{1/2}}.$$

2.b $\quad \Omega_2 = \Omega_1^c \cap \{ \eta \gtrsim |x|^{-2} \}$

$$\Rightarrow \left|\phi' - \frac{t}{x}\right| \overset{\text{tri}}{\geq} |\phi'| - \left|\frac{t}{x}\right|$$

$$\geq |\phi'| - 2\left|\phi' - \frac{t}{x}\right|$$

b/c $\quad 2\left|\phi' - \frac{t}{x}\right| \geq \left|\frac{t}{x}\right|$

$$\Rightarrow \underbrace{|\Phi' - \tfrac{t}{x}|}_{\| \; |\Phi'|} \gtrsim |\Phi'| \sim \eta^{-1/2}$$

$$\Phi'' \sim \eta^{-3/2}$$

$$\int e^{i\underbrace{x\Phi}_{\| \; \Psi}} \frac{1}{\eta^{3/4}}\, d\eta = \int \partial_\eta e^{i\Psi(\eta)} \frac{|\Phi''|^{1/2}}{i\,\partial_\eta \Psi(\eta)}$$

$$\overset{\text{IBP}}{\leq} \int \left| \partial_\eta \left(\frac{|\Phi''|^{1/2}}{\partial_\eta \Psi(\eta)} \right) \right| d\eta$$

$$\lesssim \frac{1}{|x|} \int \underbrace{\frac{1}{|\Phi''|^{1/2}} \frac{|\Phi'''|}{|\Phi' - \frac{t}{x}|}}_{\sim \eta^{-5/4}} + \underbrace{\frac{|\Phi''|^{3/2}}{|\Phi' - \frac{t}{x}|^2}}_{\sim \eta^{-5/4}} d\eta$$

$\frac{3}{2}(-\frac{3}{2}) + 2(\frac{1}{2})$

$$\sim -\frac{1}{|x|}\, \eta^{-\frac{1}{4}} \Big|_{|x|^{-2}}^{\infty} = \frac{1}{|x|^{1/2}}.$$

$\square$

(4.5) local smoothing estimates

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = f \in H^s(\mathbb{R}^d) \end{cases} \Rightarrow u(t) = S(t)f \in H^s(\mathbb{R}^d)$$

NOT smoother than f.

Prop 1 (local smoothing, $d=1$)

$$\| D^{1/2} S(t) f \|_{L^\infty_x L^2_t} \sim \| f \|_{L^2_x}$$

Pf: $D^{1/2} S(t) f(x) = \int |\xi|^{1/2} e^{ix\xi - it\xi^2} \hat{f}(\xi)\, d\xi$ $\quad \eta = \xi^2$

$$\sim \int |\eta|^{-\frac{1}{4}} e^{ix\eta^{1/2} - it\eta} \hat{f}(\eta^{1/2})\, d\eta \quad \frac{1}{2}\frac{1}{\eta^{1/2}} d\eta = d\xi$$

$$\Rightarrow \| D^{1/2} S(t) f(x) \|_{L^2_t} \overset{\text{Plancherel}}{\sim} \| |\eta|^{-\frac{1}{4}} e^{ix\eta^{1/2}} \hat{f}(\eta^{1/2}) \|_{L^2_\eta} = \left(\int \frac{1}{|\eta|^{\frac{1}{2}}} |\hat{f}(\eta^{1/2})|^2 d\eta \right)^{1/2}$$

$$\sim \left(\int |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \overset{\text{Plancherel}}{=} \| f \|_{L^2_x}$$

□

Rmk: ∃ proof w.o. F.T. (short note by T. Tao)

" A physical space proof of..

Prop 2:

$$\circledast \quad \left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| |\nabla|^{1/2} S(t) f(x) \right|^2 e^{-|x|^2} dx\, dt \right)^{1/2} \lesssim \|f\|_{L^2_x}$$

$$\circledast\circledast \quad \left(\Longrightarrow \quad \| |\nabla|^{1/2} S(t) f \|_{L^2_{t,x}(\mathbb{R}_t \times B_R)} \lesssim R^{1/2} \|f\|_{L^2_x} \right.$$

$$\left(\Longleftarrow \quad \sup_x \| |\nabla|^{1/2} S(t) f(x) \|_{L^2_t} \lesssim \|f\|_{L^2_x} \overset{\text{Prop 1 } (1-d)}{\Longleftarrow} \right.$$

Note: $\circledast \Rightarrow \circledast\circledast$

but also Prop 1 $\Rightarrow \circledast\circledast$ when $d=1$.

Schur's test:

$$Tf(x) = \int_Y K(x,y)\, f(y)\, dy$$

Suppose

$$\sup_x \int_Y K(x,y)\, \overset{q(y)}{}\, dy \le \alpha \quad p(x)$$

$$\sup_y \int_X K(x,y)\, \underset{p(x)}{}\, dx \le \beta \quad q(y)$$

$$\Longrightarrow \|T\|_{L^2_y \to L^2_x} \le \sqrt{\alpha\beta}$$

Pf: $(p \equiv q \equiv 1.)$

$$|Tf(x)|^2 = \Big|\int_Y \underset{\substack{\| \\ K^{1/2} K^{1/2}}}{K(x,y)}\, f(y)\, dy\Big|^2 \overset{C\text{-}S}{\le} \underbrace{\Big(\int_Y K(x,y)\, dy\Big)}_{\le \alpha}\Big(\int_Y K(x,y)\, f^2(y)\, dy\Big)$$

$\int \cdot\, dx$

$$\Longrightarrow \|Tf\|^2_{L^2_x} \le \alpha \int_Y \underbrace{\Big(\int_X K(x,y)\, dx\Big)}_{\le \beta} |f(y)|^2\, dy = \alpha\beta \|f\|^2_{L^2}$$

$\boxtimes$

Lec 11 22/02/16

Prop 2: $\left(\int_{\mathbb{R}_t} \int_{\mathbb{R}^d_x} |\nabla|^{1/2} S(t) f(x)|^2 e^{-|x|^2} dx\, dt \right)^{1/2} \lesssim \|f\|_{L^2_x}$

Pf: $a(x) : \mathbb{R}^d \to [0,\infty)$ $\quad a(x) = e^{-|x|^2}$

$$\int_{\mathbb{R}\times\mathbb{R}^d} |\nabla|^{1/2} S(t) f(x)|^2 a(x)\, dx\, dt$$

$$= \int_{\mathbb{R}\times\mathbb{R}^d} \iint_{\mathbb{R}^d\,\mathbb{R}^d} e^{ix\xi} e^{-it|\xi|^2} |\xi|^{1/2} \hat{f}(\xi) \cdot e^{-ix\eta} e^{it|\eta|^2} |\eta|^{1/2} \overline{\hat{f}(\eta)}\, a(x)\, d\xi d\eta\, dx\, dt$$

$e^{-ix(\eta-\xi)}$ $\qquad$ $e^{-it(|\xi|^2-|\eta|^2)} = e^{it(|\xi|+|\eta|)(|\xi|-|\eta|)}$

integrate in x and t

$$= \int_{\mathbb{R}^d_\xi \times \mathbb{R}^d_\eta} \hat{a}(\eta-\xi)\, \delta(|\xi|^2-|\eta|^2)\, |\xi|^{1/2} |\eta|^{1/2} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi d\eta$$

$\mathcal{F}(1) = \delta$ (Dirac delta) $= \mathcal{F}^{-1}(1)$

$\delta(|\xi|^2-|\eta|^2) = \delta((|\xi|+|\eta|)(|\xi|-|\eta|))$

$= \dfrac{\delta(|\xi|-|\eta|)}{|\xi|+|\eta|}$

$\delta(t)$ ← Dirac delta on 1-d

$c\delta(ct)$

as limit of $\varepsilon^{-1}\phi(\varepsilon^{-1}t)$, $\int \phi\, dt = 1$

change of var

$$= \int_{\mathbb{R}^d} \overline{\hat{f}(\eta)} \underbrace{\left(\int_{\mathbb{R}^d} K(\eta, \xi) \hat{f}(\xi) d\xi \right)} d\eta$$

$$\left(K(\eta, \xi) = \hat{a}(\eta - \xi)\, \delta(|\xi| - |\eta|) \frac{|\xi|^{1/2} |\eta|^{1/2}}{|\xi| + |\eta|} \right.$$

WANT: $L^2 \to L^2$

$$\leq \|\hat{f}\|_{L^2_\eta} \left\| \int_{\mathbb{R}^d} K(\eta, \xi) \hat{f}(\xi) d\xi \right\|_{L^2_\eta}$$

<u>Use Schur's test</u>: (K is basically symmetric in ξ & η)

$$\sup_\eta \int |K(\eta, \xi)| d\xi = \sup_\eta \int \hat{a}(\eta - \xi)\, \delta(|\xi| - |\eta|) \frac{|\xi|^{1/2} |\eta|^{1/2}}{|\xi| + |\eta|} d\xi$$

polar coord

$$\sim \int_{S^{d-1}} \int_0^\infty \underbrace{e^{-c|r\omega - \eta|^2}}_{= e^{-c|\eta|^2 \left|\omega - \frac{\eta}{|\eta|}\right|^2}} \delta(|\eta| - r) \boxed{\frac{r^{1/2} |\eta|^{1/2} r^{d-1}}{r + |\eta|}} dr\, d\sigma(\omega)$$

$\xi = r\omega$, $\omega \in S^{d-1}$

$\sigma(\omega)$: surface meas on S^{d-1}

integrate in r.

$\sim |\eta|^{d-1}$

$$\int_{S^{d-1}} e^{-c|\eta|^2 |\omega - \frac{\eta}{|\eta|}|^2} |\eta|^{d-1} \, d\sigma(\omega)$$ ← invariant under rotation

WLOG, assume $\frac{\eta}{|\eta|} = \vec{e}_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$.

$$\begin{aligned}
x_1 &= r\cos\phi_1 \\
x_2 &= r\sin\phi_1 \cos\phi_2 \\
x_3 &= r\sin\phi_1 \sin\phi_2 \cos\phi_3 \\
&\vdots \\
x_{d-1} &= r\sin\phi_1 \cdots \sin\phi_{d-2}\cos\phi_{d-1} \\
x_d &= r\sin\phi_1 \cdots \sin\phi_{d-1}
\end{aligned}$$

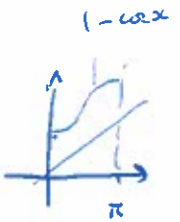
$\phi_j \in [0, \pi]$, $j = 1, \cdots, d-2$

$\phi_{d-1} \in [0, 2\pi]$

$$\int_{\phi_{d-1}=0}^{2\pi} \int_{\phi_{d-2}=0}^{\pi} \cdots \int_{\phi_1=0}^{\pi} \cdots \sin^{d-2}\phi_1 \sin^{d-3}\phi_2 \cdots \sin\phi_{d-2} \, d\phi_1 \cdots d\phi_{d-1}$$

$$\lesssim \int_{\phi_1=0}^{\pi} e^{-c|\eta|^2 |1-\cos\phi_1|^2} |\eta|^{d-1} \sin^{d-2}\phi_1 \, d\phi_1$$

Choose $c > 0$ s.t. $1 - \cos x \geq cx$ for all $x \in [0, \pi)$



$$\lesssim \int_{\phi_1=0}^{\pi} e^{-\tilde{c}|\eta|^2|\phi_1|^2} |\eta|^{d-1} \phi_1^{d-2} \, d\phi_1$$

Change of var: $\zeta = |\eta|\phi_1$

$|\eta|^{d-1} \phi_1^{d-2} \, d\phi_1 \sim \zeta^{d-2} d\zeta$

$$\lesssim \int_0^{\infty} e^{-\hat{c}\zeta^2} \zeta^{d-2} \, d\zeta \lesssim 1 \implies \text{Apply Schur's test.}$$

□

(4.6) generalized KdV (gKdV) equation

Consider $\begin{cases} \partial_t u = \partial_x^3 u + \partial_x (u^k) \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}) \end{cases}$ $x \in \mathbb{R}$.

$k=2$: KdV ← 1-d shallow water wave

$k=3$: modified KdV (mKdV)

$$s_{crit} = \frac{1}{2} - \frac{2}{k-1}$$

$k=5$: L^2- critical gKdV (quintic)

Well-posedness
$s \geq -\frac{3}{4}$ Kenig-Ponce-Vega '96
Guo, Kishimoto ~'10?

mKdV: $s \geq \frac{1}{4}$ Kenig-Ponce-Vega '93
CPAM
Comm pure applied math

- Airy equation: $\begin{cases} \partial_t u = \partial_x^3 u \\ u|_{t=0} = u_0 \end{cases}$ $x \in \mathbb{R}$

Lem 1: (local smoothing)

$$\| \partial_x S(t) u_0 \|_{L_x^\infty L_t^2} \lesssim \| u_0 \|_{L_x^2}$$

HW

↑ one full deriv.

$S(t) = e^{t\partial_x^3}$ ↑ unitary on H^s ↓ $e^{-it\frac{\partial_x^3}{3}}$

dual: $\sup_t \| \partial_x \int_{\mathbb{R}} S(t-t') F(t') \, dt' \|_{L^2_x} \lesssim \| F \|_{L^1_x L^2_t}$

$\|$

$\| \partial_x \int_{\mathbb{R}} S(-t') F(t') \, dt' \|_{L^2_x}$

$$(LHS) \overset{\text{duality}}{=} \sup_{\|f\|_{L^2_x} \le 1} \left| \int_x f(x) \overline{\int_{t'} \partial_x S(-t') F(t') \, dt' \, dx} \right|$$

$$= \sup \left| \iint \partial_x S(t) f \, \overline{F(t')} \, dt' \, dx \right|$$

$$\overset{\text{Hölder}}{\le} \sup \underbrace{\| \partial_x S(t) f \|_{L^\infty_x L^2_{t'}}}_{\le C \|f\|_{L^2_x} \lesssim 1 \text{ dx. smoothing}} \| F \|_{L^1_x L^2_{t'}}.$$

$$\lesssim \| F \|_{L^1_x L^2_{t'}}$$

$$\| \partial_x \int_0^t S(t-t') F(t') \, dt' \|_{L_t^\infty L_x^2} \overset{\text{unitarity}}{=} \| \int_0^t \partial_x S(-t') F(t') \, dt' \|_{L_t^\infty L_x^2}$$

$$= \sup_t \| \int_{\mathbb{R}} \partial_x S(-t') \, \mathbb{1}_{[0,t)}(t') \, F(t') \, dt' \|_{L_x^2}$$

$$\overset{\text{dual}}{\lesssim} \sup_t \| \mathbb{1}_{[0,t)}(t') F(t') \|_{L_x^1 L_{t'}^2} \leq \| F \|_{L_x^1 L_{t'}^2}$$

- TT^*

$$\Rightarrow \| \partial_x^2 \int_{\mathbb{R}} S(t-t') F(t') dt' \|_{L_x^\infty L_t^2} \lesssim \| F \|_{L_x^1 L_t^2}.$$

(x outside!)

$$\left((\text{LHS}) \underset{\text{loc smoothing}}{\lesssim} \| \partial_x \int_{\mathbb{R}} S(-t') F(t') \, dt' \|_{L_x^2} \overset{\text{dual}}{\lesssim} \| F \|_{L_x^1 L_t^2} \right)$$

$$\Rightarrow \| \partial_x^2 \int_0^t S(t-t') F(t') dt' \|_{L_x^\infty L_t^2} \lesssim \| F \|_{L_x^1 L_t^2}$$

↑ Non-trivial

See KPV '93 CPAM Theorem 3.5.

Lem 2 (maximal func esti) $\|S(t)u_0\|_{L_x^4 L_t^\infty} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{4}}(\mathbb{R})}$

(same index as the Schrödinger equation)

· Lem 3 - ① $\|S(t)u_0\|_{L_x^5 L_t^{10}} \lesssim \|u_0\|_{L_x^2}$

② $\left\|\int_0^t S(t-t')F(t')\,dt'\right\|_{L_x^5 L_t^{10}} \lesssim \|F\|_{L_x^{5/4} L_t^{10/9}}$.

$\Leftarrow$ Lem 1 and Lem 2. with Stein's interpolation thm.

Lec 12 24/02/16

Lem 3 ① $\| \underset{\partial_x^0}{\uparrow} S(t) u_0 \|_{L^5_x L^{10}_t} \lesssim \| u_0 \|_{L^2_x}$

② $\left\| \int_0^t S(t-t') F(t') \, dt' \right\|_{L^5_x L^{10}_t} \lesssim \| F \|_{L^{5/4}_x L^{10/9}_t}$

local smoothing : $\| \partial_x S(t) u_0 \|_{L^\infty_x L^2_t} \lesssim \| u_0 \|_{L^2_x}$

max func estimate : $\| |\partial_x|^{-\frac{1}{4}} S(t) u_0 \|_{L^4_x L^\infty_t} \lesssim \| u_0 \|_{L^2_x}$

FALSE: "interpolate" $\frac{1}{5} = \frac{1-\theta}{\infty} + \frac{\theta}{4}$ $\quad \theta = \frac{4}{5}$

$$S = (1-\theta)\,1 + \theta \cdot \left(-\frac{1}{4}\right) = \frac{1}{5} - \frac{1}{5} = 0$$

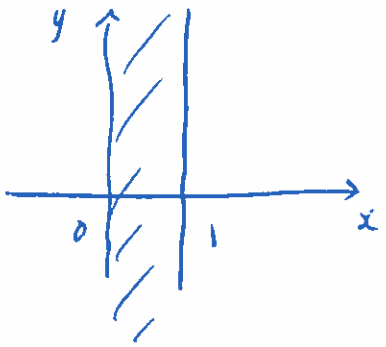
$$\frac{1-\theta}{2} + \frac{\theta}{\infty} = \frac{1}{10}$$

Three lines theorem (Phragmen-Lindelöf thm)

← Hadamard three circle thm

F conti, bdd on

$$S = \{ z = x+iy : 0 \le x \le 1 \}$$



F, analytic in $\mathring{S}$ = interior of S

Suppose $\forall y \in \mathbb{R}$,

$$|F(iy)| \le M_0, \quad |F(1+iy)| \le M_1$$

Then, $|F(x+iy)| \le M_0^{1-x} M_1^{x}$, $\forall z \in x+iy \in S$.

- **Stein's interpolation thm**: $z \in S \Rightarrow T_z$, lin operator

s.t. $(T_z f) g \in L^1(Y, \nu)$

simple $f \in L^1(X, \mu)$

simple $g \in L^1(Y, \nu)$

We say $\{T_z\}_{z \in S}$ is admissible if ① $z \longmapsto \int_Y (T_z f) g \, d\nu$ is analytic in $\mathring{S}$, conti on S

② $\exists\, a < \pi$ s.t.

$$\sup_{z \in S} e^{-a|y|} \log \left| \int_Y (T_z f) g \, d\nu \right| \le C_{f,g} < \infty$$

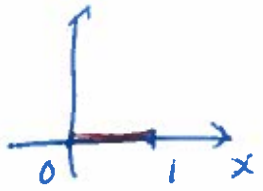
Thm (Stein) $\{T_z\}_{z \in S}$ adm.

$$\|T_{0+iy} f\|_{L^{q_0}} \le M_0(y) \|f\|_{L^{p_0}}$$

$$\|T_{1+iy} f\|_{L^{q_1}} \le M_1(y) \|f\|_{L^{p_1}}$$

$\forall$ simple f on X.

$$\sup_{y \in \mathbb{R}} e^{-b|y|} \log M_j(y) < \infty \quad \text{for some } b < \pi.$$



Let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $0 < \theta < 1$

Then, $\|T_\theta f\|_{L^q} \le M(\theta) \|f\|_{L^p}$

more complicated than Riesz–Thorin

$$M(\theta) = \exp\left\{ \frac{\sin \pi\theta}{2} \int_{\mathbb{R}} \left[\frac{\log M_0(y)}{\cosh \pi y - \cos \pi\theta} + \frac{\log M_1(y)}{\cosh \pi y + \cos \pi\theta} \right] dy \right\}$$

Pf of ① $T_z u_0 = D^{-z/4} D^{1-z} S(t) u_0$, $z \in S$

$D = |\partial_x| \sim |\xi|$

- $z = 0 + iy$

$$T_{0+iy} u_0 = \frac{\partial}{\partial x} S(t) \boxed{D_x^{-\frac{5}{4}iy} H u_0}$$

$H f(x) = p.v. \int \frac{f(y)}{x-y} dy$ ($|x-y| > \varepsilon \to \varepsilon$)

Hilbert transform: $H \sim (-i)\,\mathrm{sgn}(\xi)$

loc smoothing ⇒

$$\| T_{iy} u_0 \|_{L_x^\infty L_t^2} \lesssim \| D_x^{-\frac{5}{4}iy} H u_0 \|_{L_x^2}$$

$|\xi| = i\xi \cdot (-i)\,\mathrm{sgn}(\xi)$

$D = \partial_x H$

$|\partial_x|$

Plancherel $= \| u_0 \|_{L_x^2}$.

- $z = 1 + iy$

$$T_{1+iy} u_0 = D^{-\frac{1}{4}} S(t) D^{-\frac{5}{4}iy} u_0$$

max func esti ⇒

$$\| T_{1+iy} u_0 \|_{L_x^4 L_t^\infty} \lesssim \| u_0 \|_{L_x^2}$$

→ Interpolate (by Stein)

$\theta = 4/5$:

$$T_{4/5} u_0 = S(t) u_0 \Rightarrow \| S(t) u_0 \|_{L_x^5 L_t^{10}} \lesssim \| u_0 \|_{L_x^2}$$

□

Dual: $\left\| \int_{\mathbb{R}} S(t-t') F(t') dt' \right\|_{L^2_x} \lesssim \|F\|_{L^{5/4}_x L^{10/9}_t}$.

TT^*: $\left\| \int_{\mathbb{R}} S(t-t') F(t') dt' \right\|_{L^5_x L^{10}_t} \lesssim \|F\|_{L^{5/4}_x L^{10/9}_t}$.

↑ Replacing by $\int_0^t$ is highly non-trivial.
complicated involving $BMO_x(L^2_t)$
see Cor 3.8 in KPV '93 CPAM.

small data GWP in $L^2(\mathbb{R})$ of L^2-critical gKdV

let $\Gamma_{u_0}(u) = S(t)u_0 + \int_0^t S(t-t') \partial_x (u^5)(t') dt'$. Scrit = 0.

① Want $u = \Gamma(u)$ in $C_t L^2_x$.

$$\|\Gamma(u)\|_{L^\infty_t L^2_x} \le \|u_0\|_{L^2_x} + \left\| \partial_x \int_0^t S(t-t') u^5(t') dt' \right\|_{L^\infty_t L^2_x}$$

dual + local smoothing

$$\le \|u_0\|_{L^2_x} + C \|u^5\|_{L^1_x L^2_t}$$

$$\overset{\text{Hölder}}{\le} \|u_0\|_{L^2_x} + C \|u\|^5_{L^5_x L^{10}_t}$$

- $\|\Gamma(u)\|_{L_x^5 L_t^{10}} \overset{\text{Lem3 ①②}}{\lesssim} \|u_0\|_{L_x^2} + \underbrace{\|u^4 \partial_x u\|_{L_x^{5/4} L_t^{10/9}}}_{\text{Hölder}} \le \|u\|^4_{L_x^5 L_t^{10}} \boxed{\|\partial_x u\|_{L_x^\infty L_t^2}}$

  $\boxed{\|\partial_x u\|_{L_x^\infty L_t^2}}$: local smoothing space

- $\|\partial_x \Gamma(u)\|_{L_x^\infty L_t^2} \overset{\text{local smoothing Lem1}}{\lesssim} \|u_0\|_{L_x^2} + \|u^5\|_{L_x^1 L_t^2} \overset{\text{Hölder}}{\le} \|u_0\|_{L_x^2} + \|u\|^5_{L_x^5 L_t^{10}}$

Define $X = L_t^\infty L_x^2 \cap L_x^5 L_t^{10} \cap \{\|\partial_x u\|_{L_x^\infty L_t^2} < \infty\}$

$\Rightarrow$ ① $\|\Gamma(u)\|_X \le C_0 \|u_0\|_{L^2} + C_1 \|u\|_X^5$

By a similar argument, $u^5 - v^5 = (u-v)(u^4 + \cdots + v^4)$

② $\|\Gamma(u) - \Gamma(v)\|_X \le C_2 \left(\|u\|_X^4 + \|v\|_X^4\right) \|u - v\|_X$

Let $\|u_0\|_{L^2_x} \leq \delta$.

- $\overline{B_R} \subset X$ with $R = 2C_0\delta$. Let $u \in \overline{B_R}$

① $\|\Gamma(u)\|_X \leq C_0\delta + C_1 R^5$

$= C_0\delta + \underbrace{C_1(2C_0\delta)^5}_{\leq C_0\delta} \overset{\text{WANT}}{\leq} R = 2C_0\delta$

$\Leftarrow$ Suffices to choose $\delta \ll 1$ s.t. $C_1(2C_0\delta)^5 \leq C_0\delta$.

② $\|\Gamma(u) - \Gamma(v)\|_X \leq \underbrace{2C_2R^4}_{\| \; 2C_2(2C_0\delta)^4 \leq \frac{1}{2} \leftarrow \text{true for } \delta \ll 1} \|u - v\|_X$

$\Rightarrow$ Γ is a contraction on $\overline{B_R} \subset X$.

$\Rightarrow$ $u = \Gamma_{u_0}(u)$ is a global solution to gKdV ($k=5$),

provided $\|u_0\|_{L^2(\mathbb{R})} \ll 1$.

Lec 13 29/02/16 (Mon)

No lectures in the weeks of Mar. 14, 21.

Sec 5: Global-in-time behavior of solns to NLS

We "proved" LWP of NLS if $s \geq \max(s_{crit}, 0)$.

Q1: Does the soln exist globally (in time)? Global well-posedness
Or does it cease to exist at (before) some finite time? finite time blowup solutions

Q2: If u exists globally in time, then what is the behavior of u as $t \to \pm\infty$?

- scattering: "asymptotic linear behavior" ("$\Rightarrow$" $\|u(t)\|_{L^\infty_x} \to 0$.)

  $\exists\, u_\pm \in H^s(\mathbb{R}^d)$ s.t. $\lim_{t \to \pm\infty} \| u(t) - \underbrace{S(t) u_\pm}_{\text{lin. soln}} \|_{H^s} = 0$.

  (but $\neq S(t) u_0$)

- non scattering soln such as soliton: $u(t) = e^{it} Q(x)$

  ↑ basically keeps the same profile

Conjecture: Soliton resolution conjecture

"$u(t)$ decouples into a sum of solitons.

+ radiation (= scattering part) as $t \to \pm\infty$"

Still open: except for "integrable equations" such as KdV.

and NLW (Kenig-Merle et al. '12?)

(5.1) **Conservation laws**: $i\partial_t u + \Delta u = \pm |u|^{p-1}u$

$\left(\partial_t u = i\Delta u \mp i|u|^{p-1}u. \right)$

mass: $M(u) = \int_{\mathbb{R}^d} |u|^2 dx$ ← for KdV, some may call it "energy".

Claim: $M(u(t)) = M(u_0)$ if u is a soln to (NLS)

(smooth)

(For "rough" solns, we need to use well-posedness theory, conti dep.)

$$\partial_t \int |u|^2 = 2\,\mathrm{Re} \int \partial_t u \cdot \bar{u} = 2\,\mathrm{Re}\, i \int \Delta u \cdot \bar{u} \mp 2\,\mathrm{Re} \underbrace{i \int |u|^{p+1}}_{\text{purely imaginary}}$$

($u\bar{u}$)

$$= -2\,\mathrm{Re}\, i \int |\nabla u|^2.$$

$$= 0.$$

Rmk: mass $M \rightsquigarrow u \mapsto e^{i\theta} u$ (gauge invariance)

See HW3.

- Hamiltonian (energy): $H(u) = \underbrace{\frac{1}{2}\int |\nabla u|^2}_{\text{kinetic energy}} \pm \underbrace{\frac{1}{p+1}\int |u|^{p+1}}_{\text{potential energy}}$.

  $\frac{1}{2}\|u\|_{\dot{H}^1}^2$

  NLS can be written as a Hamil dynamics: $\partial_t u = -i \frac{\partial H}{\partial \bar{u}}$

  classical $H(p,q)$: $\partial_t p = \frac{\partial H}{\partial q}$, $\partial_t q = -\frac{\partial H}{\partial p}$ $\Rightarrow$ $\partial_t \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial H}{\partial (p,q)}$, $\sim i$

  - + sign: defocusing "lin dispersion & nonlin "cooperate" $\Rightarrow$ "expect" GWP
    $\Rightarrow H(u) \gtrsim \|u\|_{\dot{H}^1}^2$.
  - − sign: focusing $\Rightarrow$ $H(u)$ may not control $\|u\|_{\dot{H}^1}^2$.
    ↳ lin disp vs concentration by nonlinearity.

Critical regularity: $s_{crit} = \frac{d}{2} - \frac{2}{p-1}$.

- We say NLS is
  - mass-critical if $s_{crit} = 0$. $p = 1 + \frac{4}{d}$ — 1-d: quintic; 2-d: cubic
  - energy-critical if $s_{crit} = 1$. $p = 1 + \frac{4}{d-2}$ — $d = 1, 2$: no energy crit; 3-d: quintic; 4-d: cubic
  - energy-subcritical if $s_{crit} < 1$
  - energy-supercritical if $s_{crit} > 1$. (GWP of NLS/NLW is entirely open) (defocusing)

Claim: $H(u(t)) = H(u_0)$ if u is a soln to (NLS).

← related to time translation $u(t) \hookrightarrow u(t - t_0)$

- $\partial_t \frac{1}{2} \int |\nabla u|^2 = \mathrm{Re} \int \partial_t \nabla u \cdot \nabla \bar{u} \overset{\text{IBP}}{=} -\mathrm{Re} \int \partial_t u \, \Delta \bar{u}$

$= -\mathrm{Re}\, i \int |\Delta u|^2 \pm \mathrm{Re}\, i \int |u|^{p-1} u \, \Delta \bar{u}$

- $\pm \partial_t \left(\frac{1}{p+1} \int |u|^{p+1} \right) = \pm \frac{1}{2} \int |u|^{p-1} \underbrace{\partial_t (|u|^2)}_{2 \mathrm{Re}\, \partial_t u \, \bar{u}}$

$|u|^{p+1} = (u\bar{u})^{\frac{p+1}{2}}$

$= \pm \mathrm{Re}\, i \int |u|^{p-1} \bar{u} \, \Delta u$

$= \mp \mathrm{Re}\, i \int |u|^{p-1} u \, \Delta \bar{u}$

$- \mathrm{Re}\, i \int |u|^{2p}$

→ add to 0.

Aside: If $u \in C(\mathbb{R} : H^\infty)$, then u is also ∞-differentiable in time.

⇐ use $\partial_t u = \underbrace{i \Delta u}_{\in C_t H^\infty} \mp \underbrace{i |u|^{p-1} u}_{\in C_t H^\infty}$ and repeat. (by Sobolev)

$H^\infty = \bigcap_{s \in \mathbb{R}} H^s$

Momentum: $P[u] = \mathrm{Im}\int \bar{u}\nabla u \in \mathbb{R}^d$

$$= \int \frac{\bar{u}\nabla u - u\nabla\bar{u}}{2i} = -i\int \bar{u}\nabla u$$

~ related to spatial translation $u(t,x) \mapsto u(t, x+x_0)$

Claim: $P(u(t)) = P(u_0)$ if u is a soln to (NLS).

Aside: $\|u\|_{L^2_x} = 1 \Rightarrow |u(x)|^2 dx$ is a probability density.

- $\int x|u(x)|^2dx$ = "expected position".

↝ Galilean invariance → try to write it on the Fourier side

$$u(t,x) \mapsto e^{i\frac{\beta}{2}\cdot x} e^{i\frac{|\beta|^2}{4}t} u(t, x-\beta t)$$

shift by β on the Fourier side

$\beta \in \mathbb{R}^d$

- Plancherel: $\|\hat{u}\|_{L^2_\xi} = 1$

$\Rightarrow |\hat{u}(\xi)|^2 d\xi$ is also a prob. density.

$$P[u) \underset{\text{Parseval}}{=} \int \overline{\hat{u}(\xi)}\, \xi\, \hat{u}(\xi) d\xi = \int \xi |\hat{u}(\xi)|^2 d\xi = \text{"expected velocity"}$$

" momentum

(5.2) GWP and finite time blowup solns

- cubic NLS on $\mathbb{R}$. $s_{crit} = -1/2$, i.e. mass-subcritical.

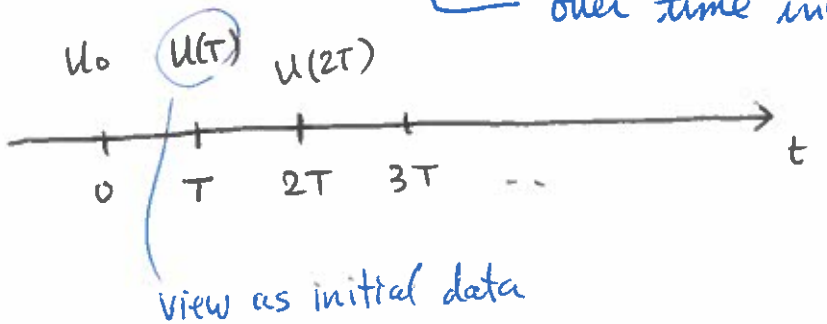
We proved LWP in $L^2(\mathbb{R})$, where local existence time $T \sim \|u_0\|_{L^2}^{-\theta}$, $\theta > 0$.
(in the subcritical sense)

But conservation of mass says

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}$$

(while the LWP argument gives $\|u(T)\|_{L^2} \leq 2\|u_0\|_{L^2}$)

$\Rightarrow$ can iterate LWP to construct global soln.
over time interval of length T.



Lec 14 02/03/16 (Wed)

- **cubic NLS on $\mathbb{R}$**, $s_{crit} = -1/2$ mass-subcritical

  GWP in $L^2(\mathbb{R})$ by iterating LWP & conservation of mass.

  $\Rightarrow$ holds for defocusing & focusing

Rmk: We proved LWP of cubic NLS on $\mathbb{R}^2$ in $L^2(\mathbb{R}^2)$

$s_{crit} = 0$ — mass-critical — critical space

local existence time $T \sim \| S(t) u_0 \|_{L_T^4 L_x^4}^{4} \; (\ll 1)$

↑ can not be quantified in terms of $\|u_0\|_{L^2_x}$

$\Rightarrow$ mass conservation alone does **not** yield GWP. (Dodson '12?)

- **cubic NLS on $\mathbb{R}^3$**, $s_{crit} = 1/2$. energy-subcritical

  defocusing case: $H(u) + M(u) = \frac{1}{2}\int |\nabla u|^2 + \underbrace{\frac{1}{4}\int |u|^4}_{\geq 0} + \int |u|^2 \gtrsim \|u\|_{H^1}^2$

  $\Rightarrow$ GWP in $H^1(\mathbb{R}^3)$. (local existence time $T \sim \|u_0\|_{H^1}^{-\theta}$

- Let $u_0 \in H^1(\mathbb{R}^3)$.

$$\Rightarrow \|u(t)\|_{H^1} \leq C\left(H(u(t)) + M(u(t))\right)^{1/2}$$

(use Sobolev)

$$\overset{\text{cons}}{=} C\left(H(u_0) + M(u_0)\right)^{1/2} =: K < \infty, \quad \forall\, t \in \mathbb{R}$$

- Choose local existence time $T \sim K^{-\theta}$.

$\Rightarrow$ Iterate LWP over intervals of length $T \rightarrow$ GWP.

but in the focusing case, $H(u) + M(u) \not\geq \|u\|_{H^1}^2$

and $\exists$ finite time blowup solns.

- quintic NLS on $\mathbb{R}$: $s_{crit} = 0$. (mass-crit but energy-subcrit.)

mass conservation alone does not yield GWP in $L^2(\mathbb{R})$ (Dodson '15?)

defocusing: conservation of mass & energy $\Rightarrow$ GWP in $H^1(\mathbb{R})$.

• energy-supercritical NLS, $s_{crit} > 1$

defocusing case : GWP is open.

(analogous to NSE on $\mathbb{R}^3$: $\dot{H}^{\frac{1}{2}}$-crit but energy $\int |u|^2 \approx L^2$-norm)

(NLS) $\quad i\partial_t u + \Delta u = -|u|^{p-1}u \qquad$ focusing

• Solitons (solitary wave solution)

$$u(t,x) = e^{it}\,\underbrace{\phi(x)}_{\text{profile}}$$

$\Leftarrow$ solves (NLS) iff ϕ solves the following elliptic PDE:

$$\circledast \qquad \Delta\phi - \phi + |\phi|^{p-1}\phi = 0, \qquad \phi \in H^1(\mathbb{R}^d)$$

FACT : $d=1$: all solns to $\circledast$ are translates of

$$Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{(p-1)x}{2}\right)} \right)^{p-1}$$

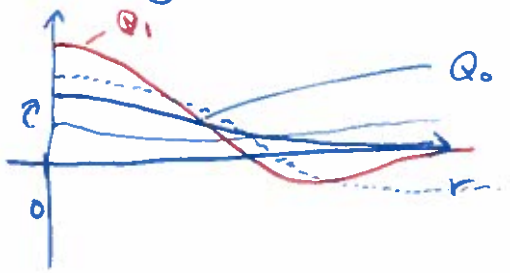
- $d \geq 2$: $\exists$ seq $\{Q_n\}_{n\geq 0}$ (radial) of real solns to (*) with increasing L^2-norms s.t. $Q_n(r)$ vanishes n times on $\mathbb{R}_+$.

  Q_0, radially sym, positive ← ground state

  existence: Berestycki-Lions-Peletier '81

  uniqueness: $\phi > 0$, $\phi \in H^1$ radial sym, C^2, exp decaying (Gidas-Ni-Nirenburg '79, Kwong '89)

  shooting method (on ODE, $r>0$)



- Ground states play an important role in elliptic PDEs, dispersive PDEs, variational problems, functional inequalities, etc.

- **mass-subcritical**: $p < 1 + \frac{4}{d}$

  scrit < 0

NLS scaling $Q^\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ ← ground state

Prop (variational characterization of Q)

$d \geq 1$, $1 < p < 1 + \frac{4}{d}$

$M > 0$ fixed.

Then, the following minimization problem

$$\min_{\|u\|_{L^2} = M} H(u)$$

has min attained at $Q^{\lambda(M)}(\cdot - x_0) e^{i\gamma_0}$, $x_0 \in \mathbb{R}^d$, $\gamma_0 \in \mathbb{R}$ (for all)

↑ rescale of Q s.t. $\|Q^{\lambda(M)}\|_{L^2} = M$

Lagrange multip problem. $\frac{d}{d\varepsilon} H(u + \varepsilon v)\big|_{\varepsilon = 0}$ ← Gâteaux deriv.

⇒ Euler-Lagrange equation: $\Delta \phi - \underset{\uparrow \text{Lag. multip}}{\lambda} \phi + |\phi|^{p-1} u = 0$

• mass-critical case : $s_{crit} = 0$. $\quad p = 1 + \frac{4}{d}$

Let $\quad J(u) = \dfrac{\left(\int |\nabla u|^2\right)\left(\int |u|^2\right)^{2/d}}{\int |u|^{2+4/d}}, \quad u \neq 0.$

Prop (i) $\min_{\substack{u \in H^1 \\ u \neq 0}} J(u)$ is attained at

$$\lambda_0^{d/2} Q(\lambda_0 x + x_0)\, e^{i\gamma_0}, \quad (\lambda_0, x_0, \gamma_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$$

↑ unique ground state.

In particular, we have the sharp Gagliardo-Nirenberg inequality:

$$\int |u|^{2+\frac{4}{d}} \leq \underline{J(Q)} \int |\nabla u|^2 \left(\int |u|^2\right)^{2/d}.$$

↑ optimal const.

$$H(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2+\frac{4}{d}} \int |u|^{2+\frac{4}{d}} dx$$

Rmk: $\forall\, u \in H^1$

(**) $$H(u) \geq \frac{1}{2} \int |\nabla u|^2 \left(1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}}\right)^{4/d}\right) dx$$

"Rigidity".

(ii) Let $u \in H^1$ s.t. $H(Q) = 0$.

$$\int |u|^2 = \int Q^2, \quad H(u) = 0.$$

Then, $u(x) = \lambda_0^{d/2} Q(\lambda_0 x - x_0) e^{i\gamma_0}$

for some $(\lambda_0, x_0, \gamma_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$.

(For mass-subcrit, $H(Q) < 0$.

Cont'd Rmk: mass-critical NLS, 1-d quintic, 2-d cubic, etc.

focusing

Let $u_0 \in H^1$ s.t. $\|u_0\|_{L^2} < \|Q\|_{L^2}$.

(**) $\Rightarrow$ $H(u) + M(u) \gtrsim \|u\|_{H^1}^2$.

↑ Implicit const depends on $\|u_0\|_{L^2}$

$\Rightarrow$ GWP in $H^1(\mathbb{R}^d)$ (also soln scatters)

provided that $\|u_0\|_{L^2} < \|Q\|_{L^2}$

Note: mass-subcritical NLS is GWP in $L^2(\mathbb{R}^d)$ by LWP & mass conservation.

Lec 15 07/03/16 (Mon)

• Focusing mass-crit NLS

① If $\|u_0\|_{L^2} < \|Q\|_{L^2}$, ($Q$ = ground state)

$M(u_0) < M(Q)$ then (NLS) is globally well-posed in $H^1(\mathbb{R}^d)$.

Moreover, all solns scatter as $t \to \pm\infty$.

$M(u_0) = M(Q)$ ② $u(t) = e^{it}Q$ is a global non-scattering soln.

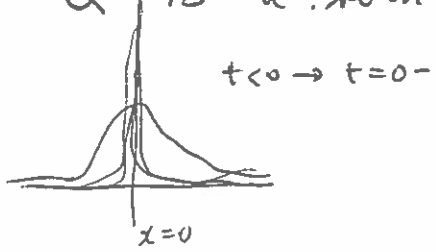
③ pseudo-conformal symmetry ($s_{crit} = 0$)

$$u(t,x) \longmapsto v(t,x) = \frac{1}{|t|^{d/2}} u\left(-\frac{1}{t}, \frac{x}{t}\right) e^{\frac{i|x|^2}{4t}} \quad (t \neq 0)$$

Apply pc transf. to $e^{it}Q$.

$$\Rightarrow Q^*(t,x) = \frac{1}{|t|^{d/2}} Q\left(\frac{x}{t}\right) e^{-\frac{i|x|^2}{4t} + \frac{i}{t}}$$

Q^* is a soln to (NLS) for $t < 0$ (and $t > 0$)



Q^* blows up at time $t=0$ (starting at $t=-1$)

- $|Q^*(t)|^2 \overset{\text{"}}{\longrightarrow}\text{"} \|Q\|_{L^2}^2 \delta_{x=0}$ as $t \nearrow 0$.
- $\|\nabla Q^*(t)\|_{L^2} \sim \frac{1}{|t|}$ $\Leftarrow$ blowup speed

Q^* not "stable"

$\Leftarrow$ Q^* is the <u>minimal mass</u> blowup soln.

- unique (Merle '93)

  i.e. if $M(u) = M(Q)$ and blows up in a finite time,

  then $u = Q^*$ (up to symmetry)

- <u>Other finite time blowup solns?</u>

$$M(Q) < M(u_0) < M(Q) + \varepsilon.$$

Merle - Raphaël '00's ~

"log log" blowup soln ~ $\sqrt{\frac{\log\log(T-t)}{T-t}}$

$\Leftarrow$ "stable"

<u>Rmk</u>: slowest possible blowup speed $\gtrsim \frac{1}{\sqrt{T-t}}$ (by scaling)

5.3 Virial identity & Morawetz estimate

Viriel = "force"

$$i\partial_t u + \Delta u = \lambda |u|^{p-1} u \qquad \lambda = \pm 1$$

$$\left(\partial_t u = i\Delta u - i\lambda |u|^{p-1} u \right)$$

Virial potential $V_a(t) = \int a(x) |u|^2 dx$ (real)

$a(x)$ "nice" func

- $a(x) = |x|^2$
- $a(x) = |x|$

$$\partial_t V_a(t) = \int a(x) 2\,\mathrm{Re}\,(\partial_t u\, \bar{u})\, dx$$

$$= -2\,\mathrm{Im} \int a \Delta u\, \bar{u}\, dx - 2\lambda \int a(x)\, \mathrm{Re}(i |u|^{p+1})\, dx$$

$$\overset{\text{IBP}}{=} 2\,\mathrm{Im} \int \nabla u \cdot \nabla(a\bar{u})\, dx$$

$$= 2\,\mathrm{Im} \int \bar{u}\, \nabla u \cdot \nabla a\, dx$$

Morawetz action

$$\mathrm{Im} \int (\nabla u \cdot \nabla \bar{u})\, a\, dx = 0$$

$|\nabla u|^2$ is real

· Compute $\partial_t^2 V_a(t)$

$$\partial_t 2\int \bar{u}\,\nabla u\cdot\nabla a\,dx = 2\int \partial_t\bar{u}\,\nabla u\cdot\nabla a + 2\int \bar{u}\,\nabla\partial_t u\cdot\nabla a$$

$$= 2\int(-i\Delta\bar{u})\,\nabla u\cdot\nabla a + 2i\int \lambda|u|^{p-1}\bar{u}\,\nabla u\cdot\nabla a$$

$$+ 2\int \bar{u}\,\nabla(i\Delta u)\cdot\nabla a - 2i\int \bar{u}\,\nabla(\lambda|u|^{p-1}u)\cdot\nabla a$$

$|u|^{p-1} = u^{\frac{p-1}{2}}\bar{u}^{\frac{p-1}{2}}$

$u^{\frac{p+1}{2}}\bar{u}^{\frac{p-1}{2}}$

$$= 2i\lambda\int |u|^{p-1}\bar{u}\,\partial_j u\,\partial_j a$$

Einstein's summation notation

$\partial_j u\,\partial_j a = \sum_{j=1}^{d}\partial_j u\,\partial_j a$

$$-2i\lambda\Big\{\int \frac{p+1}{2}|u|^{p-1}\partial_j u\,\bar{u}\,\partial_j a + \frac{p-1}{2}|u|^{p-1}u\,\partial_j\bar{u}\,\partial_j a\Big\}$$

$$= -2i\lambda\,\frac{p-1}{2}\int |u|^{p-1}(\partial_j u\,\bar{u} + \partial_j\bar{u}\,u)\,\partial_j a$$

$$= -2i\lambda\,\frac{p-1}{p+1}\int \partial_j(\underbrace{|u|^{p+1}}_{(u\bar{u})^{\frac{p+1}{2}}})\partial_j a \overset{IBP}{=} 2i\lambda\,\frac{p-1}{p+1}\int |u|^{p+1}\Delta a$$

IBP on 2nd term.

$$= -2i \int \Delta \bar{u}\, \nabla u \cdot \nabla a - 2i \int \Delta u\, \nabla \bar{u} \cdot \nabla a - 2i \int \Delta u\, \bar{u} \Delta a$$

$$= ① + ② + ③$$

$$① \overset{IBP}{=} 2i \int \partial_k \bar{u}\, \partial_k (\nabla u \cdot \nabla a) = 2i \int \partial_k \bar{u} (\partial_k \nabla u) \cdot \nabla a + 2i \int \partial_k \bar{u}\, \nabla u \cdot \partial_k \nabla a$$

$$② = 2i \int \partial_k u\, \partial_k (\nabla \bar{u} \cdot \nabla a) = 2i \int \partial_k u (\partial_k \nabla \bar{u}) \cdot \nabla a + 2i \int \partial_k u\, \nabla \bar{u} \cdot \partial_k \nabla a$$

$$\overset{IBP}{=} -2i \int \partial_k \bar{u}\, \partial_k \partial_j u\, \partial_j a - 2i \int \partial_k \bar{u}\, \partial_k u\, \Delta a$$

$$③ = 2i \int \partial_k u\, \partial_k (\bar{u} \Delta a) = 2i \int \partial_k u\, \partial_k \bar{u}\, \Delta a + 2i \int \partial_k u\, \bar{u}\, \partial_k \Delta a$$

Side comp: $2 \mathrm{Im}\, i \int \partial_k u\, \bar{u}\, \partial_k \Delta a = 2i \int (\partial_k u) \bar{u}\, \partial_k \Delta a + 2i \int u (\partial_k \bar{u}) \partial_k \Delta a$

$$= \int \partial_k (|u|^2) \partial_k \Delta a \overset{IBP}{=} - \int |u|^2 \Delta^2 a$$

$$\Rightarrow \quad \partial_t^2 V(t) = 4 \int \mathrm{Re}(\partial_k u\, \partial_j \bar{u})\, \partial_k \partial_j a + 2\lambda \frac{p-1}{p+1} \int |u|^{p+1} \Delta a - \int |u|^2 \Delta^2 a$$

IX: Virial identity.

$a(x) = |x|^2 = \sum_{j=1}^{d} x_j^2$. $\Delta a = 2d$

$\Delta^2 a = 0.$

$\partial_k \partial_j a = 2\delta_{jk}$

$$\Rightarrow \quad V(t) = \int |x|^2 |u(t,x)|^2 dx$$

and $$\partial_t^2 V(t) = 8 \int |\nabla u|^2 + 4d\lambda \frac{p-1}{p+1} \int |u|^{p+1}$$

$$= 16\, H(u) + \frac{4\lambda d}{p+1} \left(p - \left(1 + \frac{4}{d}\right) \right) \int |u|^{p+1}$$

$\left(1 + \frac{4}{d}\right)$: mass-crit power.

- focusing ($\lambda = -1$), scrit ≥ 0

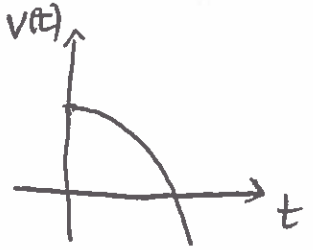
$\Rightarrow$ 2nd term ≤ 0

Suppose $u_0 \in H^1(\mathbb{R}^d)$ s.t. $H(u_0) < 0$. $\Rightarrow \partial_t^2 V(t) \leq 16\, H(u(t))$

$\leq 16 H(u_0) < 0$

$\Rightarrow V(t^*) < 0$ for some $t^* > 0$.

but $V(t) = \int |x|^2 |u(t,x)|^2 dx \geq 0$



$\Rightarrow$ u must blow up before time t^*.

(Glassey's argument, Zakharov's)

• **Morawetz estimate**: $a = |x|$, $\partial_j a = \frac{x_j}{|x|}$

$$\Rightarrow \partial_j^2 a = \frac{1}{|x|} - \frac{x_j}{|x|^2}\cdot\frac{x_j}{|x|} \Rightarrow \Delta a = \frac{d-1}{|x|}$$

$$\Rightarrow \partial_j(\Delta a) = (d-1)\frac{-x_j}{|x|^3} \Rightarrow \partial_j^2(\Delta a) = (d-1)\left(\frac{-1}{|x|^3} + \frac{3x_j}{|x|^4}\frac{x_j}{|x|}\right)$$

$$\Rightarrow \Delta^2 a = -\frac{(d-1)(d-3)}{|x|^3} \le 0 \quad \text{if } d \ge 3.$$

$d=3$: $\frac{1}{|x|}$ = fund. soln of $-\Delta$ $\Rightarrow \Delta^2 a = -8\pi\delta$.

Claim: $\partial_j \partial_k a \, \mathrm{Re}\, \partial_j u \, \partial_k \bar{u} = \frac{|\not\nabla u|^2}{|x|}$

angular component of grad

$$\not\nabla u = \nabla u - \frac{x}{|x|}\left(\frac{x}{|x|}\cdot\nabla u\right)$$

$$|\not\nabla u|^2 = |\nabla u|^2 - \left|\frac{x}{|x|}\cdot\nabla u\right|^2$$

Goal:

Morawetz: $$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d_x} \frac{|u(t,x)|^{p+1}}{|x|}\,dx\,dt \lesssim \sup_t \|u(t)\|_{\dot{H}^{1/2}}^2$$

or $M(u_0)^{1/2} H(u_0)^{1/2}$

defocusing case $a=1$

Lec 16 09/03/16 (Wed)

$$\partial_t^2 V_a(t) = \partial_t \cdot 2\,\mathrm{Im} \int \nabla u \cdot \nabla a\, \bar{u}\, dx$$

$$= 4 \int \mathrm{Re}(\partial_k u\, \partial_j \bar{u})\, \partial_k \partial_j a + 2\lambda \frac{p-1}{p+1} \int |u|^{p+1} \Delta a$$

$$- \int |u|^2 \Delta^2 a$$

$a = |x| \Rightarrow \Delta^2 a \le 0$. $(d \ge 3)$

$$\Rightarrow \partial_t\, 2\,\mathrm{Im} \int \nabla u \cdot \frac{x}{|x|} \bar{u} = \underbrace{4 \int \frac{|\nabla u|^2}{|x|} dx}_{\ge 0} + 2\lambda \frac{p-1}{p+1} \int \frac{|u|^{p+1}}{|x|} dx$$

$$\underbrace{- \int |u|^2 \Delta^2 a}_{\ge 0}$$

(*)

$\Delta^2 a = -8\pi\delta$ when $d = 3$

$\lambda = 1$ (defocusing)

$$\int_{t_0}^{t_1} \int \frac{|u|^{p+1}}{|x|} dx \lesssim \sup_{t=t_0,t_1} \left| \mathrm{Im} \int \nabla u(t) \cdot \frac{x}{|x|} \bar{u}(t)\, dx \right|$$

$$\lesssim \sup_{t_0,t_1} \|u(t)\|_{\dot{H}^{1/2}}^2$$

or $\underset{C\text{-}S}{\le} M(u_0)^{1/2} H(u_0)^{1/2}$

Take $t_0 \to -\infty$, $t_1 \to +\infty \Rightarrow$ Morawetz estimate.

Pf of Claim: $\partial_j \partial_k a \, \mathrm{Re}(\partial_j u \, \partial_k \bar{u}) = \frac{|\not\nabla u|^2}{|x|}$

$$\partial_j \partial_k a = \frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3}$$

$$\Rightarrow (\mathrm{LHS}) = \sum_{j=1}^{d} \frac{|\partial_j u|^2}{|x|} - \sum_{j,k} \mathrm{Re}\left(\frac{x_j \partial_j u}{|x|} \cdot \frac{x_k \partial_k \bar{u}}{|x|} \right) \frac{1}{x}$$

$$= \frac{|\nabla u|^2}{|x|} - \frac{1}{|x|} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 = \frac{1}{|x|} \left| \nabla u - \frac{x}{|x|} \cdot \nabla u \right|^2$$

$$= \frac{|\not\nabla u|^2}{|x|}$$

$|X|^2 - |P_v X|^2 = |X - P_v X|^2$

- Interaction Morawetz estimate (Colliander-Keel-Staffilani-Takaoka-Tao mid'00s)

Write (*) centered at y $\boxed{d=3}$

$$\partial_t \mathrm{Im} \int \nabla u(x) \cdot \frac{x-y}{|x-y|} \bar{u}(x) dx = 2 \int \frac{|\not\nabla_y u(x)|^2}{|x-y|} dx + 2\lambda \frac{p-1}{p+1} \int \frac{|u(x)|^{p+1}}{|x-y|} dx + 4\pi |u(y)|^2$$

(**)

Multiply (**) by $|u(y)|^2$ and $\int_y dy$.

$$\Rightarrow \int_{\mathbb{R}_t}\int_{\mathbb{R}_y} |u(y)|^4 dy\, dt \lesssim \sup_t \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}^{1/2}}^2$$

$$\text{or} \quad \cdot M(u_0)^{3/2} H(u_0)$$

(5.4) Scattering for (energy-subcrit) cubic NLS on $\mathbb{R}^3$.

(defocusing)

We only consider $t \to +\infty$

WTS: $\exists\, u_+ \in H^1(\mathbb{R}^3)$ s.t. $\|u(t) - S(t)u_+\|_{H^1} \to 0$ as $t \to +\infty$.

$$\| u(t) - S(t)u_+ \|_{H^1}$$
$$\| S(-t)u(t) - u_+ \|_{H^1}$$

$$S(-t)u(t) = u_0 - i\int_0^t S(-t')|u|^2 u(t')\, dt'$$

$\downarrow$? u_+

$\Rightarrow$ Suffices to make sense (in H^1) of

$$\int_0^\infty S(-t)|u|^2 u(t)\, dt.$$

Existence of wave operator: $\Omega_+ : u_+ \in H^1 \to u_0 \in H^1$.

Given u_+ in H^1, can we find $u_0 \in H^1$ s.t.

the corresp soln u scatters to $S(t)u_+$.

Rmk: If Ω_+ exists, it is injective (by the uniqueness part of WP theory.)

· if Ω_+ is invertible, we say we have **asymptotic completeness**.

· $$u_+ = u_0 - i\int_0^\infty S(-t)|u|^2 u(t)\,dt$$

$$S(-t)u(t) = u_0 - i\int_0^t S(-t')|u|^2 u(t')\,dt'$$

$\oplus \Longrightarrow$ $$\boxed{u(t) = S(t)u_+ + i\int_t^\infty S(t-t')|u|^2 u(t')\,dt'.}$$ Terminal value problem.

$S(t)u_+$: value at $t=+\infty$

Pf of existence of wave op

$$t = +\infty \longrightarrow t = T. \longrightarrow t = 0$$

① well-posedness on $[T, \infty)$

$$\widetilde{S}^1 = L^5_{t,x} \cap L^{\frac{10}{3}}_t W^{1,\frac{10}{3}}_x \qquad \left(\tfrac{10}{3}, \tfrac{10}{3}\right), \text{ adm.}$$

- $\|u\|_{L^5_{t,x}} \overset{\text{Sob}}{\lesssim} \|u\|_{L^5_t W^{1,\frac{30}{11}}_x}$ $\qquad \left(5, \tfrac{30}{11}\right)$, adm.

By Strichartz, $\|S(t)u_+\|_{\widetilde{S}^1(\mathbb{R}_t)} \lesssim \|u_+\|_{\dot{H}^1} < \infty.$

$\Rightarrow$ By MCT $\|S(t)u_+\|_{\widetilde{S}^1([T,\infty))} \leq \varepsilon.$

Define $\widetilde{\Gamma} u(t)$ by $\widetilde{\Gamma} u(t) = (\text{RHS}) \text{ of } \oplus$

$$\Rightarrow \|\widetilde{\Gamma} u\|_{\widetilde{S}^1([T,\infty))} \leq \varepsilon + C \underbrace{\|\nabla(|u|^2 u)\|_{L^{10/7}_{t,x}([T,\infty))}}$$

$$\leq \|u\|^2_{L^5_{t,x}} \|\nabla u\|_{L^{10/3}_{t,x}}$$

$$\leq \|u\|_{\widetilde{S}^1([T,\infty))}.$$

Similarly, $\| \widehat{\Gamma} u - \widehat{\Gamma} v \|_{\widetilde{S}^1([T,\infty))} \le C \Big(\|u\|^2_{\widetilde{S}^1([T,\infty))} + \|v\|^2_{\widetilde{S}^1([T,\infty))} \Big)$

$\times \| u - v \|_{\widetilde{S}^1([T,\infty))}$

$\Rightarrow$ $\widehat{\Gamma}$ is a contraction

on $\{ u : \|u\|_{\widetilde{S}^1([T,\infty))} \le 2\varepsilon \}$. $\quad \varepsilon \ll 1$.

Now, apply LWP and conservation of energy (& mass) to extend u onto $[0,T]$.

$\|u(t)\|_{H^1} \lesssim \big(H(u(T)) + M(u(T)) \big)^{1/2}$.

$\rightarrow \quad \delta \sim \big(H(u(T)) + M(u(T)) \big)^{-\theta}$.

$\square$

• Scattering (asymptotic completeness)

$$\Big\| \int_0^{\infty} S(-t)|u|^2 u(t)\,dt \Big\|_{H^1} \overset{\text{dual Str}}{\lesssim} \| \langle\nabla\rangle (|u|^2 u) \|_{L_{t,x}^{10/7}}$$

$$\lesssim \|u\|_{L_{t,x}^{5}}^2 \, \|u\|_{L_t^{10/3} W_x^{1,10/3}}$$

$$\lesssim \Big(\sup_{\substack{(q,r)\\ \text{adm}}} \|\langle\nabla\rangle u\|_{L_t^q L_x^r} \Big)^3 =: \|u\|_{S^1}^3 .$$

So, it suffices to show $\|u\|_{S^1} \lesssim 1$. → scattering.

"strong" space-time bound

Claim: "weak" space-time bd

$$\|u\|_{L_{t,x}^q} \lesssim 1 \quad \text{for some } q \in \left[\tfrac{10}{3}, 10\right]$$

implies "strong" space-time bd.: $\|u\|_{S^1} \lesssim 1$

Rmk: • Interaction Morawetz ⇒ "weak" space-time bd ($q = 4$)

• Morawetz estimate in the radial setting

$$\Rightarrow \|u\|_{L_{t,x}^5} \lesssim 1$$

$\Leftarrow$ Radial Sobolev ineq: $\| |x|^s |u| \|_{L^\infty_x(\mathbb{R}^d)} \lesssim \|u\|_{H^1}$

$\frac{d}{2} - 1 \le s \le \frac{d-1}{2}$ $\quad u$, radial.

($\Leftarrow$ localize around fixed x, apply 1-d Gagliardo-Nirenberg (in r). polar coord

$$\int_t \int_x |u|^5 = \int_t \int_x |x||u| \cdot \frac{|u|^4}{|x|} dx dt \le \underbrace{\| |x||u| \|_{L^\infty_{t,x}}}_{\text{rad Sob}} \underbrace{\int_{t,x} \frac{|u|^4}{|x|}}_{\text{Morawetz}} \le C(\|u_0\|_{H^1})$$

Pf of Claim: Given $\varepsilon > 0$, divide $\mathbb{R}_+ = \bigcup_{j=1}^{N} I_j$ s.t. $\|u\|_{L^q_{I_j} L^q_x} \le \varepsilon$. (finite)

Let $I_j = [t_j, t_{j+1})$.

$$\|u\|_{S^1(I_j)} \overset{\text{Str.}}{\lesssim} \|u(t_j)\|_{H^1} + \|u\|^2_{L^5_{t,x}} \underbrace{\|u\|_{L^{\frac{10}{3}}_t W^{1,\frac{10}{13}}_x}}_{\le \|u\|_{S^1(I_j)}}$$

$$u(t) = S(t-t_j)u(t_j) - i\int_{t_j}^t S(t-t')|u|^2 u(t') dt'$$

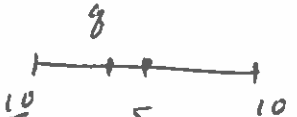
(A)

Note: $\|u\|_{L^{10/3}_{t,x}(I)} \le \|u\|_{S^0(I)}$ $\left(\frac{10}{3}, \frac{10}{3}\right)$, adm

$\le \|u\|_{S^1(I)}$

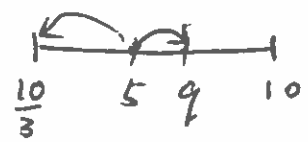
$\therefore \|u\|_{L^{10}_{t,x}(I)} \underset{Sob}{\lesssim} \|u\|_{L^{10}_I W^{1,\frac{30}{13}}_x} \le \|u\|_{S^1(I)}$

$\Rightarrow$ By interpolation, $\|u\|_{L^5_{t,x}(I_j)} \le \|u\|^{\theta}_{L^q_{t,x}(I_j)} \|u\|^{1-\theta}_{L^{10}_{t,x}}$



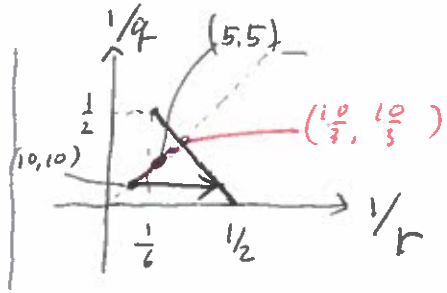
or

$\|u\|^{\theta}_{L^q_{t,x}} \|u\|^{1-\theta}_{L^{10/3}_{t,x}}$



(B) $\Rightarrow \|u\|_{L^5_{t,x}(I_j)} \le \varepsilon^{\theta} \|u\|^{1-\theta}_{S^1(I_j)}$

$\Rightarrow$ (A) & (B) $\Rightarrow \|u\|_{S^1(I_j)} \lesssim \|u(t_j)\|_{H^1} + \varepsilon^{2\theta} \|u\|^{3-2\theta}_{S^1(I_j)}$



$\Rightarrow$ continuity arg (C) $\|u\|_{S^1(I_j)} \lesssim 1 \Rightarrow$ sum over finitely many intervals

$\|u\|_{S^1([0,\infty))} \lesssim 1$

$\Rightarrow$ scattering.

© Actually, we can prove

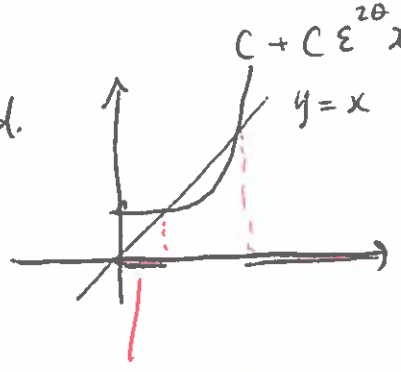
$$X(t) \lesssim 1 + \varepsilon^{2\theta} X(t)^{3-2\theta}$$

$X(t) = \|u\|_{\delta'([t_j, t])}$

$t \le t_{j+1}$

- $X(t)$ is conti.
- At $t = t_j$, ↑ is satisfied.

$$X(t_j) \lesssim 1$$



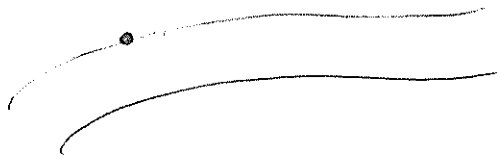
$\Rightarrow X(t) \lesssim 1$ (if $\varepsilon \ll 1$.)

Lec 17 28/03/16

Sec 6: Navier-Stokes equations

$\vec{u}$ = velocity of fluid

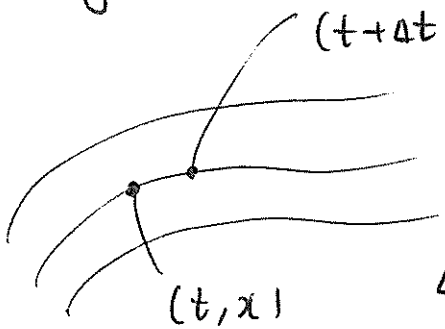
Want to take a derivative of $f(t,x)$.



① $\frac{\partial f}{\partial t}$ = usual time deriv (in Eulerian coordinates)

↑ observer stays at a fixed pt.

② Lagrangian coordinates = observer travels with a particle



$$\lim_{\Delta t \to 0} \frac{f(t+\Delta t, x+\vec{u}\Delta t) - f(t,x)}{\Delta t} =: \frac{Df}{Dt}$$

material derivative
advective deriv
convective deriv
hydrodynamic deriv
Lagrangian deriv
Stokes deriv

$$\frac{Df}{Dt} = \partial_t f + \vec{u} \cdot \nabla f$$

- Navier-Stokes equations:

$u = (u_1, u_2, u_3) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$, velocity of fluid

$p : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$, pressure.

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \Delta u + f, & t > 0 \\ \operatorname{div} u = 0, & t \geq 0 \\ u|_{t=0} = u_0 \end{cases}$$

Newton's 2nd law: $F = ma$.

(LHS) = acceleration

(RHS) = force

incompressibility

friction

4 equations, 4 unknowns

$$\begin{cases} \partial_t u_i + \sum_{k=1}^{3} u_k \partial_k u_i = -\partial_i p + \Delta u_i + f_i \\ \sum_i \partial_i u_i = 0 \end{cases}$$

$\frac{Du_i}{Dt}$

Set $f \equiv 0$.

Stokes equations : $\begin{cases} \partial_t u - \Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$

$\Longrightarrow$ Take div.

$$\underbrace{\partial_t (\operatorname{div} u) - \Delta (\operatorname{div} u)}_{=0} + \nabla \cdot \nabla p = 0$$

$\Rightarrow \Delta p = 0 \Rightarrow p$ is a harmonic function.

- Write ($\nabla \times (\nabla p) = 0$)

$$u_0 = \underbrace{v_0}_{\text{div free}} + \underbrace{\nabla p_0}_{\text{curl free}} \quad \text{s.t. } \operatorname{div} v_0 = 0$$

$$\Rightarrow -\Delta p_0 = -\operatorname{div} u_0 \quad (\Longrightarrow p_0 = -(-\Delta)^{-1} \nabla \cdot u_0$$

$$\Rightarrow \nabla p_0 = -\nabla (-\Delta)^{-1} \nabla \cdot u_0$$

$$\Rightarrow v_0 = u_0 - \nabla p_0 = \underbrace{\left(Id + \nabla (-\Delta)^{-1} \nabla \cdot \right)}_{\text{Leray projection}} u_0$$

$$V_{0i} = \sum_{j=1}^{3} \left(\delta_{ij} + \nabla_i (-\Delta)^{-1} \nabla_j \right) u_{0j}$$

$$= \mathcal{F}^{-1} \left(\underbrace{\sum_{j=1}^{3} \left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2} \right)}_{\text{matrix valued Fourier multip operator}} \widehat{u_{0j}}(\xi) \right)$$

Rmk: $\frac{i \xi_j}{|\xi|} \rightsquigarrow R_j$ = Riesz transform

"higher dimensional analogue of Hilbert transf."

($|\xi| = \sqrt{\xi_1^2 + \cdots + \xi_d^2}$)

$$R_j f(x) = c. \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy.$$

$R_j : L^p \to L^p, \quad 1 < p < \infty$

$$\sum_{j=1}^{d} R_j^2 = -Id \quad (\text{like } H^2 = -Id)$$

Helmholtz-Leray decomp / Hodge decomp.

$$u = v + \nabla p$$

v: div free, ∇p: curl free

v is locally a curl. ($\nabla \cdot (\nabla \times w) = 0$)

Here, $v = \mathcal{P} u$

$\mathcal{P}$: Leray proj.

$$\mathcal{P} = Id + \nabla (-\Delta)^{-1} \nabla \cdot$$

Apply $\mathcal{P}$ to Stokes eqns.: $\mathcal{P} \nabla p = \nabla p + \nabla (-\Delta)^{-1} \underbrace{\nabla \cdot \nabla p}_{= \Delta} p$

$$= \nabla p - \nabla p = 0.$$

$$\Rightarrow \begin{cases} \partial_t \mathcal{P} u = \Delta \mathcal{P} u \\ \mathcal{P} u|_{t=0} = v_0 \end{cases}$$

$$\begin{cases} \operatorname{div} u = 0 \\ \Rightarrow u = \mathcal{P} u \quad t > 0 \end{cases}$$

$$\Rightarrow u(t) = \mathcal{P} u(t) = K_t * \mathcal{P} u_0$$

$$= e^{t\Delta} \mathcal{P} u_0 = \mathcal{P} e^{t\Delta} u_0$$

$$= e^{tL} u_0, \quad L = \mathcal{P}\Delta$$

$$K_t(x) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}}$$

↑ Heat Kernel

$$u_i(t) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|\cdot|^2}{4t}} * \left\{ \sum_{j=1}^{3} \left(\delta_{ij} + \nabla_i (-\Delta)^{-1} \nabla_j \right) u_{0j} \right\}$$

- We need to study $e^{tL} = e^{tP\Delta}$.

Prop (L^p - L^q estimate) for heat equation $1 \le p \le q \le \infty$.

- $\| e^{t\Delta} f \|_{L^q_x} \lesssim t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \| f \|_{L^p_x}, \quad t > 0.$
- $\| \partial^\alpha (e^{t\Delta} f) \|_{L^q_x} \lesssim t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{|\alpha|}{2}} \| f \|_{L^p_x}$

α = multiindex, $|\alpha| \le 2$.

Pf: By Young's ineq,

$$\| \underbrace{e^{t\Delta} f}_{= K_t * f} \|_{L^q} \le \| K_t \|_{L^r} \| f \|_{L^p}. \qquad \frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$$

$$\|K_t\|_{L^r}^r = \frac{1}{(4\pi t)^{rd/2}} \int_{\mathbb{R}^d} e^{-\frac{r|x|^2}{4t}}\, dx$$

Change of var

$$y = \left(\frac{r}{4t}\right)^{1/2} x$$

$$dy = \left(\frac{r}{4t}\right)^{d/2} dx$$

$$\sim t^{-\frac{rd}{2}+\frac{d}{2}}$$

$$\Rightarrow \|K_t\|_{L^r} \sim t^{-\frac{d}{2}(1-\frac{1}{r})} = t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$$

- Similarly, $\|\partial^\alpha (e^{t\Delta} f)\|_{L_x^q} \le \|\partial^\alpha K_t\|_{L^r} \|f\|_{L^p}$ $\qquad \frac{1}{q}+1 = \frac{1}{r}+\frac{1}{p}$.

$\underline{\alpha = 1}$

$$\|\partial_{|x|} K_t\|_{L^r}^r = \frac{1}{(4\pi t)^{\frac{rd}{2}}} \int_{\mathbb{R}^d} \left(\frac{1}{t}\right)^{r/2} \left(\frac{|x|}{2\sqrt{t}}\right)^r e^{-\frac{r|x|^2}{4t}}\, dx$$

$$\sim t^{-\frac{rd}{2}+\frac{d}{2}-\frac{r}{2}}$$

change of var.

$$\|\partial_{|x|} K_t\|_{L^r} \sim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}$$

- $\underline{\alpha = 2}$: gain an extra factor of $\frac{1}{\sqrt{t}}$ by differentiating in $|x|$.

☑

Cor ($L^p - L^q$ estimate for the Stokes eqns). $L = P\Delta$

$$1 < p \le q < \infty \quad (\text{or } 1 < p < q = \infty)$$

$$\| e^{tL} f \|_{L^q_x} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \| f \|_{L^p_x}, \qquad t > 0$$

$$\| \partial^\alpha (e^{tL} f) \|_{L^q_x} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q}) - \frac{|\alpha|}{2}} \| f \|_{L^p_x}.$$

Pf: Note $e^{tL} f = e^{t\Delta} P f$.

Prop $\Rightarrow$ $\| e^{tL} f \|_{L^q} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \underbrace{\| Pf \|_{L^p}}$

$$\lesssim \| f \|_{L^p} \quad \text{if} \quad 1 < p < \infty.$$

P = Leray proj. = matrix valued operator

$$P_{ij} = \delta_{ij} \cdot Id + c\, R_i R_j$$

($R_i R_j$: bdd on L^p, $1 < p < \infty$)

$\square$

• Well-posedness of NSE on $\mathbb{R}^3$.

$$(NSE)\quad \begin{cases} \partial_t u + (u\cdot\nabla)u = -\nabla p + \Delta u \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

Once we construct u, take div

$$-\Delta p = \operatorname{div}(u\cdot\nabla)u$$

$$\Rightarrow p = (-\Delta)^{-1}\nabla\cdot(u\otimes u)$$

$$(u\cdot\nabla)u \overset{\text{b/c } \operatorname{div} u = 0}{=} \nabla\cdot(u\otimes u) \qquad (u\otimes u)_{ij} := u_i u_j.$$

$$(LHS)_i = \sum_{j=1}^{3} u_j \partial_j u_i + \underbrace{\sum_{j=1}^{3}(\partial_j u_j)}_{=\operatorname{div} u = 0} u_i$$

$$= \sum_{j=1}^{3} \partial_j (u_i u_j)$$

$\Rightarrow$ Apply Leray proj $\mathbb{P}$ to (NSE). (and use $u = \mathbb{P}u$)

$$\begin{cases} \partial_t u + \mathbb{P}((u\cdot\nabla)u) = Lu \\ u|_{t=0} = \mathbb{P}u_0 \end{cases}, \qquad L = \mathbb{P}\Delta.$$

Duhamel formulation (mild formulation)

$$u(t) = e^{tL} u_0 - \int_0^t e^{(t-t')L} \mathbb{P}((u\cdot\nabla)u)(t')\,dt'$$

$\Rightarrow$ We will use Fujita-Kato theory and prove small data global well-posedness in some critical spaces ($L^3(\mathbb{R}^3)$, $\dot{H}^{1/2}(\mathbb{R}^3)$,)

Lec 18 30/03/16

$$u(t) = e^{tL}u_0 - \int_0^t e^{(t-t')L} P\big((u\cdot\nabla)u\big)(t')\,dt', \quad L = P\Delta$$

Scaling: let $u^\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$

$$p^\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x)$$

$$\|u^\lambda\|_{L_t^q L_x^r(\mathbb{R}_+\times\mathbb{R}^d)} = \lambda^{1-\frac{d}{r}-\frac{2}{q}} \|u\|_{L_t^q L_x^r}$$

$\Rightarrow$ scaling inv condition: $\boxed{\frac{2}{q}+\frac{d}{r}=1}$

e.g. $L_t^\infty L_x^d$

- For p^λ, scaling inv cond: $\frac{2}{q}+\frac{d}{r}=2$

  e.g. $L_t^\infty L_x^{d/2}$

· $\underline{d=3}$ $\quad L^3(\mathbb{R}^3) \supset \dot{H}^{1/2}(\mathbb{R}^3) \quad \frac{1}{2} - \frac{1}{3} = \frac{1/2}{3}$

Notation: Let L^p_{df} = completion of $C_0^\infty(\mathbb{R}^3;\mathbb{R}^3) \cap \{\operatorname{div} f = 0\}$ under the L^p-norm. (Here, $L^p := (L^p)^3$)

$\Rightarrow P$ = Leray proj : $L^p \longrightarrow L^p_{df}$, $\quad 1 < p < \infty$

small data GWP

Theorem: (i) $\exists\, \delta > 0$ s.t. given $u_0 \in L^3(\mathbb{R}^3)$,

$\exists!$ global soln $u \in C([0,\infty); L^3) \cap C((0,\infty); W^{1,3})$

to (NSE). (Also, conti dependence.)

(ii) same in $\dot{H}^{1/2}(\mathbb{R}^3)$

Main difference from dispersive PDE:

dissipative $\longrightarrow$ $e^{t\Delta}$ (and e^{tL}) has much stronger smoothing, $t>0$

$\Rightarrow$ can be used to absorb a derivative loss (up to order 2)

(dispersive PDE: local smoothing only.)

Fujita – Kato '60's

$$\Gamma_{u_0} u(t) = e^{tL} u_0 - \int_0^t e^{(t-t')L} \mathbb{P}((u\cdot\nabla)u)(t')\,dt'.$$

$$\nabla \Gamma_{u_0} u(t) = \nabla e^{tL} u_0 - \nabla \int_0^t e^{(t-t')L} \mathbb{P}((u\cdot\nabla)u)(t')\,dt'.$$

$$\Rightarrow \|\Gamma u(t)\|_{L^3_x} \le C\|u_0\|_{L^3} + C\int_0^t (t-t')^{-1/2} \underbrace{\|(u\cdot\nabla)u(t')\|_{L^{3/2}_x}}_{p=q=3} dt'.$$

$q=3$. $p=3/2$. Also, badness of $\mathbb{P}$

$$\le \|u\|_{L^3}\|\nabla u\|_{L^3}$$

$$\le C\|u_0\|_{L^3} + C\underbrace{\int_0^t (t-t')^{-1/2}(t')^{-1/2}dt'}_{C\text{-}S} \times \|u\|_{L^\infty_t(0,t;L^3)} \times \sup_{0\le t'<t} (t')^{1/2}\|\nabla u(t')\|_{L^3_x}$$

Recall: Beta function $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$

$\mathrm{Re}\,p, \mathrm{Re}\,q > 0.$

$$\Rightarrow \int_0^t (t-t')^{-1/2}(t')^{-1/2}\,dt' = \int_0^t \left(1-\frac{t'}{t}\right)^{-1/2}\left(\frac{t'}{t}\right)^{-1/2} d\frac{t'}{t}$$

$$= \int_0^1 (1-\tau)^{-1/2}\tau^{-1/2}\,d\tau = B(1/2,1/2) < \infty$$

$$\|\nabla \Gamma u(t)\|_{L^3_x} \le C\, t^{-1/2}\|u_0\|_{L^3} + \int_0^t (t-t')^{-3/4} \underbrace{\|(u\cdot\nabla)u(t')\|_{L^2_x}}_{q=3,\ p=2}\,dt'$$

$$\le \|u(t')\|_{L^6_x}\,\|\nabla u(t')\|_{L^3_x}$$

G-N ineq (or Sobolev & interpolation) $\quad \frac{1}{3}-\frac{1}{6} = \frac{1/2}{3}$

$$\|u(t')\|_{L^6} \lesssim \||\nabla|^{1/2}u\|_{L^3} \le \|u(t')\|^{1/2}_{L^3_x}\,\|\nabla u(t')\|^{1/2}_{L^3_x}$$

$\frac{1}{2} = \theta\cdot 0 + (1-\theta)\cdot 1 \Rightarrow \theta = 1/2$

$$\Rightarrow \quad t^{1/2}\|\nabla \Gamma u(t)\|_{L^3_x} \le C\|u_0\|_{L^3_x} + C\boxed{t^{1/2}\int_0^t (t-t')^{-3/4}(t')^{-3/4}dt'} \times \left\{\sup_{t'}\|u(t)\|_{L^3_x}\right\}^{1/2} \times \left\{\sup_{t'}(t')^{1/2}\|\nabla u(t')\|_{L^3_x}\right\}^{3/2}$$

$= B(\frac{1}{4},\frac{1}{4}) < \infty$

- Define

$$\|u\|_X := \|u\|_{L^\infty_t([0,\infty);L^3_x)} + \sup_{t\ge 0} t^{1/2}\|\nabla u(t)\|_{L^3_x}$$

$$\Rightarrow \quad \|\Gamma u\|_X \le C_1\|u_0\|_{L^3_x} + C_2\|u\|_X^2$$

Similarly,

$$\|\Gamma u - \Gamma v\|_X \le C_3\left(\|u\|_X + \|v\|_X\right)\|u-v\|_X.$$

$\Rightarrow$ Γ is a contraction on $B_R \subset X$, $R \sim \|u_0\|_{L^3_x} \ll 1$.

(ii) Modification for small data GWP in $\dot{H}^{1/2}(\mathbb{R}^3)$

By interpolation,

$$\text{(*)} \qquad \| |\nabla|^{1/2} e^{tL} f \|_{L^q_x} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{4}} \| f \|_{L^p_x}$$

$$\Rightarrow \| \Gamma u \|_{\dot{H}^{1/2}} \leq C \| u_0 \|_{\dot{H}^{1/2}} + C \int_0^t (t-t')^{-1/2} \| (u\cdot\nabla) u(t') \|_{L^{3/2}_x} dt'$$

(*) with $q = 2$, $p = 3/2$

$$-\tfrac{3}{2}(\tfrac{2}{3}-\tfrac{1}{2})-\tfrac{1}{4} = -\tfrac{1}{2}$$

$$\leq C \| u_0 \|_{\dot{H}^{1/2}} + C \| u \|_{L^\infty_t \dot{H}^{1/2}} \sup_{t'>0} (t')^{1/2} \| \nabla u(t') \|_{L^3}$$

Sobolev $\dot{H}^{1/2} \subset L^3$

Same as before

and

$$\| \nabla \Gamma u \|_{L^3} \leq C t^{-1/2} \| |\nabla|^{1/2} u_0 \|_{L^2} + \text{same as before.}$$

$q = 3$, $p = 2$

$$\Rightarrow \text{use } \| u \|_X = \| u \|_{L^\infty_t (\mathbb{R}_+ ; \dot{H}^{1/2})} + \sup_{t\geq 0} t^{1/2} \| \nabla u(t) \|_{L^3_x}.$$

□

Rmk: ① LWP (for large data) also holds.

② Energy $\frac{1}{2}\int |u|^2 dx$ "satisfies"

$$\frac{1}{2}\int_{\mathbb{R}^d} |u(t)|^2 dx + \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' = \frac{1}{2}\int_{\mathbb{R}^d} |u_0|^2 dx$$

$\Rightarrow$ provides an a priori estimate only on the L^2-norm ← too weak.

i.e. (NSE) is energy-supercritical.

- Serrin-type criterion on uniqueness / (global) regularity

Leray-Hopf weak soln: Given $u_0 \in L^2_{df}$, we say that GWP for smooth solns

u is a Leray-Hopf (weak) soln if

① $u \in L^\infty([0,T]; L^2_{df}) \cap L^2([0,T]; \dot{H}^1)$

② u is a distributional soln to (NSE), i.e.

bdry term from IBP

$$\langle u(t), \phi(t)\rangle_{L^2_x} - \langle u_0, \phi(0)\rangle_{L^2_x}$$

$$= \int_0^t \langle u, \partial_t \phi\rangle\, dt' - \int_0^t \langle \nabla u, \nabla \phi\rangle\, dt' - \int_0^t \langle (u\cdot\nabla)u, \phi\rangle\, dt'.$$

for all div-free test functions $\phi \in C^\infty([0,T]; (C_0^\infty)_{df})$.

Rmk: ① Leray-Hopf soln satisfies the energy inequality:

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \|\nabla u(t')\|_{L^2_{t'}(0,t;\dot{H}^1)} \underset{\sim}{\leq} \frac{1}{2}\|u_0\|^2_{L^2}.$$

② Leray: large data global existence of L-H weak soln. (NO uniqueness.)

③ L-H weak soln u lies in $L^q_T L^r_x$, $L^q(0,T)$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \qquad 2 \leq r \leq \frac{2d}{d-2}.$$

Serrin's criterion ('60, '62)

Suppose a L-H weak soln u lies in $L_T^q L_x^r$.

$$\frac{2}{q} + \frac{d}{r} = 1 \quad , \qquad d < r \le \infty$$

$\Rightarrow$ The soln is "regular".

- $L_t^\infty L_x^3$: Escauriaza-Seregin-Šverák '03

  Gallagher-Koch-Planchon '13

  (harmonic analytic approach $\Leftarrow$ Kenig-Koch '11 on $L_t^\infty \dot{H}^{1/2}$)

Prop: $1 < p \le q < \infty$, $\gamma = \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$

(i) $\| e^{tL} f \|_{L_T^\theta L_x^q} \lesssim \| f \|_{L_T^\theta L_x^p}$

(ii) $\| \int_0^t e^{(t-t')L} f(t')\, dt' \|_{L_T^\theta L_x^q} \lesssim \| f \|_{L_T^\sigma L_x^p}$

$$\frac{1}{\sigma} - \frac{1}{\theta} = 1 - \gamma > 0.$$

(iii) $\| \int_0^t \nabla e^{(t-t')L} f(t')\, dt' \|_{L_T^\theta L^q} \lesssim \| f \|_{L_T^\beta L_x^p}$

$$\frac{1}{\beta} - \frac{1}{\theta} = \frac{1}{2} - \gamma > 0$$

Pf (i) We have $\| e^{tL} f(t) \|_{L_x^q} \lesssim t^{-\gamma} \| f(t) \|_{L_x^p}$

$\Leftarrow$ view it as a weak-type bd in time:

$$L^\infty([0,T]; L_x^p) \longrightarrow \underbrace{L^{1/\gamma,\infty}}_{\text{weak}}([0,T]; L_x^q)$$

then use Marcinkiewicz interpolation.

(ii) $\left\| \int_0^t e^{(t-t')L} f(t')\,dt' \right\|_{L_T^\theta L_x^q} \lesssim \| t^{-\gamma} * \| f(t) \|_{L_x^p} \|_{L_T^\theta}$

$\Rightarrow$ use H-L-S ineq.

(iii) same as (ii).

Back to Serrin's criterion: In the following, we make a stronger assumption on solns u and v (than being L-H weak solns) and see how Serrin's criterion $\frac{2}{q} + \frac{d}{r} = 1$, $r > d$, appears naturally.

Let u, v be mild solns to (NSE) with $u|_{t=0} = v|_{t=0} = u_0$

$$u(t) = e^{tL} u_0 - \int_0^t e^{(t-t')L} \mathbb{P}\left((u\cdot\nabla)u(t')\right) dt'$$

$$v(t) = e^{tL} u_0 - \int_0^t e^{(t-t')L} \mathbb{P}\left((v\cdot\nabla)v(t')\right) dt'.$$

Let $w(t) = u(t) - v(t)$.

Noting that $e^{(t-t')L}\mathbb{P}\left((u\cdot\nabla)u(t')\right) \overset{\text{div-free}}{=} e^{(t-t')L}\mathbb{P}\,\nabla\cdot(u\otimes u)(t')$

i.e. $e^{(t-t')L}\mathbb{P}_{ij}\left(u_k \nabla_k u_j(t')\right) = e^{(t-t')L}\mathbb{P}_{ij}\nabla_k\left(u_k u_j(t')\right)$,

we have

$$\Rightarrow \|w\|_{L_T^\theta L_x^q} \lesssim \|u \otimes w\|_{L_T^\rho L_x^p} + \|w \otimes v\|_{L_T^\rho L_x^p}$$

$$\underset{\text{Hölder}}{\leq} \left(\|u\|_{L_T^\sigma L_x^r} + \|v\|_{L_T^\sigma L_x^r} \right) \|w\|_{L_T^\theta L_x^q}.$$

Here, $\frac{1}{\rho} - \frac{1}{\theta} = \frac{1}{2} - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) > 0$ and

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}, \quad \frac{1}{\sigma} = \frac{1}{\rho} - \frac{1}{\theta}$$

$$\Rightarrow \frac{1}{2} - \frac{d}{2r} = \frac{1}{\sigma} > 0. \quad \text{i.e. } 1 = \frac{2}{\sigma} + \frac{d}{r} \quad (r > d)$$

- If $u, v \in L_T^\sigma L_x^r$, by taking T small,

$$\|w\|_{L_T^\theta L_x^q} \leq \frac{1}{2} \|w\|_{L_T^\theta L_x^q} \Rightarrow w \equiv 0.$$

Summary: Here, we assumed, u, v are mild solns and $u, v \in L_T^\sigma L_x^r$

$\Rightarrow$ uniqueness in $L_T^\theta L_x^q$ (can take $\theta = \sigma$, $q = r$.)

- One can also prove uniqueness of L-H weak solns if $u \in L_T^\sigma L_x^r$, $\frac{2}{\sigma} + \frac{d}{r} = 1$.