

Lec 17 28/03/16

Sec 6: Navier-Stokes equations

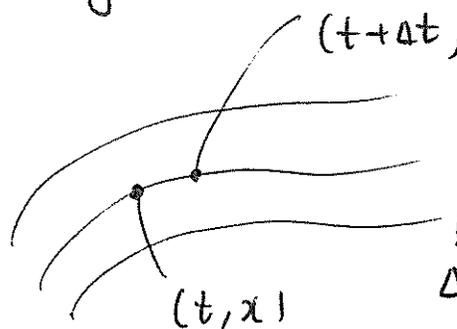
①

$\vec{u}$  = velocity of fluid

Want to take a derivative of  $f(t, x)$ .

①  $\frac{\partial f}{\partial t}$  = usual time deriv (in Eulerian coordinates)  
     $\hat{=}$  observer stays at a fixed pt.

② Lagrangian coordinates = observer travels with a particle.


$$\lim_{\Delta t \rightarrow 0} \frac{f(t+\Delta t, x + \vec{u}\Delta t) - f(t, x)}{\Delta t} =: \frac{Df}{Dt}$$

material derivative  
advective deriv  
convective deriv  
hydrodynamic deriv  
Lagrangian deriv  
Stokes deriv

$\frac{Df}{Dt} = \partial_t f + \vec{u} \cdot \nabla f$

Navier-Stokes equations:

$u = (u_1, u_2, u_3) : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , velocity of fluid

$p : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}$ , pressure.

Newton's 2<sup>nd</sup> law:  $F = ma$ .

$\partial_t u + (u \cdot \nabla) u = -\nabla p + \Delta u + f$ ,  $t > 0$

(LHS) = acceleration  
(RHS) = force

incompressibility

$div u = 0$ ,  $t \geq 0$

friction

$u|_{t=0} = u_0$

$\frac{Du_i}{Dt}$

4 equations, 4 unknowns

$\partial_t u_i + \sum_{k=1}^3 u_k \partial_k u_i = -\partial_i p + \Delta u_i + f_i$

$\sum_i \partial_i u_i = 0$

Set  $f \equiv 0$ .

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Stokes equations :

$$\begin{cases} \partial_t u - \Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

$\Rightarrow$  Take div.

$$\underbrace{\partial_t (\operatorname{div} u) - \Delta (\operatorname{div} u) + \nabla \cdot \nabla p}_{= 0} = 0$$

$\Rightarrow \Delta p = 0 \Rightarrow p$  is a harmonic function.

• Write  $u_0 = \underbrace{v_0}_{\text{div free}} + \underbrace{\nabla p_0}_{\text{curl free } (\nabla \times (\nabla p) = 0)}$  s.t.  $\operatorname{div} v_0 = 0$

$$\Rightarrow -\Delta p_0 = -\operatorname{div} u_0 \quad (\Leftrightarrow p_0 = -(-\Delta)^{-1} \nabla \cdot u_0)$$

$$\Rightarrow \nabla p_0 = -\nabla (-\Delta)^{-1} \nabla \cdot u_0$$

$$\Rightarrow v_0 = u_0 - \nabla p_0 = \underbrace{(\operatorname{Id} + \nabla (-\Delta)^{-1} \nabla \cdot)}_{\text{Leray projection}} u_0$$

Leray projection

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$$\begin{aligned}
V_{0i} &= \sum_{\bar{j}=1}^3 \left( \delta_{ij} + \nabla_i (-\Delta)^{-1} \nabla_j \right) U_{0j} \\
&= \mathcal{F}^{-1} \left( \underbrace{\sum_{\bar{j}=1}^3 \left( \delta_{ij} + \frac{\xi_i \xi_j}{|\xi|^2} \right)}_{\text{matrix valued Fourier mult operator}} \widehat{U_{0j}}(\xi) \right)
\end{aligned}$$

matrix valued Fourier mult operator

Rmk:  $i \frac{\xi_j}{|\xi|}$   $\leadsto R_j =$  Riesz transform  
 "higher dimensional analogue of Hilbert transf."

$$\sqrt{\xi_1^2 + \dots + \xi_d^2}$$

$$R_j f(x) = c. \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^{d+1}} f(y) dy.$$

$$\underline{R_j = L^p \rightarrow L^p, \quad 1 < p < \infty.}$$

$$\sum_{\bar{j}=1}^d R_j^2 = -Id \quad (\text{like } H^2 = -Id)$$

# Helmholtz-Leray decomp / Hodge decomp.

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$$u = \underbrace{v}_{\substack{\uparrow \\ \text{div free}}} + \underbrace{\nabla p}_{\substack{\uparrow \\ \text{curl free}}}$$

$v$  is locally a curl. ( $\nabla \cdot (\nabla \times w) = 0$ )

Here,  $v = \mathcal{P}u$   
 $\uparrow$  Leray proj.

$$\mathcal{P} = \text{Id} + \nabla (-\Delta)^{-1} \nabla \cdot$$

Apply  $\mathcal{P}$  to Stokes eqns.:  $\mathcal{P} \nabla p = \nabla p + \nabla (-\Delta)^{-1} \underbrace{\nabla \cdot \nabla p}_{=\Delta}$   
 $= \nabla p - \nabla p = 0.$

$$\Rightarrow \begin{cases} \partial_t \mathcal{P}u = \Delta \mathcal{P}u \\ \mathcal{P}u|_{t=0} = v_0 \end{cases} \quad \left( \begin{array}{l} \text{div } u = 0 \\ \Rightarrow u = \mathcal{P}u \quad t > 0 \end{array} \right)$$

$$\begin{aligned} \Rightarrow u(t) &= \mathcal{P}u(t) = K_t * \mathcal{P}u_0 \\ &= e^{t\Delta} \mathcal{P}u_0 = \mathcal{P} e^{t\Delta} u_0 \\ &= e^{tL} u_0, \quad \underline{L = \mathcal{P}\Delta} \end{aligned}$$

$$K_t(x) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}}$$

$\uparrow$  Heat Kernel

$$u_i(t) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}} * \left\{ \sum_{j=1}^3 (\delta_{ij} + \nabla_i (-\Delta)^{-1} \nabla_j) u_{0j} \right\}$$

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• We need to study  $e^{tL} = e^{tP\Delta}$ .

Prop ( $L^p - L^q$  estimate) <sup>for heat equation</sup>  $1 \leq p \leq q \leq \infty$ .

$$\bullet \| e^{t\Delta} f \|_{L_x^q} \lesssim t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \| f \|_{L_x^p}, \quad t > 0.$$

$$\bullet \| \partial^\alpha (e^{t\Delta} f) \|_{L_x^q} \lesssim t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{|\alpha|}{2}} \| f \|_{L_x^p}$$

$\alpha = \text{multiindex}, |\alpha| \leq 2.$

Pf: By Young's ineq,

$$\underbrace{\| e^{t\Delta} f \|_{L^q}}_{= K_t * f} \leq \| K_t \|_{L^r} \| f \|_{L^p}, \quad \frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$$

$$\|K_t\|_{L^r}^r = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{r|x|^2}{4t}} dx$$

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$$\sim t^{-\frac{rd}{2} + \frac{d}{2}}$$

change of var

$$y = \left(\frac{r}{4t}\right)^{1/2} x$$

$$dy = \left(\frac{r}{4t}\right)^{d/2} dx$$

$$\Rightarrow \|K_t\|_{L^r} \sim t^{-\frac{d}{2}(1-\frac{1}{r})} = t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$$

• Similarly,  $\|\partial^\alpha (e^{t\Delta} f)\|_{L_x^q} \leq \|\partial^\alpha K_t\|_{L^r} \|f\|_{L^p}$   $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$

$\alpha=1$

$$\|\partial_{|x|} K_t\|_{L^r}^r = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \left(\frac{1}{t}\right)^{r/2} \left(\frac{|x|}{2\sqrt{t}}\right)^r e^{-\frac{r|x|^2}{4t}} dx$$

$$\sim t^{-\frac{rd}{2} + \frac{d}{2} - \frac{r}{2}}$$

change of var.

$$\|\partial_{|x|} K_t\|_{L^r} \sim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q}) - \frac{1}{2}}$$

•  $\alpha=2$ : gain an extra factor of  $\frac{1}{\sqrt{t}}$  by differentiating in Ex 1.

□

Cor ( $L^p - L^q$  estimate for the Stokes eqns).  $L = \mathcal{P}\Delta$  (8)

$$1 < p \leq q < \infty \quad (\text{or } 1 < p < q = \infty)$$

$$\|e^{tL} f\|_{L_x^q} \lesssim t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L_x^p}, \quad t > 0$$

$$\|\partial^\alpha (e^{tL} f)\|_{L_x^q} \lesssim t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{|\alpha|}{2}} \|f\|_{L_x^p}.$$

Pf: Note  $e^{tL} f = e^{t\Delta} \mathcal{P}f$ .

$$\text{Prop} \Rightarrow \|e^{tL} f\|_{L^q} \lesssim t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \underbrace{\|\mathcal{P}f\|_{L^p}}_{\lesssim \|f\|_{L^p} \text{ if } 1 < p < \infty}$$

$\mathcal{P} =$  Leray proj. = matrix valued operator

$$\mathcal{P}_{ij} = \delta_{ij} \cdot \text{Id} + c R_i R_j$$

$\uparrow$   
bdd on  $L^p$ ,  $1 < p < \infty$

□

• Well-posedness of NSE on  $\mathbb{R}^3$ .

⑨

$$(NSE) \begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \Delta u \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

Once we construct  $u$ ,  
take  $\operatorname{div}$   
 $-\Delta p = \operatorname{div}(u \cdot \nabla) u$   
 $\Rightarrow p = (-\Delta)^{-1} \nabla \cdot (u \otimes u)$

$$(u \cdot \nabla) u \stackrel{\text{b/c } \operatorname{div} u = 0}{=} \nabla \cdot (u \otimes u)$$

$$(u \otimes u)_{ij} := u_i u_j$$

$$\begin{aligned} (LHS)_i &= \sum_{j=1}^3 u_j \partial_j u_i + \underbrace{\sum_{j=1}^3 (\partial_j u_j) u_i}_{= \operatorname{div} u = 0} \\ &= \sum_{j=1}^3 \partial_j (u_i u_j) \end{aligned}$$

$\Rightarrow$  Apply Leray proj  $\mathcal{P}$  to (NSE). (and use  $u = \mathcal{P}u$ )

$$\begin{cases} \partial_t u + \mathcal{P}((u \cdot \nabla) u) = Lu \\ u|_{t=0} = \mathcal{P}u_0 \end{cases}$$

$$L = \mathcal{P}\Delta$$

## Duhamel formulation (mild formulation)

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$$u(t) = e^{tL} u_0 - \int_0^t e^{(t-t')L} \mathcal{P}((u \cdot \nabla) u)(t') dt'$$

$\Rightarrow$  We will use Fujita-Kato theory and  
prove small data global well-posedness  
in some critical spaces ( $L^3(\mathbb{R}^3)$ ,  $\dot{H}^{1/2}(\mathbb{R}^3)$ .)