

Lec 18 30/03/16

①

$$u(t) = e^{tL} u_0 - \int_0^t e^{(t-t')L} P((u \cdot \nabla)u)(t') dt', \quad L = P\Delta$$

Scaling: let $u^\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$
 $p^\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x)$

$$\|u^\lambda\|_{L_t^q L_x^r(\mathbb{R}_+ \times \mathbb{R}^d)} = \lambda^{1 - \frac{d}{r} - \frac{2}{q}} \|u\|_{L_t^q L_x^r}$$

\Rightarrow scaling inv condition: $\frac{2}{q} + \frac{d}{r} = 1$.

e.g. $L_t^\infty L_x^d$

• For p^λ , scaling inv cond: $\frac{2}{q} + \frac{d}{r} = 2$

e.g. $L_t^\infty L_x^{d/2}$

$$\cdot \underline{d=3} \quad L^3(\mathbb{R}^3) \supset \dot{H}^{1/2}(\mathbb{R}^3) \quad \frac{1}{2} - \frac{1}{3} = \frac{1/2}{3}$$

(2)

Notation: Let $L^p_{df} =$ completion of $C_0^\infty(\mathbb{R}^3; \mathbb{R}^3) \cap \{\operatorname{div} f = 0\}$
under the L^p -norm. (Here, $L^p := (L^p)^3$)

$$\Rightarrow \mathcal{P} = \text{Leray proj} : L^p \rightarrow L^p_{df}, \quad 1 < p < \infty$$

Theorem: (i) $\exists \delta > 0$ s.t. given $u_0 \in L^3(\mathbb{R}^3)$,

$\exists!$ global soln $u \in C([0, \infty); L^3) \cap C((0, \infty); W^{1,3})$
to (NSE). (Also, conti dependence.)

(ii) same in $\dot{H}^{1/2}(\mathbb{R}^3)$

small data
GWP

Main difference from dispersive PDE:

(3)

dissipative

→ $e^{t\Delta}$ (and e^{tL}) has much stronger smoothing, $t > 0$

⇒ can be used to absorb a derivative loss (up to order 2).

(dispersive PDE: local smoothing only.)

Fujita - Kato '60's

$$\Gamma_{u_0} u(t) = e^{tL} u_0 - \int_0^t e^{(t-t')L} \underline{\underline{P}}((u \cdot \nabla) u)(t') dt'$$

$$\nabla \Gamma_{u_0} u(t) = \underline{\underline{\nabla e^{tL}}} u_0 - \underline{\underline{\nabla}} \int_0^t \underline{\underline{e^{(t-t')L}}} \underline{\underline{P}}((u \cdot \nabla) u)(t') dt'$$

$$\Rightarrow \|\Gamma u(t)\|_{L_x^3} \leq C \|u_0\|_{L_x^3} + C \int_0^t (t-t')^{-1/2} \underbrace{\|(u \cdot \nabla) u(t')\|_{L_x^{3/2}}}_{q=3, p=3/2. \text{ Also, boundedness of } \mathcal{P}} dt'$$

$$\leq C \|u_0\|_{L^3} + C \int_0^t (t-t')^{-1/2} (t')^{-1/2} dt' \leq C \|u\|_{L^3} \|\nabla u\|_{L^3}$$

$$\times \|u\|_{L_t^\infty(0,t; L^3)} \times \sup_{0 \leq t' < t} (t')^{1/2} \|\nabla u(t')\|_{L_x^3}$$

Recall: Beta function $B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt$

(4)

$\operatorname{Re} p, \operatorname{Re} q > 0.$

$$\begin{aligned} \Rightarrow \int_0^t (t-t')^{-1/2} (t')^{-1/2} dt' &= \int_0^t \left(1 - \frac{t'}{t}\right)^{-1/2} \left(\frac{t'}{t}\right)^{-1/2} d\frac{t'}{t} \\ &= \int_0^1 (1-\tau)^{-1/2} \tau^{-1/2} d\tau = B(1/2, 1/2) < \infty \end{aligned}$$

$$\begin{aligned} \|\nabla \Gamma u(t)\|_{L_x^3} &\leq C t^{-1/2} \|u_0\|_{L^3} + \int_0^t (t-t')^{-3/4} \underbrace{\|(u \cdot \nabla) u(t')\|_{L_x^2}}_{q=3, p=2} dt' \\ &\leq \underbrace{\|u(t)\|_{L_x^6}}_{\text{G-N ineq}} \|\nabla u(t)\|_{L_x^3} \end{aligned}$$

G-N ineq (or Sobolev & interpolation) $\frac{1}{3} - \frac{1}{6} = \frac{1/2}{3}$

$$\begin{aligned} \|u(t)\|_{L^6} &\lesssim \| |\nabla|^{1/2} u \|_{L^3} \leq \|u(t)\|_{L^3}^{1/2} \|\nabla u(t)\|_{L^3}^{1/2} \\ \frac{1}{2} &= \theta \cdot 0 + (1-\theta) \cdot 1 \Rightarrow \theta = \frac{1}{2} \end{aligned}$$

$$\Rightarrow t^{1/2} \|\nabla \Gamma u(t)\|_{L_x^3} \leq C \|u_0\|_{L_x^3} + C \boxed{t^{1/2} \int_0^t (t-t')^{-3/4} \underline{(t')^{-3/4}} dt'} \quad (5)$$

$$\times \left\{ \sup_{t'} \|u(t')\|_{L_x^3} \right\}^{1/2} \quad \approx B(\frac{1}{4}, \frac{1}{4}) < \infty$$

$$\times \left\{ \sup_{t'} \underline{(t')^{1/2}} \|\nabla u(t')\|_{L_x^3} \right\}^{3/2}$$

• Define

$$\|u\|_X := \|u\|_{L_t^\infty([0, \infty); L_x^3)} + \sup_{t \geq 0} \underline{t^{1/2}} \|\nabla u(t)\|_{L_x^3}$$

$$\Rightarrow \|\Gamma u\|_X \leq C_1 \|u_0\|_{L_x^3} + C_2 \|u\|_X^2$$

Similarly,

$$\|\Gamma u - \Gamma v\|_X \leq C_3 (\|u\|_X + \|v\|_X) \|u - v\|_X$$

$\Rightarrow \Gamma$ is a contraction on $B_R \subset X$, $R \sim \|u_0\|_{L_x^3} \ll 1$.

(ii) Modification for small data GWP in $\dot{H}^{1/2}(\mathbb{R}^3)$

(6)

By interpolation,

$$(*) \quad \|\nabla|^{1/2} e^{tL} f\|_{L_x^q} \lesssim t^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{4}} \|f\|_{L_x^p}$$

$$\Rightarrow \|\Gamma u\|_{\dot{H}^{1/2}} \leq C \|u_0\|_{\dot{H}^{1/2}} + C \int_0^t (t-t')^{-1/2} \|(u \cdot \nabla) u(t')\|_{L_x^{3/2}} dt'$$

$(*)$ with $q=2, p=3/2$

$$-\frac{3}{2}(\frac{2}{3} - \frac{1}{2}) - \frac{1}{4} = -\frac{1}{2}$$

$$\leq C \|u_0\|_{\dot{H}^{1/2}} + C \|u\|_{L_t^\infty \dot{H}^{1/2}} \sup_{t>0} \underbrace{(t')^{1/2} \|\nabla u(t')\|_{L^3}}_{\text{Same as before}}$$

Sobolev $\dot{H}^{1/2} \subset L^3$

and

$$\|\nabla \Gamma u\|_{L^3} \leq C \underline{t^{-1/2}} \|\nabla|^{1/2} u_0\|_{L^2} + \text{same as before.}$$

$q=3, p=2$

$$\Rightarrow \text{use } \|u\|_X = \|u\|_{L_t^\infty(\mathbb{R}_+; \dot{H}^{1/2})} + \sup_{t \geq 0} t^{1/2} \|\nabla u(t)\|_{L_x^3}$$

□

Rmk: ① LWP (for large data) also holds.

② Energy $\frac{1}{2} \int |u|^2 dx$ "satisfies"

$$\frac{1}{2} \int_{\mathbb{R}^d} |u(t)|^2 dx + \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' = \frac{1}{2} \int_{\mathbb{R}^d} |u_0|^2 dx.$$

\Rightarrow provides an a priori estimate only on the L^2 -norm
i.e. (NSE) is energy-supercritical. \uparrow
too weak.

• Serrin-type criterion on uniqueness / (global) regularity

Leray-Hopf weak soln: Given $u_0 \in L^2_{df}$, we say that GWP for smooth solns

u is a Leray-Hopf (weak) soln if

① $u \in L^\infty([0, T]; L^2_{df}) \cap L^2([0, T]; \dot{H}^1)$

② u is a distributional soln to (NSE), i.e.

$$\begin{aligned} & \langle u(t), \phi(t) \rangle_{L^2_x} - \langle u_0, \phi(0) \rangle_{L^2_x} \quad \leftarrow \text{bdry term from IBP} \\ &= \int_0^t \langle u, \partial_t \phi \rangle dt' - \int_0^t \langle \nabla u, \nabla \phi \rangle dt' - \int_0^t \langle (u \cdot \nabla) u, \phi \rangle dt'. \end{aligned} \quad (8)$$

for all div-free test functions $\phi \in C^\infty([0, T]; (C_0^\infty)_{df})$.

Rmk: ① Leray-Hopf soln satisfies the energy inequality:

$$\frac{1}{2} \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2_{t'}(0, t; \dot{H}^1)}^2 \leq \frac{1}{2} \|u_0\|_{L^2}^2.$$

② Leray: global existence of L-H weak soln. (NO uniqueness.)
 large data

③ L-H weak soln u lies in $L_T^q L_x^r$,

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq r \leq \frac{2d}{d-2}.$$

Serrin's criterion ('60, '62)

9

Suppose a L-H weak soln u lies in $L_T^q L_x^r$.

$$\frac{2}{q} + \frac{d}{r} = \underline{1}, \quad \underline{d < r \leq \infty}$$

\Rightarrow The soln is "regular".

• $L_t^\infty L_x^3$: Escauriaza - Seregin - Šverák '03.

Gallagher - Koch - Planchon '13

(harmonic analytic approach \Leftarrow Kenig - Koch '11)
on $L_t^\infty \dot{H}^{1/2}$

Prop: $1 < p \leq q < \infty, r = \frac{d}{2} (\frac{1}{p} - \frac{1}{q})$

(i) $\| e^{tL} f \|_{L_T^\theta L_x^q} \lesssim \| f \|_{L_T^\theta L_x^p}$

(ii) $\| \int_0^t e^{(t-t')L} f(t') dt' \|_{L_T^\theta L_x^q} \lesssim \| f \|_{L_T^\sigma L_x^p}$
 $\frac{1}{\sigma} - \frac{1}{\theta} = 1 - r > 0.$

(iii) $\| \int_0^t \nabla e^{(t-t')L} f(t') dt' \|_{L_T^\theta L_x^q} \lesssim \| f \|_{L_T^{\frac{p}{2}} L_x^p}$
 $\frac{1}{p} - \frac{1}{\theta} = \frac{1}{2} - r > 0.$

Pf (i) We have $\| e^{tL} f(t) \|_{L_x^q} \lesssim t^{-r} \| f(t) \|_{L_x^p}$

⇐ view it as a weak-type bd in time:

$$L^\infty([0, T]; L_x^p) \rightarrow \underbrace{L^{\frac{1}{r}, \infty}([0, T]; L_x^q)}_{\text{weak}}$$

then use Marcinkiewicz interpolation.

$$(ii) \quad \left\| \int_0^t e^{(t-t')L} f(t') dt' \right\|_{L_T^0 L_x^q} \lesssim \|t^{-\delta} * \| f(t) \|_{L_x^p} \| \|_{L_T^0} \quad (11)$$

\Rightarrow use H-L-S ineq.

(iii) same as (ii).

Back to Serrin's criterion: In the following, we make a stronger assumption on solns u and v (than being L -H weak solns) and see how Serrin's criterion $\frac{z}{q} + \frac{d}{r} = 1$, $r > d$, appears naturally.

Let u, v be mild solns to (NSE) with $u|_{t=0} = v|_{t=0} = u_0$

$$u(t) = e^{+tL} u_0 - \int_0^t e^{(t-t')L} \mathcal{P}((u \cdot \nabla) u(t')) dt'$$

$$v(t) = e^{+tL} u_0 - \int_0^t e^{(t-t')L} \mathcal{P}((v \cdot \nabla) v(t')) dt'$$

Let $w(t) = u(t) - v(t)$.

Noting that $e^{(t-t')L} \mathcal{P}((u \cdot \nabla) u(t')) \stackrel{\text{div-free}}{=} e^{(t-t')L} \mathcal{P} \nabla \cdot (u \otimes u)(t')$

$$\text{i.e.} \quad e^{(t-t')L} \mathcal{P}_{ij} (u_k \nabla_k u_j(t')) = e^{(t-t')L} \mathcal{P}_{ij} \nabla_k (u_k u_j(t')),$$

we have

$$\Rightarrow \underline{\|w\|_{L_T^\theta L_x^q}} \lesssim \|u \otimes w\|_{L_T^p L_x^p} + \|w \otimes v\|_{L_T^p L_x^p}$$

$$\leq \underset{\text{Hölder}}{\left(\|u\|_{L_T^\sigma L_x^r} + \|v\|_{L_T^\sigma L_x^r} \right)} \underline{\|w\|_{L_T^\theta L_x^q}}$$

Here, $\frac{1}{p} - \frac{1}{\theta} = \frac{1}{2} - \frac{d}{2} \left(\frac{1}{p} - \frac{1}{q} \right) > 0$ and

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}, \quad \frac{1}{\sigma} = \frac{1}{p} - \frac{1}{\theta}$$

$$\Rightarrow \frac{1}{2} - \frac{d}{2r} = \frac{1}{\sigma} > 0. \quad \text{i.e. } \underline{1 = \frac{2}{\sigma} + \frac{d}{r}} \quad (r > d)$$

• If $u, v \in L_T^\sigma L_x^r$, by taking T small,

$$\|w\|_{L_T^\theta L_x^q} \leq \frac{1}{2} \|w\|_{L_T^\theta L_x^q} \Rightarrow w \equiv 0.$$

Summary: Here, we assumed, u, v are mild solns and $u, v \in L_T^\sigma L_x^r$.

\Rightarrow uniqueness in $L_T^\theta L_x^q$ (can take $\theta = \sigma, q = r$.)

• One can also prove uniqueness of L-H weak solns if $u \in L_T^\sigma L_x^r, \frac{2}{\sigma} + \frac{d}{r} = 1$.